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Killing fields and curvatures of homogeneous Finsler manifolds

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Abstract. In this paper, we study Killing vector fields and flag curvature of homogeneous Finsler spaces. We first give a characterization of Killing vector fields of homogeneous Finsler spaces. Then we present a formula for the Riemann curvature of a homogeneous Finsler space using Killing fields, which is a generalization of the formula for homogeneous Riemannian manifolds.

1. Introduction

A Killing vector field on a manifold endowed with a Riemannian or Finsler structure is a vector field whose local flow acts by (local) isometries. In Riemannian geometry, Killing fields are closely related to isometry groups, geodesics and curvatures in various settings (see, for example, [21]). Moreover, Killing fields are typically used as symmetries in general relativity.

The study of Killing fields is also very important in Finsler geometry. Many similar problems about Killing fields in Finsler geometry have been considered as in Riemannian geometry, and many results have been generalized from Riemannian geometry to Finsler geometry. In [9], the second author studied the zero points of Killing vector fields, and generalized some interesting results of Kobayashi; see also [10]. More recently, the second author and Xu considered the Killing vector fields of constant length in [11]–[13] and used the Killing frames to get

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a formula of S-curvature of homogeneous Finsler spaces in [29]. In [17], Huang–Mo showed that the flag curvature of a Finsler metric does not change under the influence of a Killing vector field in the navigation process. On the other hand, using some techniques of Killing vector fields, we got the complete classification of homogeneous Randers spaces with isotropic S-curvature and positive flag curvature in [15]. Moreover, based on the theory of Killing fields, one can construct many interesting examples of Finsler manifolds with special curvature properties; see [27]–[28] and [5].

It is well-known that calculations in Finsler geometry are generally more complicated than in the Riemannian setting. Huang used a method based on single colored Finsler manifolds (see [16]) and got a formula of the curvature of homogeneous Finsler spaces. The popular way to deduce the curvature of homogeneous Riemannian spaces is based on using Killing fields; see [3]. In this paper, we generalize this method to homogeneous Finsler spaces and get a similar formula for their Riemann curvature.

The article is organized as follows. In Section 2, some preliminaries are presented, including Finsler spaces, the Chern connection and the affine connection associated with a nowhere zero vector field. In Section 3, we present several characterizations of Killing vector fields on a Finsler manifold. Section 4 is devoted to the study of Riemann curvature of homogeneous Finsler manifolds. Using the preparatory results of Sections 2 and 3, we deduce a direct generalization of curvature formula (7.30) in Besse's book [3].

2. Preliminaries

In this section, we present some basic definitions and facts of Finsler geometry. In particular, we recall the Chern connection, which is a useful tool in the study of geometric properties of Finsler spaces. For more details, readers are referred to [1].

2.1. Finsler spaces.

Definition 2.1. Let V be an n-dimensional real vector space. A Minkowski norm on V is a function F on V which is smooth on $V\backslash\{0\}$ and satisfies the following conditions:

(1) $F(u) \geq 0$ for all $u \in V$.

(2) $F(\lambda u) = \lambda F(u)$ for all $\lambda > 0$.

(3) For any $y \in V\backslash\{0\}$, the symmetric bilinear form g_y on V defined by

$$g_y(v,w) := \frac{1}{2}(F^2)''(y)(v,w)$$

is an inner product (i.e., positive definite scalar product).

The pair (V,F) is called a Minkowski space and the mapping

$$g : y \in V\backslash\{0\} \mapsto g_y \in T_2^0(T_yV^0) \cong T_2^0(V)$$

is the fundamental tensor of (V,F).

For example, let $\langle\,,\,\rangle$ be an inner product on V. Define $F(y) = \sqrt{\langle y,y\rangle}$. Then (V,F) is a Minkowski space, called a Euclidean Minkowski space.

It can easily be shown that for a Minkowski norm F, we have $F(u) > 0$ if $u \neq 0$. Furthermore, F is subadditive, i.e.,

$$F(u_1 + u_2) \leq F(u_1) + F(u_2),$$

where equality holds if and only if $u_2 = \alpha u_1$ or $u_1 = \alpha u_2$ for some $\alpha \geq 0$.

Let (V,F) be a Minkowski space. For any $y \neq 0$, the tensor

$$C_y := \frac{1}{4}(F^2)'''(y) \in T_3^0(V)$$

is the Cartan tensor, and $(C_y)' \in T_y^0(V)$ is the extended Cartan tensor of (V,F) at y. By abuse of notation, the latter will also be denoted by C_y. It is well-known that (V,F) is Euclidean if and only if $C_y = 0$ for all $y \in V\backslash\{0\}$.

Given a linear coordinate system $y^1,\dots,y^n$ on V, the components of the fundamental tensor, the Cartan tensor and the extended Cartan tensor of (V,F) relative to $(y^1,\dots,y^n)$ are the functions

$$g_{ij} = \frac{1}{2}[F^2]_{y^iy^j}, \qquad C_{ijk} = \frac{1}{4}[F^2]_{y^iy^jy^k}, \qquad C_{ijkl} = \frac{1}{4}[F^2]_{y^iy^jy^ky^l};$$

all of them are smooth on $V\backslash\{0\}$. The inverse of the matrix (g_{ij}) will be denoted by (g^{ij}).

Theorem 2.2 (Deicke [8], see also [1] and [4])**.** *A Minkowski space (V,F) is Euclidean if and only if $C_k := g^{ij}C_{ijk} = 0$ for all $k \in \{1,2,\dots,n\}$.*

Definition 2.3. Let M be a (connected smooth) manifold. A Finsler metric on M is a function $F\colon TM \to [0,\infty)$ such that

(1) F is C^∞ on the slit tangent bundle $TM_0 := TM\backslash\{0\}$;

(2) $F_p := F|_{T_pM}$ is a Minkowski norm for all $p \in M$.

2.2. The Chern connection and the flag curvature. Let (M,F) be a Finsler space and $(x^1,\dots,x^n)$ be a local coordinate system on an open subset U of M. Then $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}$ form a basis for the tangent space at any point in U, and $(x^1,\dots,x^n,y^1,\dots,y^n)$ is a local coordinate system of the open subset $TU\backslash\{0\}$. Therefore, on $TU\backslash\{0\}$ we have the components g_{ij} and C_{ijk}. Define

$$C^i_{jk}=g^{is}C_{sjk}.$$

The formal Christoffel symbols of the second kind are

$$\gamma^i_{jk}=g^{is}\frac{1}{2}\left(\frac{\partial g_{sj}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{ks}}{\partial x^j}\right).$$

They are smooth functions on $TU\backslash\{0\}$. We can also define some other quantities on $TU\backslash\{0\}$ by

$$N^i_j:=\gamma^i_{jk}y^k-C^i_{jk}\gamma^k_{rs}y^ry^s.$$

Now the slit tangent bundle TM_0 is a fibre bundle over the manifold M with the natural projection π. Since TM is a vector bundle over M, we have a pull-back bundle π^*TM over TM_0.

Theorem 2.4 (Chern [6], see also [1])**.** *The pull-back bundle π^*TM admits a unique linear connection, called the Chern connection, which is torsion-free and almost g-compatible. The coefficients of this connection are*

$$\Gamma^l_{jk}=\gamma^l_{jk}-g^{li}(C_{ijs}N^s_k-C_{jks}N^s_i+C_{kis}N^s_j).$$

Using the notation introduced above, let

$$\frac{\delta}{\delta x^i}:=\frac{\partial}{\partial x^i}-N^s_i\frac{\partial}{\partial y^s},\qquad \delta y^i:=dy^i+N^i_jdx^j.$$

The connection 1-forms ω^i_j and the curvature 2-forms Ω^i_j of the Chern connection are defined on TU_0 as follows:

$$\omega^i_j:=\Gamma^i_{jk}dx^k,\qquad \Omega^i_j:=d\omega^i_j-\omega^k_j\wedge\omega^i_k.$$

We have

$$\Omega^i_j=\frac{1}{2}R_j{}^i{}_{kl}dx^k\wedge dx^l+P_j{}^i{}_{kl}dx^k\wedge\delta y^l+\frac{1}{2}Q_j{}^i{}_{kl}\delta y^k\wedge\delta y^l,$$

where

$$R_j{}^i{}_{kl}=\frac{\delta\Gamma^i_{jl}}{\delta x^k}-\frac{\delta\Gamma^i_{jk}}{\delta x^l}+\Gamma^i_{hk}\Gamma^h_{jl}-\Gamma^i_{hl}\Gamma^h_{jk},\qquad P_j{}^i{}_{kl}=-\frac{\partial\Gamma^i_{jk}}{\partial y^l}.$$

For

$$u = u^i \left(\frac{\partial}{\partial x^i}\right)_p, \quad v = v^i \left(\frac{\partial}{\partial x^i}\right)_p, \quad w = w^i \left(\frac{\partial}{\partial x^i}\right)_p, \quad y = y^i \left(\frac{\partial}{\partial x^i}\right)_p \in T_pM,$$

define

$$R_y(u,v)w = R_j{}^i{}_{kl}(p,y)u^k v^l w^j \left(\frac{\partial}{\partial x^i}\right)_p, \quad P_y(u,v)w = P_j{}^i{}_{kl}(p,y)u^k v^l w^j \left(\frac{\partial}{\partial x^i}\right)_p.$$

Then it is easily seen that $g_y(P_y(y,u)v,w)$ is symmetric with respect to u, v, w, and that

$$g_y(P_y(y,u)v,y) = 0. \tag{2.1}$$

Now we can introduce the notion of flag curvature. A flag on M at $p \in M$ is a pair (σ, y), where σ is a plane in the tangent space T_pM and y is a non-zero vector in σ. The flag curvature of (σ, y) is defined to be

$$K(\sigma,y) := \frac{g_y(R_y(v,y)y,v)}{g_y(y,y)g_y(v,v) - [g_y(y,v)]^2},$$

where $v = v^i(\frac{\partial}{\partial x^i})_p$ is any nonzero vector in σ such that $\sigma = \mathrm{span}\{y,v\}$. It can be shown that $K(\sigma,y)$ is independent of the choice of v.

2.3. An affine connection along a nowhere zero vector field. Let N be an open subset of the Finsler manifold (M,F). Let $\mathfrak{X}(N)$ be the space of smooth vector fields on N, and let $\mathfrak{X}^+(N)$ be the subset of nowhere vanishing vector fields on N. Then for any $Y \in \mathfrak{X}^+(N)$, we can define an affine connection on the tangent bundle TN over N, denoted by ∇^Y, such that the following hold:

(1) ∇^Y is torsion-free:

$$\nabla^Y_U V - \nabla^Y_V U = [U,V], \quad \text{for all } U,V \in \mathfrak{X}(N).$$

(2) ∇^Y is almost metric-compatible:

$$W g_Y(U,V) = g_Y(\nabla^Y_W U, V) + g_Y(U, \nabla^Y_W V) + 2C_Y(\nabla^Y_W Y, U, V)$$

for any vector fields W, U, V on N.

In the above formulas, g_Y, resp. C_Y, is the fundamental tensor, resp. the Cartan tensor, of (M,F) along Y. In a local coordinate system, the connection ∇^Y can be defined by

$$\nabla^Y_U V = \left[U^j \frac{\partial V^i}{\partial x^j} + U^j V^k (\Gamma^i_{jk} \circ Y)\right] \frac{\partial}{\partial x^i},$$

where $U = U^i \frac{\partial}{\partial x^i}, V = V^i \frac{\partial}{\partial x^i}$, and the functions Γ^i_{jk} are the coefficients of the Chern connection. It is worthwhile to point out that the connection ∇^Y is uniquely determined by conditions (1), (2). For details, see [22] and [18].

Lemma 2.5. *For any vector fields U, V, W, Z on N, we have*

$$\begin{aligned} ZC_Y(U,V,W) &= C_Y(\nabla^Y_Z U, V, W) + C_Y(U, \nabla^Y_Z V, W) + C_Y(U, V, \nabla^Y_Z W) \\ &\quad - \frac{1}{2} g_Y(P_Y(Z,W)U,V) - \frac{1}{2} g_Y(P_Y(Z,W)V,U) \\ &\quad - C_Y(U,V,P(Y,W)Z) + C_Y(\nabla^Y_Z Y, U, V, W). \end{aligned} \tag{2.2}$$

Proof. We use the natural local coordinate system, and we write $C_{ijk}(Y) := C_{ijk} \circ Y, \Gamma^i_{jk}(Y) := \Gamma^i_{jk} \circ Y$, etc. Then

$$\begin{aligned} &\frac{\partial}{\partial x^m} C_{ijk}(Y) \\ &= \frac{\partial C_{ijk}}{\partial x^m}(Y) + C_{ijks}(Y)\frac{\partial Y^s}{\partial x^m} = \frac{1}{2}\big[g_{ij}\big]_{x^m y^k}(Y) + C_{ijks}(Y)\frac{\partial Y^s}{\partial x^m} \\ &= \frac{1}{2}\big[g_{is}\Gamma^s_{jm} + g_{sj}\Gamma^s_{im} + 2C_{ijs}\Gamma^s_{ml}y^l\big]_{y^k}(Y) + C_{ijks}(Y)\frac{\partial Y^s}{\partial x^m} \\ &= \left(C_{iks}\Gamma^s_{jm} + C_{sjk}\Gamma^s_{im} - \frac{1}{2} g_{is} P_j{}^s{}_{mk} - \frac{1}{2} g_{sj} P_i{}^s{}_{mk}\right)(Y) \\ &\qquad + C_{ijsk}(Y)\left(\Gamma^s_{ml}(Y)Y^l + \frac{\partial Y^s}{\partial x^m}\right) - C_{ijs}(Y) P_m{}^s{}_{lk}(Y)Y^l + (C_{ijs}\Gamma^s_{mk})(Y). \end{aligned}$$

In the third equation above, we have used the formula (5.29) of Shen's book [23]. Thus the lemma follows. □

Let R^Y be the curvature of the connection ∇^Y defined, as usual, by

$$R^Y(U,V)W = \nabla^Y_U \nabla^Y_V W - \nabla^Y_V \nabla^Y_U W - \nabla^Y_{[U,V]} W,$$

where $U, V, W \in \mathfrak{X}(N)$.

Lemma 2.6. *For any vector fields U, V, W on N, we have*

$$R^Y(U,V)W = R_Y(U,V)W - P_Y(W, \nabla^Y_U Y)V + P_Y(W, \nabla^Y_V Y)U. \tag{2.3}$$

Proof. To prove the lemma, we calculate the expressions of (2.3) in a natural local coordinate system $(x^1,\dots,x^n,y^1,\dots,y^n)$ of TM_0. Let

$$(U,V,W) := \left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^j}\right),\qquad Y = Y^i\frac{\partial}{\partial x^i}.$$

For simplicity, write $\widetilde{\Gamma}^k_{ij} := \Gamma^k_{ij}(Y)$. Then

$$\nabla^Y_{\frac{\partial}{\partial x^k}} Y = \left(\frac{\partial Y^s}{\partial x^k} + Y^i\widetilde{\Gamma}^s_{ik}\right)\frac{\partial}{\partial x^s},$$

and

$$\frac{\partial\widetilde{\Gamma}^k_{ij}}{\partial x^s} = \frac{\partial\Gamma^k_{ij}}{\partial x^s}(Y) + \frac{\partial\Gamma^k_{ij}}{\partial y^t}(Y)\frac{\partial Y^t}{\partial x^s}.$$

Thus

$$\begin{aligned}
&R^Y\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j}\\
&=\left(\frac{\partial\widetilde{\Gamma}^i_{jl}}{\partial x^k} - \frac{\partial\widetilde{\Gamma}^i_{jk}}{\partial x^l} + \widetilde{\Gamma}^i_{hk}\widetilde{\Gamma}^h_{jl} - \widetilde{\Gamma}^i_{hl}\widetilde{\Gamma}^h_{jk}\right)\frac{\partial}{\partial x^i}\\
&=\left(R_j{}^i{}_{kl}(Y) + \frac{\partial\Gamma^i_{jl}}{\partial y^s}\left(\frac{\partial Y^s}{\partial x^k} + N^s_k(Y)\right) - \frac{\partial\Gamma^i_{jk}}{\partial y^s}\left(\frac{\partial Y^s}{\partial x^l} + N^s_l(Y)\right)\right)\frac{\partial}{\partial x^i}\\
&= R_Y\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j} - P_Y\left(\frac{\partial}{\partial x^j},\nabla^Y_{\frac{\partial}{\partial x^k}}Y\right)\frac{\partial}{\partial x^l} + P_Y\left(\frac{\partial}{\partial x^j},\nabla^Y_{\frac{\partial}{\partial x^l}}Y\right)\frac{\partial}{\partial x^k}.
\end{aligned}$$

Here we have used the fact that $Y^i\widetilde{\Gamma}^s_{ik} = N^s_k(Y)$. Therefore, equality (2.3) holds for any natural coordinate vector fields on TM_0, which implies the result. □

3. Killing vector fields of a Finsler manifold

Definition 3.1. Let (M,F) be a Finsler manifold. A diffeomorphism φ of M onto itself is called an isometry if it satisfies

$$F(\varphi(p),\varphi_* y) = F(p,y),\quad \text{for all } p\in M, y\in T_pM.$$

Definition 3.2. A vector field X on a Finsler manifold (M,F) is called a Killing vector field if the local one-parameter group (φ_t) of M generated by X consists of local isometries.

Theorem 3.3. *Let X be a vector field on a Finsler manifold (M, F). Then the following conditions are mutually equivalent:*

(1) *X is a Killing vector field.*

(2) *For any open subset N of M, and $Y \in \mathfrak{X}^+(N)$, $U, V \in \mathfrak{X}(N)$,*

$$Xg_Y(Y, Y) = 2g_Y([X, Y], Y).$$

(3) *For N, Y, U and V as in (2),*

$$Xg_Y(U, Y) = g_Y([X, U], Y) + g_Y(U, [X, Y]).$$

(4) *For N, Y, U and V as in (2),*

$$Xg_Y(U, V) = g_Y([X, U], V) + g_Y(U, [X, V]) + 2C_Y([X, Y], U, V).$$

(5) *For N, Y, U and V as in (2),*

$$g_Y(\nabla^Y_Y X, Y) = 0.$$

(6) *For N, Y, U and V as in (2),*

$$g_Y(\nabla^Y_Y X, U) + g_Y(Y, \nabla^Y_U X) = 0.$$

(7) *For N, Y, U and V as in (2),*

$$g_Y(\nabla^Y_U X, V) + g_Y(U, \nabla^Y_V X) + 2C_Y(\nabla^Y_Y X, U, V) = 0.$$

Proof. Let (φ_t) be the local one-parameter group generated by X. Then we have

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} F(\varphi_t(x), (\varphi_t)_*Y) &= \frac{1}{2F}\frac{d}{dt}\bigg|_{t=0} g_{(\varphi_t(x),(\varphi_t)_*Y)}((\varphi_t)_*Y, (\varphi_t)_*Y) \\ &= \frac{1}{2F}(Xg_Y(Y, Y) - 2g_Y([X, Y], Y)). \end{aligned}$$

Thus the equivalence of (1) and (2) follows. For t small enough, by (2) we have

$$Xg_{Y+tU}(Y + tU, Y + tU) = 2g_{Y+tU}([X, Y + tU], Y + tU).$$

Differentiating with respect to t at $t = 0$, we obtain (3). Similarly, we can get (4) from (3). Taking $U = V = Y$, we conclude (4) $\Rightarrow$ (2).

Using the almost metric compatibility of ∇^Y, we easily get (2) $\Leftrightarrow$ (5), (3) $\Leftrightarrow$ (6) and (4) $\Leftrightarrow$ (7). This completes the proof. □

Further equivalent conditions of the Killing property can be found in [26, Corollary 1].

Theorem 3.4. *Let U, V, W be Killing vector fields on a Finsler manifold (M, F), and let Y be a nowhere zero vector field on an open subset N of M. Then on N we have*

$$\begin{aligned} &2g_Y(\nabla^Y_U V, W) \\ &\quad = g_Y([U,V],W) + g_Y([V,W],U) + g_Y([U,W],V) \\ &\qquad - 2C_Y(\nabla^Y_Y U, V, W) - 2C_Y(\nabla^Y_Y V, W, U) + 2C_Y(\nabla^Y_Y W, U, V). \end{aligned} \tag{3.4}$$

In particular,

$$g_Y(\nabla^Y_V V, W) = g_Y([V,W],V) - 2C_Y(\nabla^Y_Y V, V, W) + C_Y(\nabla^Y_Y W, V, V). \tag{3.5}$$

Proof. By Theorem 3.3, we have

$$\begin{aligned} g_Y(\nabla^Y_U V, W) &= -g_Y(\nabla^Y_W V, U) - 2C_Y(\nabla^Y_Y V, U, W) \\ &= -g_Y([W,V],U) - g_Y(\nabla^Y_V W, U) - 2C_Y(\nabla^Y_Y V, U, W). \end{aligned}$$

Similarly,

$$\begin{aligned} -g_Y(\nabla^Y_V W, U) &= g_Y([U,W],V) + g_Y(\nabla^Y_W U, V) + 2C_Y(\nabla^Y_Y W, U, V), \\ g_Y(\nabla^Y_W U, V) &= -g_Y([V,U],W) - g_Y(\nabla^Y_U V, W) - 2C_Y(\nabla^Y_Y U, W, V). \end{aligned}$$

Taking the summation of the above three equalities, we get (3.4). In particular, setting $U = V$ in (3.4), we get (3.5). □

Corollary 3.5. *Let Y, V, W be Killing vector fields on M, and suppose that Y is a nowhere vanishing vector field on an open subset N of M. Then on N we have*

$$g_Y(\nabla^Y_Y Y, V) = g_Y([Y,V],Y), \tag{3.6}$$

$$\begin{aligned} 2g_Y(\nabla^Y_Y V, W) &= -g_Y([W,V],Y) - g_Y([V,Y],W) \\ &\quad + g_Y([Y,W],V) - 2C_Y(\nabla^Y_Y Y, V, W). \end{aligned} \tag{3.7}$$

Similarly to the Riemannian case, we can prove the following results.

Proposition 3.6. *Let φ and ψ be two isometries of a connected Finsler manifold (M, F). Suppose there exists a point $p \in M$ such that $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi = \psi$.*

Theorem 3.7. *Let (M, F) be a connected Finsler manifold, and let X_1, X_2 be Killing vector fields of (M, F). If there exists a point $p \in M$ and a nowhere zero vector field Y on a neighborhood of p such that $(X_1)_p = (X_2)_p$ and $(\nabla^Y X_1)|_p = (\nabla^Y X_2)|_p$, then $X_1 = X_2$.*

4. Homogeneous Finsler manifolds

Definition 4.1. A Finsler manifold (M, F) is called homogeneous if its isometry group $I(M, F)$ acts transitively on M, i.e., for any $p, q \in M$, there exists an isometry f such that $f(p) = q$.

Suppose that (M, F) is a homogeneous Finsler manifold, and let G be a closed subgroup of $I(M, F)$ which acts transitively on M. We can assume that G is connected. Fix a point $p \in M$. Then the isotropy subgroup $H = \{f \in G | f(p) = p\}$ at the point p is compact and closed in G. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of G and H, respectively. Let Ad be the adjoint representation of G in $\mathfrak{g}$. Since $\mathrm{Ad}(H)$ is compact, there exists an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ that is complementary to $\mathfrak{h}$ in $\mathfrak{g}$, i.e.,

$$\mathfrak{g} = \mathfrak{h} \bigoplus \mathfrak{m}.$$

In this case, the coset space G/H is called reductive.

Given $y \in \mathfrak{g}$, the fundamental vector field corresponding to y, denoted by Y or $\widetilde{y}$, is defined by

$$Y_q = \left.\frac{d}{dt}\right|_{t=0} \exp(ty) \cdot q, \quad q \in M.$$

Since the transformations $\exp(tx)$ are isometries, the vector field X is a Killing vector field. Notice that for any y and z in $\mathfrak{g}$, we have

$$[Y, Z] = -\widetilde{[y, z]}.$$

Identify $\mathfrak{m}$ with T_pM by taking the values of the corresponding Killing vector fields at p. In this way, the isotropy representation of H in T_pM coincides with the restriction of the adjoint representation Ad of H to $\mathfrak{m}$. Then we have

Theorem 4.2. *Let G be a Lie group, H a compact subgroup of G which contains no nontrivial invariant subgroup of G, and $\mathfrak{m}$ an $\mathrm{Ad}H$-invariant complement to $\mathfrak{h}$ in $\mathfrak{g}$. Then there is a one-to-one correspondence between the G-invariant Finsler metrics on G/H and the Minkowski norms on $\mathfrak{m}$ satisfying*

$$F(\mathrm{Ad}(h)y) = F(y), \quad \textit{for all } h \in H, y \in \mathfrak{m}.$$

Moreover, if H is connected, then the above condition is equivalent to each of the following conditions:

(1) $g_y(y, [u, y]) = 0$ *for all* $u \in \mathfrak{h}, y \in \mathfrak{m}\backslash\{0\}$*;*

(2) $g_y(z, [u, y]) + g_y(y, [u, z]) = 0,$

(3) $g_y([u, z], w) + g_y(z, [u, w]) + 2C_y([u, y], z, w) = 0; \quad u \in \mathfrak{h}, y, z, w \in \mathfrak{m}\backslash\{0\}.$

For a proof, see [10].

Given any nonzero vector $y \in \mathfrak{m}$, we define a bilinear operation $U_y : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$g_y(U_y(u,v),w) = \frac{1}{2}(g_y([w,u]_{\mathfrak{m}},v) + g_y(u,[w,v]_{\mathfrak{m}})) + C_y([w,y]_{\mathfrak{m}},u,v), \quad (4.8)$$

where $u,v,w \in \mathfrak{m}$ and $[w,u]_{\mathfrak{m}}$ is the component of the vector $[w,u]$ in the space $\mathfrak{m}$.

In particular, if $u = v := y$, we have a vector $U_y(y,y)$, which is called a spray vector in [16]. With the help of the vector $U_y(y,y)$, we define a linear operator N_y on $\mathfrak{m}$ by

$$g_y(N_y(u),v) = C_y(U_y(y,y),u,v), \quad u,v \in \mathfrak{m}. \quad (4.9)$$

Proposition 4.3. *Let $(G/H,F)$ be a homogeneous Finsler manifold, and let $\mathfrak{m}$ be defined as in 4.2. Let Y,V be the Killing vector fields corresponding to the nonzero vectors $y,v \in \mathfrak{m}$. Then at the point $o = eH$, we have*

$$\nabla^Y_Y V|_o = -\frac{1}{2}[y,v]_{\mathfrak{m}} + U_y(y,v) - N_y(v), \quad (4.10)$$

$$\nabla^Y_V Y|_o = -\frac{1}{2}[v,y]_{\mathfrak{m}} + U_y(y,v) - N_y(v). \quad (4.11)$$

Proof. First, by (3.6), we have

$$\nabla^Y_Y Y|_o = U_y(y,y).$$

Then (4.10) and (4.11) follow from (3.7) and the torsion-freeness of ∇^Y, respectively. □

Theorem 4.4. *Let $(G/H,F)$ be a homogeneous Finsler manifold, and let $\mathfrak{m}$ be defined as in 4.2. Then for any nonzero $y,v \in \mathfrak{m}$, we have*

$$\begin{aligned} &g_y(R_y(v,y)y,v) \\ &\quad = -\frac{3}{4}g_y([y,v]_{\mathfrak{m}},[y,v]_{\mathfrak{m}}) - \frac{1}{2}g_y([y,[y,v]]_{\mathfrak{m}},v) - \frac{1}{2}g_y([v,[v,y]]_{\mathfrak{m}},y) \\ &\qquad + g_y(U_y(y,v),U_y(y,v)) - g_y(U_y(y,y),U_y(v,v)) + g_y(N_y(v),N_y(v) \\ &\qquad - 3[y,v]_{\mathfrak{m}} - 2U_y(y,v)) - C_y(U_y(y,y),U_y(y,y),v,v) - C_y([y,U_y(y,y)]_{\mathfrak{m}},v,v) \\ &\qquad - 2C_y(U_y(y,U_y(y,y)) - N_y(U_y(y,y)),v,v). \end{aligned} \quad (4.12)$$

Proof. Let Y and V be the (fundamental) Killing vector fields on G/H corresponding to y and v. Then, by (2.3) and the torsion-freeness and almost metric compatibility of ∇^Y, we have

$$\begin{aligned}
&g_y(R_y(v,y)y,v)\\
&\quad = g_y(\nabla^Y_V\nabla^Y_Y Y - \nabla^Y_Y\nabla^Y_V Y - \nabla^Y_{[V,Y]}Y, V) - g_y(P_Y(Y,\nabla^Y_Y Y)V,V)\\
&\qquad + g_y(P_Y(Y,\nabla^Y_V Y)Y,V) + g_y(\nabla^Y_V Y, \nabla^Y_Y V)\\
&\quad = Vg_y(\nabla^Y_Y Y, V) - g_y(\nabla^Y_Y Y, \nabla^Y_V V) - Yg_y(\nabla^Y_V Y, V) + g_y(\nabla^Y_V Y, [V,Y])\\
&\qquad + 2C_y(\nabla^Y_Y Y, [V,Y], V) - g_y(P_Y(Y,\nabla^Y_Y Y)V,V)\\
&\quad = Vg_y(\nabla^Y_Y Y, V) - Yg_y(\nabla^Y_V Y, V) - g_y(\nabla^Y_Y Y, \nabla^Y_V V) + g_y(\nabla^Y_V Y, \nabla^Y_V Y)\\
&\qquad - 2g_y(N_y(v), [v,y]_{\mathfrak{m}}) - g_y(P_Y(Y,\nabla^Y_Y Y)V,V). \qquad (4.13)
\end{aligned}$$

Hereinafter $Vg_y(U,W)$ means $Vg_Y(U,W)|_o$. We calculate the first four terms in the last formula (4.13) above. Using (3.6), we get

$$\begin{aligned}
Vg_y(\nabla^Y_Y Y, V) &= Vg_y([Y,V],Y) = g_y([V,[Y,V]],Y) + g_y([Y,V],[V,Y])\\
&= -g_y([v,[v,y]]_{\mathfrak{m}}, y) - g_y([y,v]_{\mathfrak{m}}, [y,v]_{\mathfrak{m}}). \qquad (4.14)
\end{aligned}$$

Since Y is a Killing vector field, by Theorem 3.3, we have

$$g_Y(\nabla^Y_V Y, V) = -C_Y(\nabla^Y_Y Y, V, V).$$

Then by (2.2), we obtain

$$\begin{aligned}
-Yg_y(\nabla^Y_V Y, V) &= YC_y(\nabla^Y_Y Y, V, V)\\
&= C_y(\nabla^Y_Y\nabla^Y_Y Y, V, V) + 2C_y(\nabla^Y_Y Y, \nabla^Y_Y V, V)\\
&\quad + C_y(\nabla^Y_Y Y, \nabla^Y_Y Y, V, V) - g_y(P_Y(Y,\nabla^Y_Y Y)V,V)\\
&= C_y(\nabla^Y_Y\nabla^Y_Y Y, v, v) + 2g_y(N_y(v), \nabla^Y_Y V)\\
&\quad + C_y(U_y(y,y), U_y(y,y), v, v) - g_y(P_Y(Y,\nabla^Y_Y Y)V,V). \qquad (4.15)
\end{aligned}$$

Using (3.5), one gets

$$\begin{aligned}
&- g_y(\nabla^Y_Y Y, \nabla^Y_V V)\\
&\quad = -g_y(\widetilde{U}_y(y,y), \nabla^Y_V V)\\
&\quad = -g_y([V,\widetilde{U}_y(y,y)], V) + 2C_y(\nabla^Y_Y V, \widetilde{U}_y(y,y), V) - C_y(\nabla^Y_Y \widetilde{U}_y(y,y), V, V)
\end{aligned}$$

$$
\begin{aligned}
&= g_y([v, U_y(y,y)]_{\mathfrak{m}}, v) + 2C_y(\nabla_Y^Y V, U_y(y,y), v) - C_y(\nabla_Y^Y \widetilde{U}_y(y,y), v, v) \\
&= -g_y(U_y(y,y), U_y(v,v)) + C_y([U_y(y,y), y]_{\mathfrak{m}}, v, v) \\
&\quad + 2g_y(\nabla_Y^Y V, N_y(v)) - C_y(\nabla_Y^Y \widetilde{U}_y(y,y), v, v). \qquad (4.16)
\end{aligned}
$$

Moreover, taking into account (4.11) and the invariance of ad$\mathfrak{h}$, we find that

$$
\begin{aligned}
& g_y(\nabla_V^Y Y, \nabla_V^Y Y) \\
&= g_y\left(\frac{1}{2}[v,y]_{\mathfrak{m}} - U_y(y,v) + N_y(v), \frac{1}{2}[v,y]_{\mathfrak{m}} - U_y(y,v) + N_y(v)\right) \\
&= \frac{1}{4} g_y([y,v]_{\mathfrak{m}}, [y,v]_{\mathfrak{m}}) + g_y([y,v]_{\mathfrak{m}}, U_y(y,v)) + g_y(U_y(y,v), U_y(y,v)) \\
&\quad + g_y(N_y(v), N_y(v) - [y,v]_{\mathfrak{m}} - 2U_y(y,v)) \\
&= \frac{1}{4} g_y([y,v]_{\mathfrak{m}}, [y,v]_{\mathfrak{m}}) + \frac{1}{2} g_y([[y,v],y]_{\mathfrak{m}}, v) + \frac{1}{2} g_y([[y,v],v]_{\mathfrak{m}}, y) \\
&\quad + g_y(U_y(y,v), U_y(y,v)) + g_y(N_y(v), N_y(v) - [y,v]_{\mathfrak{m}} - 2U_y(y,v)). \qquad (4.17)
\end{aligned}
$$

On the other hand, for any nonzero $z \in \mathfrak{m}$, we have

$$
\begin{aligned}
& g_y(\nabla_Y^Y \nabla_Y^Y Y, z) - g_y(\nabla_Y^Y \widetilde{U}_y(y,y), z) \\
&\quad = Y g_y(\nabla_Y^Y Y, Z) - Y g_y(\widetilde{U}_y(y,y), Z) \\
&\quad = g_y([Y,[Y,Z]], Y) - g_y([Y, \widetilde{U}_y(y,y)], Z) - g_y(\widetilde{U}_y(y,y), [Y,Z]) \\
&\quad = g_y([y,[y,z]]_{\mathfrak{m}}, y) + g_y([y, U_y(y,y)]_{\mathfrak{m}}, z) + g_y(U_y(y,y), [y,z]_{\mathfrak{m}}) \\
&\quad = g_y([y, U_y(y,y)]_{\mathfrak{m}}, z).
\end{aligned}
$$

Thus we obtain

$$
[\nabla_Y^Y \nabla_Y^Y Y - \nabla_Y^Y \widetilde{U}_y(y,y)]|_o = [y, U_y(y,y)]_{\mathfrak{m}}. \qquad (4.18)
$$

Notice that differentiating $\nabla_U^Y V$ with respect to Y along the direction W, one gets $-P_Y(U,W)V$. By (3.7), differentiating with respect to Y along the direction V, we get

$$
\begin{aligned}
g_Y(P_Y(Y,V)V, W) &= g_Y([W,V],V) + g_Y(\nabla_V^Y V, W) + 4C_Y(\nabla_Y^Y V, V, W) \\
&\quad + 2C_Y([V,Y],V,W) - C_Y([Y,W],V,V) + C_Y(\nabla_Y^Y Y, V, V, W).
\end{aligned}
$$

Then, by (4.15) and (4.10),

$$\begin{aligned} & g_y(P_Y(Y,V)V, \nabla^Y_Y Y) \\ & = g_y(P_Y(Y,V)V, \widetilde{U}_y(y,y)) \\ & = g_y(N_y(v), 2U_y(y,v) + [y,v]_{\mathfrak{m}} - 2N_y(v) + \frac{1}{2}C_y([y,U_y(y,y)]_{\mathfrak{m}}, v, v) \\ & \quad + C_y(U_y(y,y), U_y(y,y), v, v) + C_y(U_y(y, U_y(y,y)) - N_y(U_y(y,y)), v, v). \quad (4.19) \end{aligned}$$

Plugging (4.14), (4.15), (4.16), (4.17), (4.19) into (4.13), and taking into account equations (4.18) and (4.10), we obtain (4.12). □

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