Publ. Math. Debrecen
46 / 3-4 (1995), 301–314

Spaces having approximate resolutions consisting of finite-dimensional polyhedra

By VLASTA MATIJEVIĆ (Split)

Abstract. Finitistic spaces are characterized as spaces which admit approximate resolutions consisting of finite-dimensional polyhedra, having PL-bonding maps and dense projections.

§1. Preliminaries

The notion of a finitistic space was introduced by R. G. Swan ([21]) for purposes in cohomological dimension theory. Some topological properties of finitistic spaces are given in [1], [2], [5], [6] and [19]. For instance, in [19] it is shown that for paracompact finitistic spaces, integral cohomological dimension is preserved under the Stone–Čech compactification. The purpose of this paper is to give more information about finitistic spaces using the techniques of approximate resolutions of spaces. The theory of approximate resolutions of spaces was introduced by S. Mardešić and L. R. Rubin ([11]) and further developed by S. Mardešić and T. Watanabe ([15]) in an attempt to overcome some defects in the theory of polyhedral inverse systems of compact non-metric and non-compact spaces.

We shall use the same terminology and notions as in [15]. A *normal* or *numerable* (*open*) *covering* of a (topological) space X is an open covering $\mathcal{U}$ of X which admits a subordinate partition of unity. The set of all normal coverings of X is denoted by $\mathrm{Cov}(X)$. For any subset $A \subseteq X$ and any $\mathcal{U} \in \mathrm{Cov}(X)$, the subset $\bigcup\{U \in \mathcal{U} : \ U \cap A \neq \emptyset\} \subseteq X$ is denoted by $\mathrm{st}(A,\mathcal{U})$ and called the *star* of A (with respect to $\mathcal{U}$). If $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} \leq \mathcal{U}$. For two maps $f, g : Y \to X$ which are

Mathematics Subject Classification: 54B25.
Key words and phrases: Approximate system, approximate resolution, pseudocompact space, covering dimension, finitistic space, $\mathcal{P}$-like space.

$\mathcal{U}$-near, i.e. for every $y \in Y$ there exists a $U \in \mathcal{U}$ with $f(y), g(y) \in U$, we write $(f,g) \leq \mathcal{U}$. If $\mathcal{U} \in \mathrm{Cov}(X)$ and $A \subseteq X$ are given, the normal covering $\{U \cap A : U \in \mathcal{U}\}$ of the subspace A is denoted by $\mathcal{U} \mid A$. We say that a subset $A \subseteq X$ is normally embedded in a space X provided every $\mathcal{V} \in \mathrm{Cov}(A)$ admits a normal covering $\mathcal{U} \in \mathrm{Cov}(X)$ such that $\mathcal{U} \mid A \leq \mathcal{V}$. The order of $\mathcal{U}$, in notation $\mathrm{ord}(\mathcal{U})$, is the largest integer n such that $\mathcal{U}$ contains n elements with non-empty intersection, or ∞ if no such integer exists. We say that $\dim X \leq n$ provided, for any $\mathcal{U} \in \mathrm{Cov}(X)$, there is a $\mathcal{V} \in \mathrm{Cov}(X)$ such that $\mathcal{V} \leq \mathcal{U}$ and $\mathrm{ord}(\mathcal{V}) \leq n+1$.

A space X is called *finitistic* if for each normal covering $\mathcal{U} \in \mathrm{Cov}(X)$ there are a positive integer n and a normal covering $\mathcal{V} \in \mathrm{Cov}(X)$ such that $\mathcal{V} \leq \mathcal{U}$ and $\mathrm{ord}(\mathcal{V}) \leq n$. This means that for each $\mathcal{U} \in \mathrm{Cov}(X)$ of a finitistic space X there exists a refinement $\mathcal{V} \in \mathrm{Cov}(X)$ such that $|N(\mathcal{V})|$ is a finite-dimensional polyhedron, where $N(\mathcal{V})$ denotes the nerve of the covering $\mathcal{V}$. The dimension of $|N(\mathcal{V})|$ depends on $\mathcal{U}$. By the definition of finitistic spaces, it is clear that every compact space and every finite-dimensional space is finitistic. Finitistic spaces need not be finite-dimensional; any compact infinite-dimensional space provides an example.

We now quote from [15] the main definitions and results concerning approximate resolutions.

An *approximate inverse system* $\mathcal{X}$ is a collection $\{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ consisting of

(i) a preordered indexing set $\Lambda = (\Lambda, \leq)$ (it need not be antisymmetric), which is directed and unbounded (i.e. has no upper bound).

(ii) for each $\lambda \in \Lambda$, X_λ is a topological space and $\mathcal{U}_\lambda \in \mathrm{Cov}(X_\lambda)$.

(iii) for any two related indices $\lambda \leq \lambda'$, $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ is a (continuous) map ($p_{\lambda\lambda} = \mathrm{id}_{X_\lambda}$ is the identity map on X_λ).

Furthermore, the following three conditions must be satisfied:

(A1) For any three related indices $\lambda \leq \lambda' \leq \lambda''$,

$$(p_{\lambda\lambda'} p_{\lambda'\lambda''}, p_{\lambda\lambda''}) \leq \mathcal{U}_\lambda\,;$$

(A2) For each $\lambda \in \Lambda$ and each $\mathcal{U} \in \mathrm{Cov}(X_\lambda)$, there exists a $\lambda' \geq \lambda$ such that

$$(p_{\lambda\lambda_1} p_{\lambda_1\lambda_2}, p_{\lambda\lambda_2}) \leq \mathcal{U}, \quad \text{whenever} \quad \lambda_2 \geq \lambda_1 \geq \lambda'\,;$$

(A3) For each $\lambda \in \Lambda$ and each $\mathcal{U} \in \mathrm{Cov}(X_\lambda)$, there exists a $\lambda' \geq \lambda$ such that

$$\mathcal{U}_{\lambda''} \leq p^{-1}_{\lambda\lambda''}\mathcal{U} = \left\{p^{-1}_{\lambda\lambda''}(U);\ U \in \mathcal{U}_\lambda\right\} : \quad \text{whenever} \quad \lambda'' \geq \lambda'\,.$$

An *approximate mapping* $\boldsymbol{p}=\{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *of a topological space* X *into an approximate inverse system* $\mathcal{X}$ is a family of maps $p_\lambda : X \to X_\lambda,\ \lambda \in \Lambda$, such that the following condition holds:

(AS) For any $\lambda \in \Lambda$ and any $\mathcal{U} \in \mathrm{Cov}(X_\lambda)$, there exists a $\lambda' \geq \lambda$ such that

$$(p_{\lambda\lambda''}p_{\lambda''}, p_\lambda) \leq \mathcal{U}, \quad \text{for every} \quad \lambda'' \geq \lambda' .$$

Let POL denote the collection of all polyhedra (endowed with the CW-topology).

An *approximate resolution of a space* X is an approximate mapping $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ of X into an approximate system $\mathcal{X}$ satisfying the following two conditions:

(R1) For any $P \in$ POL, $\mathcal{V} \in \mathrm{Cov}(P)$ and mapping $f : X \to P$ there is a $\lambda \in \Lambda$ such that, for every $\lambda' \geq \lambda$, there exists a mapping $g : X_{\lambda'} \to P$ satisfying $(gp_{\lambda'}, f) \leq \mathcal{V}$.

(R2) For every $P \in$ POL and $\mathcal{V} \in \mathrm{Cov}(P)$ there is a $\mathcal{V}' \in \mathrm{Cov}(P)$ such that for any $\lambda \in \Lambda$ and any two maps $g, g' : X_\lambda \to P$, for which $(gp_\lambda, g'p_\lambda) \leq \mathcal{V}'$, there exists a $\lambda' \geq \lambda$ such that $(gp_{\lambda\lambda''}, g'p_{\lambda\lambda''}) \leq \mathcal{V}$ for any $\lambda'' \geq \lambda'$.

An approximate resolution of a space X can be characterized by conditions of a different kind. Instead of (R1) and (R2), which are often difficult to verify, more convenient are the following two equivalent conditions ([15], Theorem 2.8):

(B1) For every $\mathcal{U} \in \mathrm{Cov}(X)$ there is a $\lambda \in \Lambda$ such that $p_{\lambda'}^{-1}\mathcal{U}_{\lambda'} \leq \mathcal{U}$, for every $\lambda' \geq \lambda$.

(B2) For each $\lambda \in \Lambda$ there is a $\lambda' \geq \lambda$ such that, for every $\lambda'' \geq \lambda'$, $p_{\lambda\lambda''}(X_{\lambda''}) \subseteq \mathrm{st}(p_\lambda(X), \mathcal{U}_\lambda)$.

§2. Spaces having compact approximate resolutions

First, we shall describe a special class of finitistic spaces using approximate resolutions.

A topological space X is called *pseudocompact* if X is a Tychonoff space and every continuous real-valued function defined on X is bounded.

Theorem 2.1. *Let X be a Tychonoff space. Then the following statements are equivalent.*

(i) *X is pseudocompact.*

(ii) *X admits an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$, where all X_λ are compact polyhedra, all bonding maps $p_{\lambda\lambda'}$ are surjective and Λ is cofinite.*

(iii) *X admits an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$, where all X_λ are compact Hausdorff spaces.*

Our proof of Theorem 2.1 uses a result from [12] and the following two propositions:

Proposition 2.2. *Let $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an approximate resolution of a space X and let $A \subseteq X$. If A is a dense subset of X, normally embedded in X, then $\boldsymbol{p} \mid A = \{p_\lambda \mid A : \lambda \in \Lambda\} : A \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is an approximate resolution of A. Conversely, if $\boldsymbol{p} \mid A = \{p_\lambda \mid A : \lambda \in \Lambda\} : A \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is an approximate resolution of A, then A is normally embedded in X. Furthermore, A is a dense subset of X if X is a Tychonoff space.*

Proof. Assume that A is a dense subset of X, normally embedded in X. We have to prove that $\boldsymbol{p} \mid A = \{p_\lambda \mid A : \lambda \in \Lambda\} : A \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is an approximate resolution of A. Consequently, we need to verify (B1) and (B2) or equivalently (B1)* and (B2)* (see 2.9, 2.10 of [15]).

(B1)* Choose any $\mathcal{U} \in \operatorname{Cov}(A)$. We need to find an index $\lambda \in \Lambda$ and a covering $\mathcal{V} \in \operatorname{Cov}(X_\lambda)$ such that $(p_\lambda \mid A)^{-1}\mathcal{V} \leq \mathcal{U}$. Since $A \subseteq X$ is normally embedded in X, there exists $\mathcal{U}_1 \in \operatorname{Cov}(X)$ such that

$$\mathcal{U}_1 \mid A \leq \mathcal{U} \ . \tag{1}$$

By assumption, $\boldsymbol{p}$ is an approximate resolution of X and therefore, by (B1)*, there exist $\lambda \in \Lambda$ and $\mathcal{V} \in \operatorname{Cov}(X_\lambda)$ such that

$$p_\lambda^{-1}\mathcal{V} \leq \mathcal{U}_1 \ . \tag{2}$$

We claim that $(p_\lambda \mid A)^{-1}\mathcal{V} \leq \mathcal{U}$. Let $V \in \mathcal{V}$ be given. Then, by (1) and (2), $(p_\lambda \mid A)^{-1}(V) = p_\lambda^{-1}(V) \cap A \subseteq U_1 \cap A \subseteq U$ for some, $U_1 \in \mathcal{U}_1$ and $U \in \mathcal{U}$, which establishes $(p_\lambda \mid A)^{-1}\mathcal{V} \leq \mathcal{U}$.

(B2)* Choose any $\lambda \in \Lambda$ and $\mathcal{U} \in \operatorname{Cov}(X_\lambda)$. We need to find an index $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \operatorname{st}(p_\lambda(A), \mathcal{U})$. By assumption, $\boldsymbol{p}$ is an approximate resolution and therefore, by (B2)* for $\boldsymbol{p}$, there exists a $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \operatorname{st}(p_\lambda(X), \mathcal{U})$. Since $\bar{A} = X$, we get

$$p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \operatorname{st}(p_\lambda(X), \mathcal{U}) = \operatorname{st}(p_\lambda(\bar{A}), \mathcal{U}) \subseteq \operatorname{st}(\overline{p_\lambda(A)}, \mathcal{U}) = \operatorname{st}(p_\lambda(A), \mathcal{U}) \,,$$

which establishes (B2)*.

Assume, now, that $\boldsymbol{p} \mid A$ is an approximate resolution of A. We must prove that A is normally embedded in X. Let $\mathcal{V}$ be a normal covering of A.

Since $\boldsymbol{p} \mid A$ is an approximate resolution, by (B1)* (which is equivalent to (B1)), there exist a $\lambda \in \Lambda$ and a $\mathcal{W} \in \mathrm{Cov}(X_\lambda)$ such that

$$(3) \qquad (p_\lambda \mid A)^{-1}\,\mathcal{W} \leq \mathcal{V}\ .$$

Put $\mathcal{U} = p_\lambda^{-1}\,\mathcal{W} \in \mathrm{Cov}(X)$. We claim that $\mathcal{U} \mid A \leq \mathcal{V}$. Let $U \cap A$, $U \in \mathcal{U}$, be any member of the normal covering $\mathcal{U} \mid A$. Then by (3), $U \cap A = p_\lambda^{-1}(W) \cap A = (p_\lambda \mid A)^{-1}(W) \subseteq V$, for some $W \in \mathcal{W}$ and $V \in \mathcal{V}$, which establishes $\mathcal{U} \mid A \leq \mathcal{V}$. This shows that A is normally embedded in X. Furthermore, let X be a Tychonoff space. We shall prove that A is dense in X. Assume that A is not dense in X, i.e. there exists a point $x \in X \setminus \bar{A}$. Since X is a Tychonoff space, there exists a mapping $f : X \to I = [0,1]$ such that $f(x) = 0$ and $f(\bar{A}) = 1$. Let $\mathcal{W} = \{[0,1), (0,1]\}$. Then $\mathcal{U} = f^{-1}\,\mathcal{W}$ is a normal covering of X such that $x \notin \mathrm{st}(\bar{A},\mathcal{U}) = \mathrm{st}(A,\mathcal{U})$. Since $\boldsymbol{p}$ is an approximate resolution of X, by (B1)*, there exist a $\lambda \in \Lambda$ and a normal covering $\mathcal{V}$ of X_λ such that $p_\lambda^{-1}\,\mathcal{V} \leq \mathcal{U}$. We claim that

$$(4) \qquad p_\lambda(x) \notin \mathrm{st}(p_\lambda(A), \mathcal{V})\,.$$

Indeed, if a member $V \in \mathcal{V}$ meets $p_\lambda(A)$, then $p_\lambda^{-1}(V)$ meets A, and therefore it is contained in $f^{-1}((0,1])$. Since $x \notin f^{-1}((0,1])$, $p_\lambda(x) \notin V$, which establishes (4). Choose a normal covering $\mathcal{V}_1$ of X_λ such that $\mathrm{st}\,\mathcal{V}_1 \leq \mathcal{V}$. By (AS) for $\boldsymbol{p}$ and (B2)* for $\boldsymbol{p} \mid A$, there exists a $\lambda' \in \Lambda$ such that

$$(5) \qquad (p_{\lambda\lambda'}p_{\lambda'}, p_\lambda) \leq \mathcal{V}_1 \quad \text{and}$$

$$(6) \qquad P_{\lambda\lambda'}(X_{\lambda'}) \subseteq \mathrm{st}(p_\lambda(A), \mathcal{V}_1)\,.$$

Now (5) and (6) imply

$$\begin{aligned} p_\lambda(x) \in p_\lambda(X) &\subseteq \mathrm{st}(p_{\lambda\lambda'}p_{\lambda'}(X), \mathcal{V}_1) \subseteq \mathrm{st}(p_{\lambda\lambda'}(X_{\lambda'}), \mathcal{V}_1) \subseteq \\ &\subseteq \mathrm{st}(\mathrm{st}(p_\lambda(A), \mathcal{V}_1), \mathcal{V}_1) \subseteq \mathrm{st}(p_\lambda(A), \mathrm{st}\,\mathcal{V}_1) \subseteq \mathrm{st}(p_\lambda(A), \mathcal{V})\,, \end{aligned}$$

which contradicts (4). This completes the proof of the proposition. □

Proposition 2.3. *Let X be a Tychonoff space. X is pseudocompact if and only if it is normally embedded in its Stone–Čech compactification βX.*

Proof. Let $\mathcal{U} \in \mathrm{Cov}(X)$ be a normal covering of the space X. We need to find a normal covering $\mathcal{V} \in \mathrm{Cov}(\beta X)$ such that $\mathcal{V} \mid X \leq \mathcal{U}$. Since $\mathcal{U}$ is a normal covering of X, there exist a metric space Y, a continuous mapping $f : X \to Y$ and an open covering $\mathcal{W} \in \mathrm{Cov}(Y)$ such that $f^{-1}W \leq \mathcal{U}$. The space $f(X) \subseteq Y$ is pseudocompact and metric and therefore, it is a compact space ([4], Theorems 3.10.21 and 5.1.20). Consequently, $f : X \to f(X)$ is a continuous map of the Tychonoff space X to the compact space $f(X)$. Therefore, f is extendable to a mapping $\bar{f} : \beta X \to f(X) \subseteq Y$. Put $\mathcal{V} = (\bar{f})^{-1}\,\mathcal{W} \in \mathrm{Cov}(\beta X)$. Then, $\mathcal{V} \mid X = (\bar{f})^{-1}\,\mathcal{W} \mid X = f^{-1}\,\mathcal{W} \leq \mathcal{U}$, which shows that X is normally embedded in βX.

Let a Tychonoff space X be a normally embedded in its Stone–Čech compactification βX. Let $f : X \to \mathbb{R}$ be any real-valued function. Consider the open covering $\mathcal{U} = \{(i, i+2) : i \in \mathbb{Z}\} \in \mathrm{Cov}(\mathbb{R})$. Then $f^{-1}\,\mathcal{U}$ is a normal covering of X. Since X is normally embedded in βX, there exists a finite covering $\mathcal{V} = \{V_1, \ldots, V_n\} \in \mathrm{Cov}(\beta X)$ such that $\mathcal{V} \mid X \leq f^{-1}\,\mathcal{U}$. Now, we obtain $X = X \cap (V_1 \cup \cdots \cup V_n) \subseteq f^{-1}U_1 \cup \cdots \cup f^{-1}U_n = f^{-1}(U_1 \cup \cdots \cup U_n)$, where each U_i is some $(k, k+2) \subseteq \mathbb{R}$. This implies that $f(X) \subseteq U_1 \cup \cdots \cup U_n$, i.e. $f(X)$ is bounded. $\square$

Proof of Theorem 2.1. $(i) \implies (ii)$ Let X be a pseudocompact space and βX its Stone–Čech compactification. Since βX is a compact Hausdorff space it admits an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$, where all X_λ are compact polyhedra, all $p_{\lambda\lambda'}$ are (irreducible) surjections and Λ is cofinite ([12], Theorem 1). Actually, Mardešić and Rubin constructed, for each compact Hausdorff space Y, such an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : Y \to \mathcal{X} = \{X_\lambda, \varepsilon_\lambda, p_{\lambda\lambda'}, \Lambda\}$ with numerical meshes ε_λ. Since X_λ are metric compacta and Λ is cofinite, it is possible to replace the numerical meshes by open coverings $\mathcal{U}_\lambda$ and thus obtain an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\}$: $Y \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ ([13], Theorem 1 and Remark 1.). Now Proposition 2.2 and Proposition 2.3 imply that $\boldsymbol{p} \mid X = \{p_\lambda \mid X : \lambda \in \Lambda\}$: $X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is the desired approximate resolution of X. Note that there exist compact Hausdorff spaces Y which do not admit (commutative) polyhedral resolutions with surjective bonding maps ([17], [18]).

$(ii) \implies (iii)$ is obvious.

$(iii) \implies (i)$ Let X be a Tychonoff space, which admits an approximate resolution consisting of compact spaces and let $f : X \to \mathbb{R}$ be any real-valued function. Let $\mathcal{U} = \{(i, i+2) : i \in \mathbb{Z}\} \in \mathrm{Cov}(\mathbb{R})$. By (R1)*, there exist a $\lambda \in \Lambda$ and a mapping $g : X_\lambda \to \mathbb{R}$ such that $(f, gp_\lambda) \leq \mathcal{U}$. Then $f(X) \subseteq \mathrm{st}(gp_\lambda(X), \mathcal{U}) \subseteq \mathrm{st}(g(X_\lambda), \mathcal{U})$. Since X_λ is a compact space $g(X_\lambda) \subseteq \mathbb{R}$ is a compact subset of $\mathbb{R}$ and it is therefore bounded. The choice of $\mathcal{U}$ guaranties the boundedness of $\mathrm{st}(g(X_\lambda), \mathcal{U})$ and thus, also the boundedness of $f(X)$. $\square$

Remark 2.4. If we omit the requirements on surjectivity of the bonding maps, then Theorem 2.1 is true already for commutative resolutions ([7] and [9], Theorem 1).

Corollary 2.5. *Every pseudocompact space X is finitistic.*

PROOF. Let X be pseudocompact and $\mathcal{U} \in \mathrm{Cov}(X)$ arbitrary. By Theorem 2.1, X admits an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ consisting of compact spaces. By (B1)* there exist an index λ and a finite covering $\mathcal{V} \in \mathrm{Cov}(X)$ such that $p_\lambda^{-1}\mathcal{V} \leq \mathcal{U}$. Then $p_\lambda^{-1}\mathcal{V}$ is the desired finite-dimensional refinement. $\square$

Note that Corollary 2.5 also easily follows from Proposition 2.3.

Using Corollary 2.5, it is easy to answer in the negative Hattori's Question 1. from [6]: Is every normal finitistic space paracompact?

Let $X = [0, \omega_1)$ be the space of all countable ordinal numbers. X is normal pseudocompact space, but not compact. Consequently, it is a finitistic space. However, X is not paracompact because paracompact pseudocompact spaces are compact ([4], Theorem 5.1.20 and 3.10.21).

§3. Spaces having finite-dimensional approximate resolutions

Let K be a simplicial complex and let $f, g : X \to |K|$ be mappings into its geometric realization $|K|$ endowed with the CW-topology. We say that g is a K-*modification* of f if, for every point $x \in X$ and (closed) simplex $\sigma \in K$, $f(x) \in \sigma$ implies $g(x) \in \sigma$. Note that the relation "to be a K-modification" is reflexive and transitive but, generally, not symmetric or antisymmetric.

Let K be a simplicial complex. We say that a simplex $\sigma \in K$ is *principal* if it is not a proper face of any other simplex of K.

Let K be a simplicial complex, $|K|$ its geometric realization and let $x \in |K|$. By $\mathrm{st}(x, K)$ we denote the open set $\bigcup\limits_{x \in \sigma} \mathrm{Int}\,\sigma \subseteq |K|$.

Proposition 3.1. *Let K be a simplicial complex and let $\{x_a : a \in A\} \subseteq |K|$ be a family of points x_a such that $\mathrm{st}(x_a, K) \cap \mathrm{st}(X_{a'}, K) = \emptyset$, for each $a \neq a'$. Then there exists a retraction $\varphi : |K| \setminus \{x_a : a \in A\} \to |K| \setminus \Big(\bigcup\limits_{a \in A} \mathrm{st}(x_a, K)\Big)$, which is a K-modification of the inclusion map $i : |K| \setminus \{x_a : a \in A\} \to |K|$.* $\square$

Proposition 3.2. *Let X be a topological space, K a finite-dimensional simplicial complex and $f : X \to |K|$ a map. Then there exist a subcomplex L of K and a dense K-modification $g : X \to |L|$ of f.*

PROOF. If $\overline{f(X)} = |L|$, where L is a subcomplex of K, then L and $g = f$ satisfy the proposition (if K is 0-dimensional, g is a surjection). So, without loss of generality, we may assume that $\dim K = n \geq 1$ and

$\overline{f(X)}$ is a proper subset of $|M|$, where $|M|$ is the carrier of $\overline{f(X)}$ (i.e. M is the minimal subcomplex of K, such that $|M|$ contains $\overline{f(X)}$). Since $\overline{f(X)} \subset |M|$ there exists a principal simplex $\sigma \in M$ such that $\operatorname{Int}\sigma \setminus f(X) \neq \emptyset$, for otherwise $\overline{f(X)}=|M|$. Let $\mathcal{A}_n$ be the family of all simplexes $\sigma \in M$ such that $\dim\sigma = n$ and $\operatorname{Int}\sigma \setminus f(X) \neq \emptyset$. If $\mathcal{A}_n$ is non-empty, we can choose points $x_\sigma \in \operatorname{Int}\sigma \setminus f(X)$ for each $\sigma \in \mathcal{A}_n$. Then $\{x_\sigma : \sigma \in \mathcal{A}_n\}$ satisfies the conditions in Propositions 3.1. Therefore, there exist a proper subcomplex M_n of M and a mapping $\varphi_n : |M| \setminus \{x_\sigma : \sigma \in \mathcal{A}_n\} \to |M_n| = |M| \setminus \Big(\bigcup_{\sigma\in\mathcal{A}_n} \operatorname{st}(x_\sigma, M)\Big)$, which is an M-modification of the inclusion $f : |M| \setminus \{x_\sigma : \sigma \in \mathcal{A}_n\} \to |M|$. So, $\varphi_n f$ is a K-modification of f. M_n is a subcomplex of M having the property that $\varphi_n f(X)$ contains the interiors of all n-dimensional simplexes of M_n. Let $\mathcal{A}_{n-1}$ be the family of all principal simplexes $\sigma \in M_n$ such that $\dim\sigma = n-1$ and $\operatorname{Int}\sigma \setminus \varphi_n f(X) \neq \emptyset$. If $\mathcal{A}_{n-1}$ is a nonempty family, we can repeat the procedure and obtain a subcomplex M_{n-1} of M_n and a mapping $\varphi_{n-1} : |M_n| \setminus \{x_\sigma : \sigma \in \mathcal{A}_{n-1}\} \to |M_{n-1}|$ which is an M_n-modification of the inclusion $i : |M_n| \setminus \{x_\sigma : \sigma \in \mathcal{A}_{n-1}\} \to |M_n|$. So, $\varphi_{n-1}\varphi_n f$ is a M_n-modification of $\varphi_n f$ and also a K-modification of f. M_{n-1} is a subcomplex of M_n having the property that $\varphi_{n-1}\varphi_n f(X)$ contains the interiors of all n-dimensional and all $(n-1)$-dimensional simplexes of M_{n-1}. In the same manner repeating n times this procedure, we obtain subcomplexes $M_1 \leq M_2 \leq \cdots \leq M$ of M and a mapping $\varphi_1\varphi_2 \circ \cdots \circ \varphi_{n-1}\varphi_n f : X \to |M_1|$ having the property that $\varphi_1\varphi_2 \circ \cdots \circ \varphi_{n-1}\varphi_n f(X)$ contains the interiors of all principal simplexes of M_1 and $\varphi_1\varphi_2 \circ \cdots \circ \varphi_{n-1}\varphi_n f$ is a K-modification of f. Hence, $L = M_1$ and $g = \varphi_1\varphi_2 \circ \cdots \circ \varphi_{n-1}\varphi_n f : X \to |M_1|$ have the required properties. □

Remark 3.3. Let X be a space, let $\mathcal{U}$ be a normal covering of X with the property $\operatorname{ord}\mathcal{U} \leq n$, let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$ and let $f : X \to |N(\mathcal{U})|$ be a canonical map. Then there exist a subcomplex N of $N(\mathcal{U})$ and a dense $N(\mathcal{U})$-modification $g : X \to |N|$ of f. The mapping $g : X \to |N| \subseteq |N(\mathcal{U})|$ is also canonical.

Lemma 3.4. *Let X be a finitistic space, let $P_1, \ldots, P_n$ be polyhedra, let $f_1 : X \to P_1, \ldots, f_n : X \to P_n$ be mappings and let $\mathcal{U}_1 \in \operatorname{Cov}(P_1), \ldots, \mathcal{U}_n \in \operatorname{Cov}(P_n)$ be open coverings. Then there exist a finite-dimensional polyhedron P, a dense map $f : X \to P$ and PL-mappings $p_1 : P \to P_1, \ldots, p_n : P \to P_n$ such that $(f_i, p_i f) \leq \mathcal{U}_i$, for $i = 1, \ldots, n$.*

Proof. For each $i = 1, \ldots, n$ choose a triangulation K_i of P_i so fine that the covering $\bar{\mathcal{S}}_i$ formed by all the closures of the members of $\mathcal{S}_i = \{\operatorname{st}(v, K_i) : v \in K_i^0\} \in \operatorname{Cov}(|K_i| = P_i)$ refines $\mathcal{U}_i$, i.e. $\bar{\mathcal{S}}_i \leq \mathcal{U}_i$ (see, e.g., [14], Theorem 4, Appendix 1). Let $\mathcal{V} \in \operatorname{Cov}(X)$ be a normal covering of X such that $\mathcal{V} \leq f_i^{-1}(\mathcal{S}_i),\ i = 1, \ldots, n$. Since X is a finitistic space, there exist an integer n and a normal covering $\mathcal{U} \in \mathrm{Cov}(X)$ such that $\mathcal{U} \leq \mathcal{V}$ and $\mathrm{ord}(\mathcal{U}) \leq n$. Let $g : X \to |N(\mathcal{U})|$ be a canonical map. By Proposition 3.2 there exist a finite-dimensional polyhedron $P = |N|$ and a dense mapping $f : X \to |N| \subseteq |N(\mathcal{U})|$ which is an $N(\mathcal{U})$-modification of g.

Now, we will define mappings $\pi_i : N(\mathcal{U})^0 = \mathcal{U} \to K_i^0,\ i = 1, 2, \ldots, n$, in the following way. To a vertex $U \in N(\mathcal{U})^0$ we assign a vertex $v = \pi_i(U) \in K_i^0$ such that $U \subseteq f_i^{-1}(\mathrm{st}(v, K_i)),\ i = 1, \ldots, n$.

Claim 1. For each $i = 1, \ldots, n,\ \ \pi_i : N(\mathcal{U})^0 \to K_i^0$ is a simplicial mapping.

Indeed, let $U_1, \ldots, U_m$ be vertices of $N(\mathcal{U})^0$, which span a simplex of $N(\mathcal{U})$. Then $U_1 \cap U_2 \cap \ldots U_m \neq \emptyset$ and therefore, $\emptyset \neq U_1 \cap \cdots \cap U_m \subseteq f_i^{-1}(\mathrm{st}(\pi_i(U_1), K_i^0)) \cap \cdots \cap f_i^{-1}(\mathrm{st}(\pi_i(U_m), K_i^0)) \subseteq f_i^{-1}(\mathrm{st}(\pi_i(U_1), K_i^0) \cap \cdots \cap \mathrm{st}(\pi_i(U_m), K_i^0))$. Now, we obtain $\mathrm{st}(\pi_i(U_1), K_i^0) \cap \cdots \cap \mathrm{st}(\pi_i(U_m), K_i^0) \neq \emptyset$, which shows that the vertices $\pi_i(U_1), \ldots, \pi_i(U_m)$ span a simplex of K_i. For each $i = 1, \ldots, n$, the mapping π_i induces a mapping $|\pi_i| : |N(\mathcal{U})| \to |K_i|$. Put $p_i = |\pi_i| \mid (|N| = P) : P \to |K_i| = P_i$. Note that each p_i is a PL-mapping.

Claim 2. For each $i = 1, \ldots, n$, $p_i f : X \to |K_i|$ is a K_i-modification of f_i.

Let $x \in X$ be an arbitrary point of X and $\sigma = [v_1, \ldots, v_k] \in K_i$ a simplex of K_i such that $f_i(x) \in \sigma$. We need to prove that $p_i f(x) \in \sigma$. Let $U_1, \ldots, U_s$ be all the members of the covering $\mathcal{U}$ which contain x. Then $g(x) \in \tau = [U_1, \ldots, U_s] \in N(\mathcal{U})$. Since $f : X \to |N|$ is an $N(\mathcal{U})$-modification of $g, f(x) \in \tau \cap |N|$ and therefore, there exists a simplex $\tau' = [U_{j_1}, \ldots, U_{j_r}] \leq \tau\ (r \leq s)$ such that $f(x) \in \tau'$ and $U_{j_1}, \ldots, U_{j_r} \in N^0$. Then $p_i f(x) \in [p_i(U_{j_1}), \ldots, p_i(U_{j_r})] = [\pi_i(U_{j_1}), \ldots, \pi_i(U_{j_r})]$. Put $\pi_i(\{U_{j_1}, \ldots, U_{j_r}\}) = \{w_1, \ldots, w_t\} \subseteq K_i^0,\ t \leq r$. Then, $p_i f(x) \in [w_1, \ldots, w_t]$. Since $x \in U_1 \cap \cdots \cap U_s \subseteq f_i^{-1}(\mathrm{st}(\pi_i(U_1), K_i^0)) \cap \cdots \cap f_i^{-1}(\mathrm{st}(\pi_i(U_s), K_i^0))$, we obtain $f_i(x) \in \mathrm{st}(w_1, K_i^0) \cap \cdots \cap \mathrm{st}(w_t, K_i^0)$. Now we have $f_i(x) \in \sigma \cap \mathrm{st}(w_1, K_i^0) \cap \cdots \cap \mathrm{st}(w_t, K_i^0)$ and therefore, $[v_1, \ldots, v_k] \cap \mathrm{st}(w_j, K_i^0) \neq \emptyset$, for each $j = 1, \ldots, t$. But this implies that each w_j is some v_1, i.e. $\{w_1, \ldots, w_t\} \subseteq \{v_1, \ldots, v_k\}$. This means that $[w_1, \ldots, w_t] \leq [v_1, \ldots, v_k]$ and therefore, $p_i f(x) \in \sigma$.

Claim 3. $(p_i f, f_i) \leq \mathcal{U}_i$.

Because of Claim 2., for each $x \in X$ there exists a simplex $\sigma \in K_i$ such that $p_i f(x),\ f_i(x) \in \sigma$. Then, $p_i f(x),\ f_i(x) \in \overline{\mathrm{st}(v, K_i^0)}$, where v is any vertex of σ. Since $\bar{\mathcal{S}}_i \leq \mathcal{U}_i$ there exists a $U \in \mathcal{U}_i$ such that $p_i f(x),\ f_i(x) \in U$. This completes the proof of the lemma. $\square$

Theorem 3.5. *For a topological space X the following statements are equivalent.*

(i) *X is finitistic.*

(ii) *X admits an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ where all X_λ are finite-dimensional polyhedra, all bonding maps $p_{\lambda\lambda'}$ are PL, all projections p_λ are dense and Λ is cofinite.*

(iii) *X admits an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ consisting of finite-dimensional spaces.*

Proof. $(i) \implies (ii)$. Let X be a finitistic space. The required approximate resolution is obtained by repeating the proof of Theorem 1.7 of [10] using Lemma 3.4 instead of Lemma 3.1.

$(ii) \implies (iii)$ is obvious.

$(iii) \implies (i)$. Let $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an approximate resolution such that each X_λ is a finite-dimensional space. Let $\mathcal{U} \in \mathrm{Cov}(X)$ be an arbitarary normal covering of X. By (B2)*, there exist an index λ and a covering $\mathcal{V} \in \mathrm{Cov}(X_\lambda)$ such that $p_\lambda^{-1}\mathcal{V} \leq \mathcal{U}$. Since X_λ is finite-dimensional, there exist an integer n and a covering $\mathcal{W} \in \mathrm{Cov}(X_\lambda)$ such that $\mathrm{ord}(\mathcal{W}) \leq n$ and $\mathcal{W} \leq \mathcal{V}$. Then $p_\lambda^{-1}\mathcal{W} \in \mathrm{Cov}(X)$ is the required finite-dimensional refinement of $\mathcal{U}$. $\square$

Remark 3.6. Let X be a topological space with dimension $\dim X \leq m$. Then one can achieve the polyhedron P in Lemma 3.4 has $\dim P \leq m$ and therefore, X admits an approximate resolution $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ such that each polyhedron X_λ has $\dim X_\lambda \leq m$, all projections p_λ are dense, all bonding maps $p_{\lambda\lambda'}$ are PL and Λ is cofinite (see [23] Theorem 1.).

Remark 3.7. Recently the following result has been obtained: If X is a Tychonoff finitistic space then X admits a commutative resolution consisting of finite-dimensional polyhedra ([20], Theorem 1.8). This is a weaker form of the implication $(i) \implies (ii)$ of Theorem 3.5.

§4. Approximate resolutions of direct products

Using the techniques of approximate resolutions of spaces one can obtain some results concerning the product of finitistic spaces.

Let $\mathcal{U}$ be any open covering of a space X, and for every $U \in \mathcal{U}$ let $\mathcal{V}_U$ be an open covering of a fixed space Y. Then the family $\mathcal{S} = \{U \times V : U \in \mathcal{U}, V \in \mathcal{V}_U\}$ is an open covering of the product space $X \times Y$, called a *stacked covering* of $X \times Y$ (over $\mathcal{U}$).

It is well-known fact that, for every space X, every compact Hausdorff space Y and every normal covering $\mathcal{W}$ of $X \times Y$, there exist a normal covering $\mathcal{U}$ of X and a stacked covering $\mathcal{S}$ of $X \times Y$ (over $\mathcal{U}$), which refines $\mathcal{W}$ and is normal ([3], pp. 357, 361). We shall denote this stacked covering $\mathcal{S}$ by $\mathcal{U} \times (\mathcal{V}_U)_{U \in \mathcal{U}}$.

Theorem 4.1. *Let $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an approximate resolution of a space X and let Y be a compact Hausdorff space. Then there exist an approximate resolution $\boldsymbol{q} = \{q_\mu : \mu \in M\} : X \times Y \to \mathcal{Z} = \{Z_\mu, \mathcal{V}_\mu, q_{\mu\mu'}, M\}$ of $X \times Y$, such that each Z_μ is some $X_\lambda \times Y$, each $q_{\mu\mu'}$ is some $p_{\lambda\lambda'} \times 1$ and each q_μ is some $p_\lambda \times 1$.*

Proof. First, we shall prove that $\boldsymbol{p} \times 1 = \{p_\lambda \times 1 : \lambda \in \Lambda\} : X \times Y \to \mathcal{X} \times Y = \{X_\lambda \times Y, p_{\lambda\lambda'}, \times 1, \Lambda\}$ satisfies conditions (A2), (AS), (B1)* and (B2)*.

(A2) Let $\lambda \in \Lambda$ and let $\mathcal{W} \in \mathrm{Cov}(X_\lambda \times Y)$ be a normal covering of $X_\lambda \times Y$. Choose a stacked covering $\mathcal{U} \times (\mathcal{V}_U)_{U \in \mathcal{U}} \in \mathrm{Cov}(X_\lambda \times Y)$ refining $\mathcal{W}$, where $\mathcal{U} \in \mathrm{Cov}(X_\lambda)$. By (A2) for $\mathcal{X}$, there is a $\lambda' \geq \lambda$, such that, for all $\lambda_2 \geq \lambda_1 \geq \lambda'$, $(p_{\lambda\lambda_1} p_{\lambda_1\lambda_2}, p_{\lambda\lambda_2}) \leq \mathcal{U}$ holds. Now, we obtain

$$((p_{\lambda\lambda_1} \times 1)(p_{\lambda_1\lambda_2} \times 1), p_{\lambda\lambda_2} \times 1) = ((p_{\lambda\lambda_1} p_{\lambda_1\lambda_2}) \times 1, p_{\lambda\lambda_2} \times 1) \leq$$
$$\leq \mathcal{U} \times (\mathcal{V}_U)_{U \in \mathcal{U}} \leq \mathcal{W},$$

which establishes (A2).

(AS) is proved in the same manner as (A2).

(B1)* Let $\mathcal{W} \in \mathrm{Cov}(X \times Y)$ be given. Choose a stacked covering $\mathcal{U} \times (\mathcal{V}_U)_{U \in \mathcal{U}} \in \mathrm{Cov}(X \times Y)$ refining $\mathcal{W}$, where $\mathcal{U} \in \mathrm{Cov}(X)$. Applying (B1)* for $\boldsymbol{p}$ to $\mathcal{U}$, we get $\lambda \in \Lambda$ and $\mathcal{U}_1 \in \mathrm{Cov}(X_\lambda)$ such that $p_\lambda^{-1} \mathcal{U}_1 \leq \mathcal{U}$. We now put $\mathcal{W}_1 = \mathcal{U}_1 \times (\mathcal{V}'_{U'})_{U' \in \mathcal{U}_1} \in \mathrm{Cov}(X_\lambda \times Y)$, where $\mathcal{V}'_{U'} = \mathcal{V}_U$, for some $U \in \mathcal{U}$ with property $p_\lambda^{-1}(U') \subseteq U$. Then $(p_\lambda \times 1)^{-1} \mathcal{W}_1 \leq \mathcal{U} \times (\mathcal{V}_U)_{U \in \mathcal{U}} \leq \mathcal{W}$, which proves (B1)*.

(B2)* Let $\lambda \in \Lambda$ and let $\mathcal{W} \in \mathrm{Cov}(X_\lambda \times Y)$ be a normal covering of $X_\lambda \times Y$. Choose a stacked covering $\mathcal{U} \times (\mathcal{V}_U)_{U \in \mathcal{U}} \in \mathrm{Cov}(X_\lambda \times Y)$ refining $\mathcal{W}$, where $\mathcal{U} \in \mathrm{Cov}(X_\lambda)$. Obviously, $\mathrm{st}((p_\lambda \times 1)(X \times Y), \mathcal{U} \times (\mathcal{V}_U)_{U \in \mathcal{U}}) = \mathrm{st}(p_\lambda(X), \mathcal{U}) \times Y$. By (B2)* for $\boldsymbol{p}$, there is a $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \mathrm{st}(p_\lambda(X), \mathcal{U})$. Consequently,

$$(p_{\lambda\lambda'} \times 1)(X_{\lambda'} \times Y) \subseteq \mathrm{st}(p_\lambda(X), \mathcal{U}) \times Y =$$
$$= \mathrm{st}((p_\lambda \times 1)(X \times Y), \mathcal{U} \times (\mathcal{V}_U)_{U \in \mathcal{U}}) \subseteq \mathrm{st}((p_\lambda \times 1)(X \times Y), \mathcal{W}).$$

We have just proved that $\boldsymbol{p} \times 1$ is an ungauged approximate resolution in the sense of [8]. Using the construction described in ([8], Theorem 2 and

Theorem 7) we can associate with $\boldsymbol{p}\times 1$ an induced approximate resolution, which satisfies all the required conditions. $\square$

Remark 4.2. Theorem 4.1 improves the main theorem in [22], since there are no assumptions on $\boldsymbol{p}$. However, the connection between the meshes $\mathcal{U}_\lambda$ and $\mathcal{V}_\mu$ gets lost.

It is well-known fact that the product of two finitistic spaces need not be finitistic ([2], Example 2.1.). However, the following holds.

Corollary 4.3. *Let X be a finitistic space and Y a finite-dimensional compact Hausdorff space. Then $X\times Y$ is a finitistic space.*

Proof. Let X be a finitistic space. Then, by Theorem 3.5, X admits an approximate resolution $\boldsymbol{p}=\{p_\lambda:\lambda\in\Lambda\}:X\to\mathcal{X}=\{X_\lambda,\mathcal{U}_\lambda,p_{\lambda\lambda'},\Lambda\}$, where all X_λ are finite-dimensional polyhedra. Since Y is a compact Hausdorff space there exists an approximate resolution $\boldsymbol{q}=\{q_\mu:\mu\in M\}:X\times Y\to\mathcal{Z}=\{Z_\mu,\mathcal{V}_\mu,q_{\mu\mu'},M\}$, such that each Z_μ is some $X_\lambda\times Y$, each $q_{\mu\mu'}$ is some $p_{\lambda\lambda'}\times 1$ and each q_μ is some $p_\lambda\times 1$. Each X_λ is a finite-dimensional paracompact space and Y is a finite-dimensional compact Hausdorff space which implies $\dim(X_\lambda\times Y)\le\dim X_\lambda+\dim Y$ ([16], Corollary 26-5). Hence, the approximate resolution $\boldsymbol{q}=\{q_\mu:\mu\in M\}:X\times Y\to\mathcal{Z}=\{Z_\mu,\mathcal{V}_\mu,q_{\mu\mu'},M\}$ consists of finite-dimensional spaces, which implies that $X\times Y$ is finitistic. $\square$

For paracompact finitistic spaces, the assertion of Corollary 4.3 was proved in [2] (Theorem 2.2).

Corollary 4.4. *Let X be a space with $\dim X=m$ and let Y be a compact Hausdorff space with $\dim Y=n$. Then $\dim(X\times Y)\le m+n$.* $\square$

§5. Characterization of some classes of $\mathcal{P}$-like spaces

Let $\mathcal{P}$ be a (nonempty) class of Hausdorff spaces. A space X is called $\mathcal{P}$-*like* provided for every normal covering $\mathcal{U}\in\mathrm{Cov}(X)$, there exist a member $P\in\mathcal{P}$, a normal covering $\mathcal{V}\in\mathrm{Cov}(P)$ and a mapping $f:X\to P$ such that $f^{-1}\mathcal{V}\le\mathcal{U}$ and $\overline{f(X)}=P$.

Proposition 5.1. *Let $\mathcal{P}$ be a class of spaces and let $\boldsymbol{p}=\{p_\lambda:\lambda\in\Lambda\}:X\to\mathcal{X}=\{X_\lambda,\mathcal{U}_\lambda,p_{\lambda\lambda'},\Lambda\}$ be an approximate resolution of a space X consisting of $\mathcal{P}$-like spaces. If $\overline{p_\lambda(X)}=X_\lambda$, for all $\lambda\in\Lambda$, then X is a $\mathcal{P}$-like space.*

Proof. Let $\mathcal{U}\in\mathrm{Cov}(X)$. By property (B1)*, there exist a $\lambda\in\Lambda$ and a $\mathcal{V}\in\mathrm{Cov}(X_\lambda)$ such that $p_\lambda^{-1}\mathcal{V}\le\mathcal{U}$. Since X_λ is $\mathcal{P}$-like, there exist a

$P \in \mathcal{P}$, a $\mathcal{W} \in \mathrm{Cov}(P)$ and a mapping $g : X_\lambda \to P$ such that $g^{-1}\mathcal{W} \le \mathcal{V}$ and $\overline{g(X_\lambda)} = P$. Put $f = gp_\lambda : X \to P$. Clearly, $f^{-1}\mathcal{W} \le \mathcal{U}$ and $\overline{f(X)} = P$. □

Proposition 5.2. *Let $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an approximate resolution, where all the spaces X_λ are normal. If $\overline{p_{\lambda\lambda'}(X_{\lambda'})} = X_\lambda$, for all $\lambda \le \lambda'$, then also $\overline{p_\lambda(X)} = X_\lambda$, for all $\lambda \in \Lambda$.*

Proof. Assume that for a certain $\lambda \in \Lambda$ we have $\overline{p_\lambda(X)} \neq X_\lambda$. Choose a point $x \in X_\lambda \setminus \overline{p_\lambda(X)}$ and put $U = X_\lambda \setminus \{x\}$. U is an open set, which contains $\overline{p_\lambda(X)}$. Since X_λ is normal, there exists an open set $V \subseteq X_\lambda$ such that $p_\lambda(X) \subseteq \overline{p_\lambda(X)} \subseteq V \subseteq \overline{V} \subseteq U$. By (B3)* (which, for normal spaces, is equivalent to (B2)), there is a $\lambda' \ge \lambda$ such that $p_{\lambda\lambda'}(X_{\lambda'}) \subseteq V$. Then $\overline{p_{\lambda\lambda'}(X_{\lambda'})} \subseteq \overline{V} \subseteq X_\lambda \setminus \{x\}$, which contradicts the assumption $\overline{p_{\lambda\lambda'}(X_{\lambda'})} = X_\lambda$. □

Remark 5.3. Let $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an approximate resolution of a pseudocompact space X, where all the spaces X_λ are compact polyhedra and all the bonding maps are surjections. Then all the projections p_λ are also surjections.

An immediate consequence of Propositions 5.1 and 5.2 is the following proposition.

Proposition 5.4. *Let $\mathcal{P}$ be a class of normal spaces and let $\boldsymbol{p} = \{p_\lambda : \lambda \in \Lambda\} : X \to \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an approximate resolution of a topological space X, where each $X_\lambda \in \mathcal{P}$ and all the bonding maps are dense. Then X is a $\mathcal{P}$-like space.*

Theorem 5.5. *The following characterizations hold.*

(i) A Tychonoff space X is $\mathcal{P}$-like, $\mathcal{P}$ being the class of all compact polyhedra, if and only if X is pseudocompact.

(ii) A space X is $\mathcal{P}$-like, $\mathcal{P}$ being the class of all polyhedra P with $\dim P \le n$, if and only if X has $\dim X \le n$.

(iii) A space X is $\mathcal{P}$-like, $\mathcal{P}$ being the class of all finite-dimensional polyhedra, if and only if X is finitistic.

References

[1] S. Deo and A. R. Pears, A completely finitistic space is finite-dimensional, *Bull. London Math. Soc.* **17** (1985), 49–51.

[2] S. Deo and M. Singh, On certain constructions in finitistic spaces, *Internat. J. Math. and Math. Sci.* **6** (1983), 477–482.

[3] A. Dold, Lectures on algebraic topology, *Springer Verlag*, *Berlin-Heidelberg-New York*, 1972.

[4] R. Engelking, General Topology, Monografie Matematyczne 60, *Polish Scientific Publishers*, *Warszawa*, 1977.
[5] P. Fletcher, R. A. McCoy and R. Slover, On boundedly metacompact and boundedly paracompact spaces, *Proc. Amer. Math. Soc.* **25** (1970), 335–342.
[6] Y. Hattori, A note on finitistic spaces, *Quest. and Answ. in Gen. Top.* **3** (1985), 47–55.
[7] I. Lončar, Some results on resolutions of spaces, *Rad JAZU, Matem. znan.* **428 (6)** (1987), 37–49.
[8] S. Mardešić, On approximate inverse systems and resolutions, *Fund. Math.* **142** (1993), 243–255.
[9] S. Mardešić, Strong shape of the Stone-Čech compactification, *Comment. Math. Univ. Carolinae* **33, 3** (1992), 533–539.
[10] S. Mardešić and V. Matijević, $\mathcal{P}$-like spaces are limits of approximate $\mathcal{P}$-resolutions, *Topology Appl.* **45** (1992), 189–202.
[11] S. Mardešić and L. R. Rubin, Approximate inverse systems of compacta and covering dimension, *Pacific J. Math.* **138** (1989), 129–144.
[12] S. Mardešić and L. R. Rubin, Cell-like mappings and nonmetrizable compacta of finite cohomological dimension, *Trans. Amer. Math. Soc.* **313** (1989), 53–79.
[13] S. Mardešić, L. R. Rubin and N. Uglešić, A note on approximate systems of metric compacta, *Topology Appl.* (*to appear*).
[14] S. Mardešić and J. Segal, Shape Theory, *North-Holland Publ. Comp.*, *Amsterdam*, 1982.
[15] S. Mardešić and T. Watanabe, Approximate resolutions of spaces and mappings, *Glasnik Mat.* **24 (44)** (1989), 587–637.
[16] K. Nagami, Dimension Theory, *Academic Press*, *New York and London*, 1970.
[17] B. A. Pasynkov, On the absence of polyhedral spectra for bicompacta (Russian), *Dokl. Akad. Nauk USSR 1942* (1962), 546–549.
[18] B. A. Pasynkov, On spectra and dimension of topological spaces (Russian), *Mat. Sbornik* **57** (1962), 449–476.
[19] L. R. Rubin and P. J. Schapiro, Compactifications which preserve cohomological dimension, *Glasnik. Mat.* **28 (48)** (1993), 155–165.
[20] J. Segal, S. Spiez and B. Günther, Strong shape of uniform spaces, *Topology Appl.* **49** (1993), 237–249.
[21] R. G. Swan, A new method in fixed point theory, *Comm. Math. Helv.* **34** (1960), 1–16.
[22] N. Uglešić and V. Matijević, An approximate resolution of the product with a compact factor, *Tsukuba J. Math.* **16** (1992), 75–84.
[23] T. Watanabe, Approximate resolutions and covering dimension, *Topology Appl.* **38** (1991), 147–154.

VLASTA MATIJEVIĆ
DEPARTMENT OF MATHEMATICS
58000 SPLIT, TESLINA 12/III
CROATIA

(Received April 5, 1994; revised September 19, 1994)