The Cotton tensor in the projective geometry of sprays

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Abstract. I define, by analogy with the projective theory of linear connections, a Cotton tensor for any spray, and discuss its role in the projective geometry of sprays.

1. Introduction

It may seem that there is little new to be said about the projective geometry of sprays. After all the subject has a long history, dating back as it does to Douglas and Berwald; has been extensively covered in standard texts by ANTONELLI *et al.* [1] and SHEN [15]; and has been brought up to date by J. SZILASI *et al.* [16], Z. SZILASI [17], and YOUSSEF *et al.* [18].

My excuse for writing another paper on the topic is that it seems to me that there is a curious lacuna in the literature on the projective geometry of sprays, which I can best explain by comparing it to the special, but longer established, case of the projective geometry of symmetric linear connections or quadratic sprays. In the latter a projectively invariant tensor, the Weyl projective curvature tensor, is defined, and it is shown that in dimension 3 or more the vanishing of the Weyl tensor is the necessary *and sufficient* condition for the linear connection to be locally projectively flat, that is, locally projectively equivalent to the flat connection. If one consults the literature on the projective geometry of general sprays, however, one finds that a great deal of effort has been put into defining a generalized Weyl projective curvature tensor, but almost none, so far as I can see, into explaining what it is for, in particular, into deriving the

consequences of its vanishing. The one exception to these criticisms that I know of is, I have to confess, a paper of my own [9]: and on reflection, I now think that it does not really do the matter justice.

Let me explain my title. In the projective theory of symmetric linear connections there is, in addition to the Weyl tensor, a second equally important but less prominent tensor called the projective Cotton tensor. This comes into its own in the case of 2-dimensional manifolds with symmetric linear connection. In 2 dimensions the Weyl tensor vanishes identically, and the necessary and sufficient condition for projective flatness is the vanishing of the Cotton tensor. But in fact, the Cotton tensor does a lot of the work in the proof of the result that in dimension 3 or more the vanishing of the Weyl tensor entails projective flatness. It so happens that the vanishing of the Weyl tensor in such higher dimensions implies that the Cotton tensor vanishes, and this latter fact is the key to the proof. Searching for a mention of a generalized Cotton tensor in the projective theory of sprays should be an indirect way of finding a proof of the missing results. It produces, for me at least, no hits. (The generalized Cotton tensor to be introduced below does actually appear in my paper [9] referred to earlier, but was not named because I did not recognise its significance at the time. A version also appears in a section of Shen's book [15] entitled 'Berwald–Weyl Curvature'; but Shen, following BERWALD [3], really deals only with the 2-dimensional case. I shall discuss Berwald's work in my final section.) The aim of this paper is to rectify the deficiences described above.

In the projective theory of general sprays there is, of course, in addition to the Weyl and Cotton tensors, a projectively invariant tensor called the Douglas tensor which has no counterpart in the linear theory, whose vanishing is the necessary and sufficient condition for the Berwald connection to be projectively equivalent to a linear connection. And it is well known that in dimension 3 or more the vanishing of both the Douglas tensor and the Weyl tensor is necessary and sufficient for the spray to be projectively flat. But the argument for this assertion first uses the fact that the vanishing of the Douglas tensor ensures that the Berwald connection is projectively linear; then the Weyl tensor is just that of any projectively equivalent linear connection, so one can use the result for the linear case. I want to emphasise that the question of interest in this paper is what happens when the Weyl tensor is zero but the Douglas tensor is not.

There is another way of looking at this theory, which emphasises the role of systems of second-order differential equations, such as those which define the geodesics of a spray, and investigates the invariants of such systems under so-called point transformations (in which the independent variable is treated on the

same footing as the dependent ones). FELS [12] solved this problem for a system of two or more equations (which is locally equivalent to a spray over a base manifold of at least 3 dimensions), and found invariants which are versions of the Douglas and Weyl tensors (see [11] for the details). (The case of a single second-order differential equation – a 2-dimensional spray – had been studied long before by CARTAN [5]: see also [7].) Those who approach the theory from this starting point ([6], for example) refer to the collection of those systems for which the equivalent of the Douglas tensor vanishes (but not necessarily the Weyl tensor) as the projective branch, those for which the equivalent of the Weyl tensor vanishes (but not necessarily the Douglas tensor) as the conformal branch. The results to be established in this paper are concerned with the conformal branch, in this (slightly mysterious) terminology.

It is not very difficult to guess what the results in question might be. Associated with the Berwald connection of any spray, there are two tensors which are usually described as curvatures: the hv-curvature, which is usually called the Berwald curvature, and the hh-curvature, which is called the Riemann curvature. A Berwald connection whose Riemann curvature vanishes, but whose Berwald curvature is not necessarily zero, is said to be R-flat (the terminology is that used by Shen in [15]). Then, as I show below, over a manifold of dimension at least 3, the vanishing of its Weyl projective curvature tensor is the necessary and sufficient condtion for a spray to be locally projectively equivalent to one which is R-flat; and over a manifold of dimension 2, the vanishing of its projective Cotton tensor is the necessary and sufficient condition for a spray to be locally projectively equivalent to one which is R-flat. In fact, if one were to go through the basic results of the linear theory subsituting 'Berwald connection of a spray' for 'symmetric linear connection' and 'R-flat' for 'flat', one would obtain correct results for the general theory.

The main aims of this paper are the identification of the Cotton tensor and the proofs of the sufficiency parts of the theorems just described. The paper also aims to provide a clear, comprehensive and definitive account of the fundamentals of the projective geometry of sprays, so that much of what is covered will of necessity be familiar in one guise or another; but I hope that the methods used will be of interest.

(The relevant results for linear connections are as follows: see, for example, [14], or [2] for a modern account which names the Cotton tensor. The Weyl projective curvature tensor is given by

$$W^h_{kij} = R^h_{kij} + s_{ki}\delta^h_j - s_{kj}\delta^h_i + (s_{ji} - s_{ij})\delta^h_k,$$

where

$$s_{ij} = \frac{1}{(n^2 - 1)}(nR_{ij} + R_{ji})$$

with $R_{ij} = R^h_{ihj}$, the Ricci tensor: s is often called the Schouten tensor. Notice that

$$s_{ij} - s_{ji} = \frac{1}{(n+1)}(R_{ij} - R_{ji}).$$

The Weyl projective curvature tensor is projectively invariant. The projective Cotton tensor is

$$s_{ki;j} - s_{kj;i} = c_{kij},$$

where the semicolon indicates a covariant derivative. The Cotton tensor is not in general projectively invariant, but satisfies

$$(n - 2)c_{kij} = -W^h_{kij;h}.$$

A linear connection is projectively flat if and only if, in dimension 3 or more the Weyl tensor vanishes, and in dimension 2 the Cotton tensor vanishes.)

2. Background

I gather together here basic facts and formulae and establish notations.

I denote by $\tau : T^\circ M \to M$ the slit tangent bundle of a manifold M, with projection the restriction of the usual tangent bundle projection, and with local coordinates x^i, y^i where the y^i are the canonical fibre coordinates corresponding to local coordinates x^i on M.

In the Introduction, I have been somewhat cavalier with my use of the term 'tensor' when referring to geometric objects associated with general sprays. The usual approach to the geometry of sprays is to work with $\tau^*(TM) \to T^\circ M$, the pullback of TM to $T^\circ M$: this is of course a vector bundle over $T^\circ M$ whose fibre at $(x, y) \in T^\circ M$ is isomorphic to $T_x M$. One may then form tensor products of copies of $\tau^*(TM)$ and its dual, and a tensor field on $T^\circ M$ is a section of such a tensor product bundle over $T^\circ M$. In short, when referring to a tensor, I expected the reader to supply the phrase 'along the tangent bundle projection'.

There is however an alternative, which is to replace $\tau^*(TM)$ with $V(T^\circ M)$, the subbundle of $T(T^\circ M)$ consisting of vertical vectors, the kernel bundle of $\tau_* : T(T^\circ M) \to T^\circ M$. Its fibre at $(x, y) \in T^\circ M$ is the vertical subspace of $T_{(x,y)}T^\circ M$; it too is isomorphic to $T_x M$, and indeed the two bundles $\tau^*(TM)$ and $V(T^\circ M)$ are naturally isomorphic. One can equally well take 'tensor' to mean a section

of a tensor bundle formed from $V(T^\circ M)$. For most of the paper, it makes little difference which interpretation is used. However, there is one section, Section 4, where there is a distinct advantage in using the $V(T^\circ M)$ approach.

I denote by Δ the Liouville vector field on $T^\circ M$,

$$\Delta = y^i \frac{\partial}{\partial y^i};$$

it generates dilations in the fibres, and is fundamental to the definition of positive homogeneity. Any spray Γ is of course positively homogeneous:

$$[\Delta, \Gamma] = \Gamma.$$

For a spray on $T^\circ M$, conventionally expressed as

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i},$$

where the coefficients Γ^i are positively-homogeneous of degree 2, I denote by H_i the horizontal lift to $T^\circ M$, with respect to the horizontal distribution it defines, of the basic coordinate field $\partial/\partial x^i$:

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}, \qquad \Gamma_i^j = \frac{\partial \Gamma^j}{\partial y^i}.$$

It is often convenient to write V_i for the vertical lift of $\partial/\partial x^i$, that is,

$$V_i = \frac{\partial}{\partial y^i}.$$

The connection coefficients of the corresponding Berwald connection are given by

$$\Gamma_{ij}^h = \frac{\partial \Gamma_i^h}{\partial y^j} = \Gamma_{ji}^h; \qquad \Gamma_i^j = y^k \Gamma_{ik}^j.$$

Covariant derivatives with respect to the Berwald connection are expressed as follows: for a vector field with components U^i, the components of its covariant derivatives with respect to H_j and V_j respectively are

$$U_{;j}^i = H_j(U^i) + \Gamma_{jk}^i U^k, \qquad U_{,j}^i = V_j(U^i).$$

Note that $,j$ simply stands for partial differentiation with respect to y^j, and will be extensively used with this meaning. Since the Γ_{ij}^h are symmetric in their lower indices,

$$[H_i, V_j] = \Gamma_{ij}^h V_h = [H_j, V_i].$$

The Berwald curvature of the connection is defined componentwise as follows:

$$B_{ijk}^h = V_i(\Gamma_{jk}^h) = \frac{\partial^3 \Gamma^h}{\partial y^i \partial y^j \partial y^k}.$$

The Riemann curvature comes in three versions. It may be defined in terms of the geodesic deviation equation or Jacobi equation, as a type $(1,1)$ tensor with components R_k^h, so that the equation for a Jacobi field J is

$$\nabla_\Gamma^2 J^h + R_k^h J^k = 0.$$

Secondly, there is the type $(1,2)$ tensor with components R_{ij}^h which measures the failure of the horizontal distribution to be involutive:

$$[H_j, H_k] = -R_{jk}^h V_h, \qquad R_{jk}^h = H_j(\Gamma_k^h) - H_k(\Gamma_j^h).$$

And thirdly, there is the usual definition in terms of the covariant derivative, here applied to covariant differentiation with respect to the Berwald connection in horizontal directions, giving a type $(1,3)$ tensor

$$R_{ijk}^h = H_j(\Gamma_{ik}^h) - H_k(\Gamma_{ij}^h) + \Gamma_{jl}^h \Gamma_{ik}^l - \Gamma_{kl}^h \Gamma_{ij}^l.$$

These three tensors are to be regarded as different presentations of the same geometric object. They are related as follows:

$$\begin{aligned}
R_k^h &= y^j R_{jk}^h = y^i y^j R_{ijk}^h, \\
R_{jk}^h &= \tfrac{1}{3}(R_{k,j}^h - R_{j,k}^h) = y^i R_{ijk}^h, \\
R_{ijk}^h &= R_{jk,i}^h = \tfrac{1}{3}(R_{k,ij}^h - R_{j,ik}^h).
\end{aligned}$$

In referring to them, I shall differentiate between them by their types. I shall most often use the type $(1,2)$ Riemann curvature. This is because the involutivity of the horizontal distribution will be of importance. When the Berwald connection is actually linear, so that R_{ijk}^h is independent of y, R_{jk}^h depends linearly on y; and so in making comparisons with the linear case, one has to mentally differentiate with respect to y^i (say).

These curvatures have the following properties. The coefficient B_{ijk}^h of the Berwald curvature is positively-homogeneous of degree -1; the coefficient R_{jk}^h of the type $(1,2)$ Riemann curvature is positively-homogeneous of degree 1. One of the Ricci identities for repeated covariant differentiation will be needed, here exemplified in terms of a vector field with components U^h:

$$U_{;i,j}^h - U_{;j,i}^h = B_{ijk}^h U^k.$$

The type $(1,1)$ Riemann curvature satisfies $R_k^h y^k = 0$. The type $(1,3)$ Riemann curvature satisfies the algebraic Bianchi identity or cyclic identity

$$R_{ijk}^h + R_{jki}^h + R_{kij}^h = 0.$$

The type $(1,2)$ Riemann curvature satisfies the differential Bianchi identity

$$R_{ij;k}^h + R_{jk;i}^h + R_{ki;j}^h = 0.$$

(There are other Bianchi identities but they will not be needed.)

The Ricci tensor is defined by $R_{ij} = R_{ihj}^h$. It is not in general symmetric: rather, from the cyclic identity we have

$$R_{ij} - R_{ji} = R_{hij}^h.$$

3. Isotropy

Proposition 1. *The following conditions on the Riemann curvatures of a spray are equivalent:*

(1) *there is a function ρ, positively-homogeneous of degree 2, and covector τ, of degree 1, such that*
$$R_k^h = \rho \delta_k^h + \tau_k y^h;$$

(2) *there is a covector ξ, of degree 1, and skew-symmetric type $(0,2)$ tensor η, of degree 0, such that*

$$R_{jk}^h = \xi_j \delta_k^h - \xi_k \delta_j^h + \eta_{jk} y^h.$$

Furthermore, when such conditions hold, we must have

$$\rho + \tau_k y^k = 0,$$

and for $\dim M > 2$,

$$\eta_{jk} = \xi_{j,k} - \xi_{k,j}.$$

PROOF. Using the relations between the variants of the Riemann curvature quoted in the previous section, one finds that with $R_k^h = \rho \delta_k^h + \tau_k y^h$,

$$R_{jk}^h = \tfrac{1}{3} \left((\rho_{,j} - \tau_j)\delta_k^h - (\rho_{,k} - \tau_k)\delta_j^h - (\tau_{j,k} - \tau_{k,j})y^h \right);$$

while if $R_{jk}^h = \xi_j \delta_k^h - \xi_k \delta_j^h + \eta_{jk} y^h$, then

$$R_k^h = (\xi_j y^j)\delta_k^h + (\eta_{jk} y^j - \xi_k)y^h.$$

Now $R_k^h y^k = 0$, so $\rho + \tau_k y^k = 0$. If $R_{jk}^h = \xi_j \delta_k^h - \xi_k \delta_j^h + \eta_{jk} y^h$, then also

$$R_{jk}^h = \tfrac{1}{3}\left((\rho_{,j} - \tau_j)\delta_k^h - (\rho_{,k} - \tau_k)\delta_j^h - (\tau_{j,k} - \tau_{k,j})y^h\right)$$

with $\rho = \xi_j y^j$ and $\tau_k = \eta_{jk} y^j - \xi_k$. It is easy to see (and proved in the next section) that if $\pi_j \delta_k^h - \pi_j \delta_k^h + \sigma_{jk} y^h = 0$ for some π and skew-symmetric σ, then provided dim $M > 2$, $\pi = 0$ and $\sigma = 0$. So

$$\xi_j = \tfrac{1}{3}(\rho_{,j} - \tau_j) \quad \text{and} \quad \eta_{jk} = -\tfrac{1}{3}(\tau_{j,k} - \tau_{k,j}),$$

from which evidently $\eta_{jk} = \xi_{j,k} - \xi_{k,j}$.   □

When dim $M = 2$, however, it is possible to have $\pi_j \delta_k^h - \pi_j \delta_k^h + \sigma_{jk} y^h = 0$ with π and σ nonzero, since $-y^2 \delta_2^h - y^1 \delta_1^h + y^h = 0$; and so the final argument does not work. In fact, the formula for R_{jk}^h in terms of ρ and τ becomes, on substitution for ρ and τ in terms of ξ and η,

$$R_{12}^h = \xi_1 \delta_2^h - \xi_2 \delta_1^h + \eta_{12} y^h + \left(\eta_{12} - (\xi_{1,2} - \xi_{2,1})\right)(-y^2 \delta_2^h - y^1 \delta_1^h + y^h),$$

and gives no new information.

A spray is said to be isotropic, or to have isotropic Riemann curvature, if its curvature satisfies the condition of Proposition 1 in either of its forms, and if in addition, $\eta_{jk} = \xi_{j,k} - \xi_{k,j}$ when dim $M = 2$ (the case when this does not occur automatically). The condition for a spray to be isotropic will most often be taken in the type $(1, 2)$ form

$$R_{jk}^h = \xi_j \delta_k^h - \xi_k \delta_j^h + (\xi_{j,k} - \xi_{k,j})y^h,$$

where ξ is a covector field which is positively-homogeneous of degree 1. (Other authors, with Shen [15] as the prime example, tend to use the type $(1, 1)$ version.)

The properties of type $(1, 2)$ tensor fields constructed from covectors in the above manner will be of major importance later. I pause to establish the properties that will be needed.

Let q_i be the components of a covector, positively-homogeneous of degree 1. Consider the type $(1, 2)$ tensor Q, skew-symmetric in its lower indices, whose components are given by

$$Q_{ij}^h = q_i \delta_j^h - q_j \delta_i^h + (q_{i,j} - q_{j,i})y^h.$$

I shall call Q the type $(1, 2)$ isotropic tensor constructed from q, and say that such a tensor is of isotropic type.

Lemma 1. *The tensor Q satisfies the cyclic identity*

$$Q^h_{ij,k} + Q^h_{jk,i} + Q^h_{ki,j} = 0.$$

PROOF. We have

$$Q^h_{ij,k} = q_{i,k}\delta^h_j - q_{j,k}\delta^h_i + (q_{i,j} - q_{j,i})\delta^h_k + (q_{i,jk} - q_{j,ik})y^h,$$

from which the result is almost self-evident. □

Lemma 2. *With Q as above,*

$$q_i = \frac{1}{(n^2 - 1)}\big((n-1)Q_i + (Q_k y^k)_{,i}\big),$$

where $Q_i = Q^j_{ij}$ is the trace of Q. In particular, if Q is trace-free, then $q = 0$, that is, a type $(1,2)$ tensor of isotropic type whose trace vanishes itself vanishes.

PROOF. We have

$$Q_i = nq_i - q_i + (q_{i,j} - q_{j,i})y^j = nq_i - ((q_j y^j)_{,i} - q_i) = (n+1)q_i - (q_j y^j)_{,i}.$$

Now $q_j y^j$ is positively-homogeneous of degree 2, so

$$(n-1)(q_j y^j) = Q_i y^i,$$

from which the formula follows. Clearly, if $Q_i = 0$, then $q_i = 0$, and so $Q^h_{ij} = 0$. □

In the case of an isotropic spray, or indeed for the type $(1,2)$ Riemann curvature in all generality, the formula derived in the previous lemma may be expressed in terms of the Ricci tensor.

Lemma 3. *If R^h_{ij} are the components of the Riemann curvature of a spray, then with $R_i = R^j_{ij}$,*

$$(n-1)R_i + (R_k y^k)_{,i} = (n-1)R_i + R^k_{k,i} = -(nR_{ki} + R_{ik})y^k,$$

where R_{ij} are the components of the Ricci tensor. Furthermore,

$$\big((n-1)R_i + R^k_{k,i}\big)_{,j} - \big((n-1)R_j + R^k_{k,j}\big)_{,i} = (n-1)(R_{ij} - R_{ji}).$$

PROOF. Firstly, $R_{kl}^h y^k = R_l^h$, so $R_k y^k = R_k^k$. Furthermore,

$$(n-1)R_i + (R_k y^k)_{,i} = nR_i + R_{k,i} y^k.$$

The Ricci tensor is given by $R_{ki} = R_{kji}^j$. Now $R_{ij}^h = y^k R_{kij}^h$, thus $R_{ij}^j = y^k R_{kij}^j = -y^k R_{ki}$. Moreover, $R_{jk,i}^h = R_{ijk}^h$, and therefore $R_{k,i} = R_{kj,i}^j = R_{ikj}^j = -R_{ik}$. Thus

$$nR_i + R_{k,i} y^k = -(nR_{ki} + R_{ik})y^k.$$

Finally,

$$\big((n-1)R_i + R_{k,i}^k\big)_{,j} - \big((n-1)R_j + R_{k,j}^k\big)_{,i} = (n-1)(R_{i,j} - R_{j,i}) = (n-1)(R_{ij} - R_{ji})$$

as claimed. $\qquad\square$

I can now deal more fully with the case dim $M = 2$.

Proposition 2. *Any spray over a 2-dimensional base is isotropic.*

PROOF. Set

$$\xi_i = \tfrac{1}{3}(R_i + (R_k y^k)_{,i}) = -\tfrac{1}{3}(2R_{ki} + R_{ik})y^k$$

with $R_i = R_{ij}^j$, the values predicted by Lemmas 2 and 3 with $n = 2$ if the type $(1,2)$ Riemann tensor is to be given by

$$R_{ij}^h = \xi_i \delta_j^h - \xi_j \delta_i^h + (\xi_{i,j} - \xi_{j,i})y^h.$$

Since the type $(1,2)$ Riemann tensor is skew, when dim $M = 2$, it has just two components which could be nonzero, namely R_{12}^1 and R_{12}^2. Now

$$R_{12}^1 = y^1 R_{112}^1 + y^2 R_{212}^1, \qquad R_{12}^2 = y^1 R_{112}^2 + y^2 R_{212}^2.$$

On the other hand,

$$R_{11} = R_{111}^1 + R_{121}^2 = -R_{112}^2, \qquad R_{12} = R_{112}^1 + R_{122}^2 = R_{112}^1,$$
$$R_{21} = R_{211}^1 + R_{221}^2 = -R_{212}^2, \qquad R_{22} = R_{212}^1 + R_{222}^2 = R_{212}^1.$$

Note that

$$y^1 R_{12} + y^2 R_{22} = R_{12}^1, \qquad y^1 R_{11} + y^2 R_{21} = -R_{12}^2.$$

So

$$R_{12}^1 - \xi_1 \delta_2^1 + \xi_2 \delta_1^1 - (\xi_{1,2} - \xi_{2,1})y^1$$
$$= R_{12}^1 - \tfrac{1}{3}(2R_{k2} + R_{2k})y^k - \tfrac{1}{3}(R_{12} - R_{21})y^1$$
$$= R_{12}^1 - \tfrac{1}{3}(2R_{12} + R_{21} + R_{12} - R_{21})y^1 - \tfrac{1}{3}(2R_{22} + R_{22})y^2$$
$$= R_{12}^1 - (R_{12}y^1 + R_{22}y^2) = 0;$$

and similarly, for R_{12}^2. $\qquad\square$

4. Aside: vector-valued forms

There is a nice way of viewing several of the results in the previous sections which seems worth discussing, though it is really a side issue. It involves the notion of a vector-valued form. Here it will be advantageous to use $V(T^\circ M)$ rather than $\tau^*(TM)$.

Observe first of all that the base manifold plays no essential role in many of our results: we could regard the coordinates x^i as mere parameters. So let us fix a point $x \in M$, and confine our attention to $T^\circ_x M$. Now the curvatures may be considered to be $V(T^\circ M)$ tensors, which means that they can be considered as defining at any $x \in M$ tensor fields on the manifold $T^\circ_x M$. In particular, tensors of type $(1, r)$ take their values in $T_x M$, and these values may therefore be expressed in the form

$$T^h_{i_1 i_2 \ldots i_r} \frac{\partial}{\partial y^h},$$

where the $\partial/\partial y^h$, or indeed any vertical lifts, may be regarded as constant vector fields.

A tensor of type $(1, r)$ which is skew-symmetric in its covariant indices defines a vector-valued r-form

$$T^h_{i_1 i_2 \ldots i_r} dy^{i_1} \wedge dy^{i_2} \wedge \cdots \wedge dy^{i_r} \otimes \frac{\partial}{\partial y^h}.$$

Any vector-valued r-form Θ may be expressed as

$$\Theta = \theta^h \otimes \frac{\partial}{\partial y^h}$$

where each θ^h is an ordinary r-form. Any (vertical) vector field on $T^\circ_x M$ is a vector-valued 0-form; any type $(1, 1)$ tensor field is a vector-valued 1-form; in particular,

$$dy^h \otimes \frac{\partial}{\partial y^h}$$

is the identity tensor I, whose components are δ^h_i.

One may extend the wedge product to a product of p-forms and vector-valued q-forms on $T^\circ_x M$ in an obvious way:

$$\alpha \wedge \left(\beta^h \otimes \frac{\partial}{\partial y^h} \right) = (\alpha \wedge \beta^h) \otimes \frac{\partial}{\partial y^h},$$

the result being a vector-valued $(p + q)$-form. Likewise, the interior product extends to vector-valued forms:

$$i_U \left(\theta^h \otimes \frac{\partial}{\partial y^h} \right) = (i_U \theta_h) \otimes \frac{\partial}{\partial y^h},$$

where U is a vector field on $T_x^\circ M$ (or a vertical vector field on $T^\circ M$); when applied to a vector-valued r-form, $r \geq 1$, this yields a vector-valued $(r - 1)$-form.

Again, one may make the exterior derivative act on vector-valued forms by

$$d \left(\theta^h \otimes \frac{\partial}{\partial y^h} \right) = (d\theta^h) \otimes \frac{\partial}{\partial y^h}.$$

It is crucial here to note that differentiation is with respect to the variables y^i only, and that the $\partial/\partial y^h$ are to be treated as constants. The upper indices such as h on θ^h are tensorial, but transform with the Jacobian of a coordinate transformation on the base, that is, effectively with a constant matrix, and it is this which makes this manoeuvre valid – indeed, all the extensions of exterior algebra and calculus considered so far.

A vector-valued form Θ such that $d\Theta = 0$ is of course said to be closed.

The type $(1, 2)$ Riemann curvature tensor, whose components are R_{jk}^h, defines a vector-valued 2-form

$$R_{jk}^h dy^j \wedge dy^k \otimes \frac{\partial}{\partial y^h}.$$

The type $(1, 3)$ Riemann curvature tensor, with components R_{ijk}^h, does not define a vector-valued form. Consider, however,

$$d \left(R_{jk}^h dy^j \wedge dy^k \otimes \frac{\partial}{\partial y^h} \right) = R_{ijk}^h dy^i \wedge dy^j \wedge dy^k \otimes \frac{\partial}{\partial y^h} :$$

this vanishes, when $\dim M > 2$, precisely because R_{ijk}^h satisfies the cyclic identity

$$R_{ijk}^h + R_{jki}^h + R_{kij}^h = 0.$$

That is, the cyclic identity is equivalent to the property of the vector-valued 2-form defined by R_{jk}^h that it is closed. (When $\dim M = 2$, the vector-valued 2-form defined by R_{ij}^h is closed simply for dimensional reasons.)

A vector-valued r-form Θ is positively homogeneous if there is some integer N such that

$$\mathcal{L}_\Delta \Theta = N\Theta = \mathcal{L}_\Delta \left(\theta^h \otimes \frac{\partial}{\partial y^h} \right) = \left(\mathcal{L}_\Delta \theta^h \right) \otimes \frac{\partial}{\partial y^h} - \theta^h \otimes \frac{\partial}{\partial y^h};$$

that is, if each r-form θ^h satisfies

$$\mathcal{L}_\Delta \theta^h = (N+1)\theta^h;$$

and therefore if each of its coefficients $\theta^h_{i_1 i_2 \ldots i_r}$ is positively-homogeneous of degree $N+1-r$. If in addition to being homogeneous Θ is closed, we have

$$(N+1)\Theta = d(i_\Delta \theta^h) \otimes \frac{\partial}{\partial y^h} = d(i_\Delta \Theta).$$

Now the vector-valued 2-form defined by the type $(1,2)$ Riemann curvature is positively-homogeneous with $N=2$ (the coefficients R^h_{jk} are of degree 1) and closed; in this case, the formula above gives back the previously established relationship

$$3R^h_{jk} = R^h_{k,j} - R^h_{j,k}.$$

In particular, the curvature vector-valued 2-form is exact.

Isotropic type $(1,2)$ tensors have a special structure; in the first place, as vector-valued forms they may be written as

$$\xi \wedge I + \eta \otimes \Delta,$$

where ξ is a 1-form and η a 2-form.

I shall show that for $n > 2$, if $\pi \wedge I + \sigma \otimes \Delta = 0$, then $\pi = 0$, $\sigma = 0$. At $y \in T^\circ_x M$ we may choose vectors $v_1, v_2 \in T_y T^\circ_x M$ which are linearly independent of Δ_y and of each other. Evaluate the vector-valued 2-form on v_1, v_2:

$$\pi_y(v_1)v_2 - \pi_y(v_2)v_1 + \sigma_y(v_1, v_2)\Delta_y = 0.$$

Linear independence then implies that π_y vanishes on any vector linearly independent of Δ_y, and σ_y vanishes on any pair of such vectors. Now evaluate the vector-valued 2-form on Δ_y, v with v linearly independent of Δ_y:

$$\pi_y(\Delta_y)v + \sigma_y(\Delta_y, v)\Delta_y = 0,$$

which completes the argument. (For any 2-form σ, we get $-i_\Delta \sigma \wedge I + \sigma \otimes \Delta = 0$ if $n=2$.)

More generally, we may consider vector-valued forms $\xi \wedge I + \eta \otimes \Delta$, where ξ is an $(r-1)$-form, $r \geq 1$, η an r-form, and the whole is a vector-valued r-form. It may be shown, by an extension of the argument used above, that provided that

$r < n$, if $\pi \wedge I + \sigma \otimes \Delta = 0$, then $\pi = 0$, $\sigma = 0$. But if $r = n$, then for any n-form σ, $(-1)^{(n-1)} i_\Delta \sigma \wedge I + \sigma \otimes \Delta = 0$.

I now consider the exterior derivative of such a vector-valued r-form, say

$$d(\xi' \wedge I + \eta' \otimes \Delta) = (d\xi' + (-1)^r \eta') \wedge I + d\eta' \otimes \Delta.$$

That is, the exterior derivative has the same structure:

$$d(\xi' \wedge I + \eta' \otimes \Delta) = \xi \wedge I + \eta \otimes \Delta,$$

with

$$\xi = d\xi' + (-1)^r \eta', \qquad \eta = d\eta' = (-1)^r d\xi.$$

This explains the final result of Proposition 1. We have already established that the curvature vector-valued 2-form is the exterior derivative of a vector-valued 1-form. So if it can be expressed as

$$(\xi_j \delta_k^h - \xi_k \delta_j^h + \eta_{jk} y^h) dy^j \wedge dy^k \otimes \frac{\partial}{\partial y^h} = (2\xi_k dy^k) \wedge I + (\eta_{jk} dy^j \wedge dy^k) \otimes \Delta,$$

then we must have (with $r = 1$, $n > 2$)

$$\eta_{jk} dy^j \wedge dy^k = -d(2\xi_k dy^k) = (\xi_{j,k} - \xi_{k,j}) dy^j \wedge dy^k.$$

5. The projective geometry of sprays

The considerations which lead to the study of projective transformations of sprays may be outlined as follows.

Let $\sigma : S(TM) \to M$ be the sphere bundle of M, that is, the quotient of $T^\circ M$ by the action induced by Δ, so that a point s of $S(TM)$ is an equivalence class $[y]$ of nonzero vectors y in $T_x M$, $x = \sigma(s)$, where the equivalence relation is multiplication by a positive scalar; in other words, $s = [y]$ is the ray through y. Suppose given a spray Γ. The fact that it is positively homogeneous means that the 2-dimensional distribution $\mathcal{D}$ on $T^\circ M$ spanned by it and Δ is involutive; the leaves of $\mathcal{D}$ project down to $S(TM)$ to define on it a smooth foliation by oriented 1-dimensional submanifolds $\mathfrak{S}$ which satisfy the so-called second-order property, namely that if $\mathfrak{S}_s$ is the submanifold through s, the (oriented) tangent space to $\sigma(\mathfrak{S}_s)$ at x coincides with the ray s. Such a structure is called a path geometry on M, and captures the idea of a collection of paths – unparametrised but oriented 1-dimensional submanifolds – with one path through each point

of M in each direction. It may be shown that every path geometry on M may be constructed from a spray on $T^\circ M$ in this way. In fact, starting from $\mathfrak{S}$, we may define a distribution $\mathcal{D}$ on $T^\circ M$ by stipulating that $v \in T_{(x,y)}T^\circ M$ belongs to $\mathcal{D}_{(x,y)}$ if the projection of v to $S(TM)$ is tangent to $\mathfrak{S}_s$, $s = [y]$. Then $\mathcal{D}$ is a two-dimensional distribution on $T^\circ M$ containing Δ. It may be shown that $\mathcal{D}$ contains a spray (see [10]): indeed, it contains many, any two of which must differ by a suitable multiple of Δ, say

$$\tilde{\Gamma} = \Gamma - 2P\Delta,$$

where Γ and $\tilde{\Gamma}$ are sprays in $\mathcal{D}$, P is a function on $T^\circ M$, which must be positively-homogeneous of degree 1, and the numerical coefficient is chosen so that $\tilde{\Gamma}^i = \Gamma^i + Py^i$.

Here is a reminder of the standard terminology. Two sprays related in this way are projectively equivalent; one is obtained from the other by a projective transformation; and a set of sprays so related is a projective (equivalence) class. Geometric objects defined from a spray which do not change under a projective transformation are projectively invariant.

5.1. The Douglas tensor. With

$$\tilde{\Gamma}^i = \Gamma^i + Py^i,$$

where P is positively-homogeneous of degree 1, we have

$$\tilde{B}^i_{jkl} = B^i_{jkl} + P_{,kl}\delta^i_j + P_{,jl}\delta^i_k + P_{,jk}\delta^i_l + P_{,jkl}y^i.$$

With $B_{kl} = B^i_{ikl}$, the mean Berwald curvature, we have

$$\tilde{B}_{kl} = B_{kl} + (n+1)P_{,kl},$$

since $P_{,kl}$ is positively-homogeneous of degree -1. It follows that if we set

$$D^i_{jkl} = B^i_{jkl} - \frac{1}{(n+1)}\left(B_{kl}\delta^i_j + B_{jl}\delta^i_k + B_{jk}\delta^i_l + B_{jk,l}y^i\right),$$

then $\tilde{D}^i_{jkl} = D^i_{jkl}$.

Proposition 3. *The tensor whose components are the D^i_{jkl} is projectively invariant.*

It is the Douglas tensor of the projective class of sprays.

Now $B_{kl} = \Gamma^j_{j,kl}$, so given any spray, there is locally – that is to say, over any coordinate neighbourhood in M – a projectively equivalent one for which $\tilde{B}_{kl} = 0$, namely the one with

$$P = -\frac{1}{(n+1)}\Gamma^j_{\ j}.$$

For this spray, $\tilde{B}^i_{jkl} = D^i_{jkl}$.

If we suppose that M is connected and admits a global nowhere-vanishing n-form, then we can find for any spray a projectively-equivalent globally-defined spray whose Berwald curvature coincides with its Douglas tensor.

For convenience, I suppose that the n-form is given in (positively oriented) coordinates by

$$\sqrt{\nu(x)}(dx)^n = \sqrt{\nu(x)}\,dx^1 \wedge \cdots \wedge dx^n.$$

It determines a global $2n$-form Ω on TM, whose expression in coordinates is

$$\Omega(x,y) = \nu(x)(dx)^n \wedge (dy)^n.$$

Proposition 4. *Given such a $2n$-form Ω and any spray Γ, there is a projectively equivalent spray $\tilde{\Gamma}$ such that $\mathrm{div}_\Omega\tilde{\Gamma} = 0$; and the Berwald curvature of $\tilde{\Gamma}$ coincides with the Douglas tensor of the projective class.*

PROOF. Recall that $\mathcal{L}_{\tilde{\Gamma}}\Omega = \mathrm{div}_\Omega\tilde{\Gamma} = d(i_{\tilde{\Gamma}}\Omega)$. Take $\tilde{\Gamma} = \Gamma - 2P\Delta$, where P is positively-homogeneous of degree 1. I first compute $d(i_{P\Delta}\Omega)$:

$$d(i_{P\Delta}\Omega) = d(Pi_\Delta\Omega) = dP \wedge (i_\Delta\Omega) + Pd(i_\Delta\Omega)$$
$$= -i_\Delta(dP \wedge \Omega) + \Delta(P)\Omega + nP\Omega = (n+1)P\Omega.$$

So if we take

$$P = \frac{1}{2(n+1)}\mathrm{div}_\Omega\Gamma,$$

then $\mathrm{div}_\Omega\tilde{\Gamma} = 0$. But

$$d(i_{\tilde{\Gamma}}\Omega) = d\left(\nu\sum_{k=1}^n \left((-1)^{k+1}y^k dx^1 \cdots \widehat{dx^k} \cdots \wedge dx^n \wedge (dy)^n\right.\right.$$
$$\left.\left. - 2(-1)^{n+k+1}\tilde{\Gamma}^k(dx)^n \wedge dy^1 \cdots \widehat{dy^k} \cdots \wedge dy^n\right)\right)$$
$$= \left(y^k\frac{\partial \log \nu}{\partial x^k} - 2\tilde{\Gamma}^k_k\right)\Omega.$$

Thus, when $\mathrm{div}_\Omega\tilde{\Gamma} = 0$,

$$\tilde{\Gamma}^k_k = \tfrac{1}{2}y^k\frac{\partial\log\nu}{\partial x^k},$$

and therefore,

$$\tilde{B}_{ij} = \frac{\partial^2\tilde{\Gamma}^k_k}{\partial y^i\partial y^j} = 0 \quad\text{and}\quad D^i_{jkl} = \tilde{B}^i_{jkl}. \qquad\square$$

Evidently, if a projective class of sprays contains one whose Berwald connection is linear, then the Douglas tensor of the class vanishes. The converse is also clear: if the Douglas tensor vanishes, any spray whose Berwald curvature coincides with the Douglas tensor has linear Berwald connection.

Theorem 1. *If the Douglas tensor of a projective class of sprays vanishes, then the class contains local sprays whose Berwald connections are linear, and if further the base manifold is connected and admits a global volume form, global ones.*

There is an improved version of the global aspect of this result in [16], in which the $2n$-form Ω on TM is replaced by what is called there a vertically-invariant volume form, and it is shown that the tangent manifold of any manifold admits a vertically-invariant volume form: so, in fact, the vanishing of the Douglas tensor is the necessary and sufficient condition for the existence of a global spray in the projective class whose Berwald connection is linear.

5.2. The Weyl tensor. For the horizontal lifts of basic coordinate vector fields defined by projectively related sprays, we have

$$\tilde{H}_i = \frac{\partial}{\partial x^i} - \tilde{\Gamma}^j_i\frac{\partial}{\partial y^j} = H_i - PV_i - V_i(P)\Delta,$$

so that

$$
\begin{aligned}
[\tilde{H}_i, \tilde{H}_j] &= [H_i - PV_i - V_i(P)\Delta, H_j - PV_j - V_j(P)\Delta]\\
&= [H_i, H_j] - \big(H_i(P) - PV_i(P)\big)V_j + \big(H_j(P) - PV_j(P)\big)V_i\\
&\quad - \big(H_i(V_j(P)) - H_j(V_i(P))\big)\Delta,
\end{aligned}
$$

since $[H_i, V_j] = \Gamma^k_{ij}V_k = [H_j, V_i]$. Notice that

$$
\begin{aligned}
H_i(V_j(P)) - H_j(V_i(P)) &= V_j(H_i(P)) - V_i(H_j(P))\\
&= V_j\big(H_i(P) - PV_i(P)\big) - V_i\big(H_j(P) - PV_j(P)\big).
\end{aligned}
$$

So with

$$p_i = H_i(P) - PV_i(P) = P_{;i} - PP_{,i},$$

we have

$$\tilde{R}^h_{ij} = R^h_{ij} + p_i\delta^h_j - p_j\delta^h_i + (p_{i,j} - p_{j,i})y^h;$$

note that p_i is positively-homogeneous of degree 1. That is to say, the difference of the type $(1,2)$ Riemann curvatures is the type $(1,2)$ isotropic tensor formed from p.

Denote by r the covector whose components, constructed (apart from numerical factors) from R^h_{ij} in the manner of Lemma 3, are given by

$$r_i = -\frac{1}{(n^2-1)}\big((n-1)R_i + R^k_{k,i}\big) = \frac{1}{(n^2-1)}(nR_{ki} + R_{ik})y^k.$$

Proposition 5. *The tensor*

$$W^h_{ij} = R^h_{ij} + r_i\delta^h_j - r_j\delta^h_i + (r_{i,j} - r_{j,i})y^h$$

is projectively invariant.

PROOF. Denote by $\tilde{r}_i$ the similar covector constructed from $\tilde{R}^h_{ij}$. When R^h_{ij} and $\tilde{R}^h_{ij}$ are projectively related,

$$\tilde{R}^h_{ij} - R^h_{ij} = p_i\delta^h_j - p_j\delta^h_i + (p_{i,j} - p_{j,i})y^h.$$

So $p_i = r_i - \tilde{r}_i$, by Lemma 2 with $Q^h_{ij} = \tilde{R}^h_{ij} - R^h_{ij}$ and Lemma 3, and therefore

$$\tilde{R}^h_{ij} + \tilde{r}_i\delta^h_j - \tilde{r}_j\delta^h_i + (\tilde{r}_{i,j} - \tilde{r}_{j,i})y^h = R^h_{ij} + r_i\delta^h_j - r_j\delta^h_i + (r_{i,j} - r_{j,i})y^h,$$

as claimed. □

The tensor W with components W^h_{ij} is the Weyl projective curvature tensor of the projective class of sprays. The expression for the covector field r in terms of the Ricci tensor shows that it corresponds to the Schouten tensor of the linear theory: indeed, if the connection is linear, $r_i = s_{ki}y^k$.

We shall rarely have to use an explicit expression for r_i: in fact, it is the structure of the additional terms that is important, as the following proposition reveals.

Proposition 6. *The trace W^j_{ij} of W vanishes. Moreover, W is uniquely determined by the facts that*

(1) *the tensor $W^h_{ij} - R^h_{ij}$ is of isotropic type;*

(2) *the trace of W vanishes.*

PROOF. It is almost evident from the definitions that the trace of W vanishes, but in detail we have

$$W_{ij}^j = R_i + (n+1)r_i - (r_j y^j)_{,i}$$
$$= R_i - \frac{1}{(n-1)}\big((n-1)R_i + R_{k,i}^k\big) + \frac{1}{(n^2-1)}\big((n-1)R_{k,i}^k + 2R_{k,i}^k\big) = 0.$$

If r and $\hat{r}$ are both covector fields, positively-homogeneous of degree 1, such that $W_{ij}^j = \hat{W}_{ij}^j = 0$, where

$$W_{ij}^h = R_{ij}^h + r_i \delta_j^h - r_j \delta_i^h + (r_{i,j} - r_{j,i})y^h$$
$$\hat{W}_{ij}^h = R_{ij}^h + \hat{r}_i \delta_j^h - \hat{r}_j \delta_i^h + (\hat{r}_{i,j} - \hat{r}_{j,i})y^h,$$

then the trace of

$$W_{ij}^h - \hat{W}_{ij}^h = (r_i - \hat{r}_i)\delta_j^h - (r_j - \hat{r}_j)\delta_i^h + \big((r_i - \hat{r}_i)_{,j} - (r_j - \hat{r}_j)_{,i}\big)y^h$$

vanishes, and so $\hat{r} = r$ by Lemma 2. $\qquad \square$

Just as is the case for the Riemann curvature, the Weyl projective curvature exists in several equivalent forms. Set

$$W_k^h = y^j W_{jk}^h, \qquad W_{ijk}^h = W_{jk,i}^h.$$

Proposition 7. *For the Weyl projective curvature of any spray,*

$$W_{jk}^h = \tfrac{1}{3}(W_{k,j}^h - W_{j,k}^h) = y^i W_{ijk}^h.$$

PROOF. We know that such relationships hold for R_{ij}^h. The key point is that they also hold for the additional terms collectively. The cyclic identity for R_{ijk}^h plays an important role in the corresponding results for the Riemann curvature: that it holds for the additional terms is established in Lemma 1. $\qquad \square$

Proposition 8. *The type $(1,3)$ Weyl projective curvature tensor satisfies the cyclic identity*

$$W_{ijk}^h + W_{jki}^h + W_{kij}^h = 0$$

and is completely trace-free.

PROOF. The cyclic identity follows immediately from Lemma 1. We have $W_{ijk}^k = (W_{jk}^k)_{,i} = 0$ by Proposition 6, and of course, $W_{ikj}^k = 0$ also. The vanishing of W_{kij}^k then follows from the cyclic identity. $\qquad \square$

5.3. The Cotton tensor. With r the covector field defined in terms of R as before, I set

$$r_{ij} = r_{j;i} - r_{i;j},$$

and call it, by analogy with the corresponding object in the projective theory of linear connections, the projective Cotton tensor. It is evidently skew, and positively-homogeneous of degree 1. Notice that by the symmetry of Γ_{ij}^k, we can also express r_{ij} as

$$r_{ij} = H_i(r_j) - H_j(r_i).$$

Recall that the type $(1, 2)$ Riemann curvature satisfies a Bianchi identity involving its horizontal covariant derivative:

$$R_{ij;k}^h + R_{jk;i}^h + R_{ki;j}^h = 0.$$

I next examine the consequences of this for the Weyl and Cotton tensors.

Proposition 9. *We have*

$$W_{ij;k}^h + W_{jk;i}^h + W_{ki;j}^h = r_{ij}\delta_k^h + r_{jk}\delta_i^h + r_{ki}\delta_j^h + (r_{ij,k} + r_{jk,i} + r_{ki,j})y^h.$$

PROOF. Using the Bianchi identity quoted above, one finds that

$$W_{ij;k}^h + W_{jk;i}^h + W_{ki;j}^h$$
$$= r_{ij}\delta_k^h + r_{jk}\delta_i^h + r_{ki}\delta_j^h + (r_{i,j;k} - r_{j,i;k} + r_{j,k;i} - r_{k,j;i} + r_{k,i;j} - r_{i,k;j})y^h.$$

Now $r_{j,k;i} - r_{j;i,k} = r_h B_{jik}^h$, so that (for example)

$$r_{j,k;i} - r_{i,k;j} = r_{j;i,k} - r_{i;j,k} = r_{ij,k},$$

by the symmetry of B_{ijk}^h in its lower indices. □

This relation can be solved for r_{ij} rather in the manner of Lemma 2.

Proposition 10. *Provided $n > 2$,*

$$r_{ij} = \frac{1}{(n-2)}W_{ij;k}^k + \frac{1}{(n-2)(n+1)}(B_{ik}W_j^k - B_{jk}W_i^k),$$

where $B_{ij} = B_{kij}^k$ is the mean Berwald curvature. In particular, for $n > 2$, if the Weyl projective curvature vanishes, then so does the projective Cotton tensor.

PROOF. Let me temporarily write W_{ij} for $W_{ij;k}^k$. Summing the expressions in Proposition 9 over h and k gives

$$W_{ij} = (n-1)r_{ij} + (r_{jk,i} - r_{ik,j})y^k.$$

Contract with y^j, and use the skew symmetry of r_{jk} and homogeneity:

$$W_{ij}y^j = (n-2)r_{ij}y^j.$$

It follows that

$$(n-2)(r_{jk,i} - r_{ik,j})y^k = (n-2)\big((r_{jk}y^k)_{,i} + r_{ij} - (r_{ik}y^k)_{,j} + r_{ij}\big)$$
$$= \big((W_{jk}y^k)_{,i} - (W_{ik}y^k)_{,j}\big) + 2(n-2)r_{ij}.$$

Now,

$$W_{ik}y^k = W_{ik;l}^l y^k = (W_{ik}^l y^k)_{;l} = -W_{i;l}^l,$$

so

$$(n-2)(n+1)r_{ij} = (n-2)W_{ij;k}^k + W_{j;k,i}^k - W_{i;k,j}^k.$$

But

$$W_{i;k,j}^k = W_{i,j;k}^k + B_{kjl}^k W_i^l - B_{ijk}^l W_l^k,$$

and therefore,

$$W_{j;k,i}^k - W_{i;k,j}^k = W_{j,i;k}^k - W_{i,j;k}^k + B_{ik}W_j^k - B_{jk}W_i^k$$
$$= 3W_{ij;k}^k + B_{ik}W_j^k - B_{jk}W_i^k.$$

It follows that

$$(n-2)(n+1)r_{ij} = (n+1)W_{ij;k}^k + B_{ik}W_j^k - B_{jk}W_i^k,$$

which gives the stated formula when $n > 2$.   □

When the Berwald connection is linear (or more generally, when the mean Berwald curvature vanishes) this reduces to $(n-2)r_{ij} = W_{ij;k}^k$, which is equivalent to the key property of the Cotton tensor in the projective geometry of linear connections.

The Cotton tensor is not projectively invariant. I now derive, for later use, its transformation law under a projective transformation: that is, I find the relationship between r_{ij} and $\tilde{r}_{ij}$ where $\tilde{R}_{ij}^h = R_{ij}^h + p_i\delta_j^h - p_j\delta_i^h + (p_{i,j} - p_{j,i})y^h$ with $p_i = H_i(P) - PV_i(P)$ and $\tilde{H}_i = H_i - PV_i - V_i(P)\Delta$.

Proposition 11.
$$\tilde{r}_{ij} = r_{ij} + W_{ij}^h P_{,h}.$$

PROOF. It was shown in the proof of Proposition 5 that

$$\tilde{r}_i = r_i - p_i.$$

In order to obtain $\tilde{r}_{ij} = \tilde{H}_i(\tilde{r}_j) - \tilde{H}_j(\tilde{r}_i)$, we shall need $\tilde{H}_i(p_j) - \tilde{H}_j(p_i)$. I shall first compute $H_i(p_j) - H_j(p_i)$:

$$H_i(p_j) - H_j(p_i) = [H_i, H_j](P) - H_i(PV_j(P)) + H_j(PV_i(P)).$$

Now,

$$
\begin{aligned}
H_i(PV_j(P)) &= H_i(P)V_j(P) + P\big([H_i, V_j](P) + V_j(H_i(P))\big) \\
&= p_i V_j(P) + PV_j(p_i) + 2PV_i(P)V_j(P) + PV_j(V_i(P)) + P[H_i, V_j](P)
\end{aligned}
$$

(substituting $PV_i(P) + p_i$ for $H_i(P)$). Recall that $[H_i, V_j] = \Gamma_{ij}^k V_k = [H_j, V_i]$. Then

$$
\begin{aligned}
H_i(p_j) &- H_j(p_i) \\
&= -R_{ij}^h V_h(P) - p_i V_j(P) + p_j V_i(P) - (p_{i,j} - p_{j,i})P = -\tilde{R}_{ij}^h V_h(P).
\end{aligned}
$$

Since reversing the projective transformation requires one simply to change the sign of P, it follows directly that

$$\tilde{H}_i(p_j) - \tilde{H}_j(p_i) = R_{ij}^h V_h(P).$$

Thus

$$
\begin{aligned}
\tilde{H}_i(\tilde{r}_j) - \tilde{H}_j(\tilde{r}_i) &= \tilde{H}_i(r_j + p_j) - \tilde{H}_j(r_i + p_i) \\
&= H_i(r_j) - H_j(r_i) - PV_i(r_j) + PV_j(r_i) \\
&\quad - V_i(P)r_j + V_j(P)r_i + R_{ij}^h V_h(P) \\
&= H_i(r_j) - H_j(r_i) + \big(R_{ij}^h + r_i\delta_j^h - r_j\delta_i^h + (r_{i,j} - r_{j,i})y^h\big)P_{,h} \\
&= H_i(r_j) - H_j(r_i) + W_{ij}^h P_{,h},
\end{aligned}
$$

as claimed.   □

6. The main theorems

There are some preliminary observations to be made before we come to the proofs of the main theorems.

It was noted in Subsection 5.1 that given any spray there is a projectively equivalent one for which the mean Berwald curvature vanishes. The key feature of the Berwald connection of such a spray, established in Proposition 4, is that Γ_i^i takes the form

$$\Gamma_i^i = y^k \frac{\partial \varphi}{\partial x^k}$$

for some function φ independent of y, or, in other words, that

$$\Gamma_{ik}^i = \frac{\partial \varphi}{\partial x^k}.$$

A spray for which this holds has another important property: its Ricci tensor is symmetric. In fact,

$$R_{kj} = R_{kij}^i = H_i(\Gamma_{jk}^i) - H_j(\Gamma_{ik}^i) + \Gamma_{il}^i \Gamma_{jk}^l - \Gamma_{jl}^i \Gamma_{ik}^l,$$

whence

$$R_{jk} - R_{kj} = H_k(\Gamma_{ij}^i) - H_j(\Gamma_{ik}^i) = \frac{\partial}{\partial x^k}\left(\frac{\partial \varphi}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial \varphi}{\partial x^k}\right) = 0.$$

Proposition 12. *Given any spray, there is a projectively equivalent one whose Ricci tensor is symmetric. For such a spray,*

- $(n-1)r_i = R_{ij}y^j$;
- $r_{i,j} = r_{j,i}$;
- $W_{ij}^h = R_{ij}^h + r_i\delta_j^h - r_j\delta_i^h$;
- $(n-2)r_{ij} = W_{ij;k}^k$.

Recall that a spray has isotropic Riemann curvature if and only if there is a covector field ξ, positively-homogeneous of degree 1, such that

$$R_{ij}^h = \xi_i\delta_j^h - \xi_j\delta_i^h + (\xi_{i,j} - \xi_{j,i})y^h.$$

With Lemma 2 to hand, it is easy to see that this is equivalent to the vanishing of the Weyl tensor: as before, it is simply a matter of the form of the expression.

Proposition 13. *A spray has vanishing Weyl projective curvature tensor if and only if it has isotropic Riemann curvature.*

PROOF. If $R_{ij}^h = \xi_i \delta_j^h - \xi_j \delta_i^h + (\xi_{i,j} - \xi_{j,i})y^h$, then the Weyl tensor also has isotropic form:

$$W_{ij}^h = (\xi_i + r_i)\delta_j^h - (\xi_j + r_j)\delta_i^h + \big((\xi_i + r_i)_{,j} - (\xi_j + r_j)_{,i}\big)y^h;$$

but W is trace-free, so by Lemma 2 it vanishes. Conversely, if $W_{ij}^h = 0$, then R_{ij}^h has the stated form with $\xi_i = -r_i$. $\qquad\square$

Corollary 1. *For two projectively equivalent sprays, if one is isotropic, so is the other.*

Theorem 2. *For* $\dim M > 2$, *the following properties of a spray are equivalent:*

(1) *it is isotropic;*

(2) *its Weyl projective curvature vanishes;*

(3) *it is locally projectively equivalent to one which is R-flat.*

PROOF. The equivalence of the first two statements has already been established in Proposition 13.

Under a projective transformation, $\tilde{R}_{ij}^h = R_{ij}^h + p_i \delta_j^h - p_j \delta_i^h + (p_{i,j} - p_{j,i})y^h$, where p_i is positively-homogeneous of degree 1. So if there is a projective transformation making $\tilde{R}_{ij}^h = 0$, R_{ij}^h is isotropic.

For the converse, I assume that the Weyl projective curvature vanishes, so that $R_{ij}^h = -r_i \delta_j^h + r_j \delta_i^h - (r_{i,j} - r_{j,i})y^h$. I have to show that there is a locally-defined function P, positively-homogeneous of degree 1, such that $p_i = H_i(P) - PV_i(P) = r_i$, for then the projective transformation with such P makes $\tilde{R}_{ij}^h = 0$. We know from Proposition 10 that the projective Cotton tensor vanishes, since $\dim M > 2$. Furthermore, by Proposition 12 we can, and shall, assume that the Ricci tensor is symmetric

Consider the manifold $T^\circ M \times \mathbb{R}$, with coordinate z on the final factor. On this manifold I define a modified horizontal distribution, spanned by the vector fields

$$\hat{H}_i = H_i - zV_i + r_i \frac{\partial}{\partial z}, \quad i = 1, 2, \ldots, n.$$

I compute the bracket of two such vector fields:

$$\left[H_i - zV_i + r_i \frac{\partial}{\partial z}, H_j - zV_j + r_j \frac{\partial}{\partial z} \right]$$

$$= -R_{ij}^h V_h - r_i V_j + r_j V_i + \big(z(r_{i,j} - r_{j,i}) + H_i(r_j) - H_j(r_i) \big) \frac{\partial}{\partial z}$$

$$= -W_{ij}^h V_h + r_{ij} \frac{\partial}{\partial z} + (r_{i,j} - r_{j,i}) \left(\Delta + z \frac{\partial}{\partial z} \right) = 0,$$

since, by assumption, $W_{ij}^h = 0$, $r_{ij} = 0$, and $r_{i,j} - r_{j,i} = 0$. Furthermore,

$$\left[\Delta + z\frac{\partial}{\partial z}, H_i - zV_i + r_i\frac{\partial}{\partial z}\right] = zV_i - zV_i + r_i\frac{\partial}{\partial z} - r_i\frac{\partial}{\partial z} = 0.$$

Now $\hat{\Delta} = \Delta + z\partial/\partial z$ is the generator of dilations in the fibres of $T^\circ M \times \mathbb{R} \to M$, so the vector fields $\hat{H}_i$ are dilation invariant.

The distribution $\mathcal{D}$ on $T^\circ M \times \mathbb{R}$ spanned by the $\hat{H}_i$ is evidently regular and n-dimensional. I have just shown that it is involutive, and invariant by $\hat{\Delta}$. So $\mathcal{D}$ is integrable. Its integral submanifolds are n-dimensional, and mapped one to another by dilations of the fibres of $T^\circ M \times \mathbb{R} \to M$. They are moreover everywhere transverse to the fibration $T^\circ M \times \mathbb{R} \to T^\circ M$. A section of this latter fibration will be given by $z = P$, where P is a function on $T^\circ M$; and $\mathcal{D}$ will be tangent to the image of the section if and only if

$$\hat{H}_i(z - P) = H_i(z - P) - zV_i(z - P) + r_i\frac{\partial}{\partial z}(z - P) = -H_i(P) + zV_i(P) + r_i = 0,$$

where $z = P$, that is, if and only if

$$H_i(P) - PV_i(P) = r_i.$$

Moreover,

$$\Delta(z - P) + z\frac{\partial}{\partial z}(z - P) = z - \Delta(P),$$

and so P will be positively-homogeneous of degree 1 if and only if $\hat{\Delta}$ is also tangent to the image of the section $z = P$.

So a local section of $T^\circ M \times \mathbb{R} \to T^\circ M$ with the property that both $\mathcal{D}$ and $\hat{\Delta}$ are tangent to its image will provide a function P with the required properties. It remains to show how to construct such a local section. Take any point $x_0 \in M$. Let P_0 be any function on $T_{x_0}^\circ M \times \mathbb{R}$ which is positively-homogeneous of degree 1, and let $\sigma_0 \subset T_{x_0}^\circ M$ be the graph of P_0, that is, the section $z = P_0$ of the fibration $T_{x_0}^\circ M \times \mathbb{R} \to T_{x_0}^\circ M$. Then σ_0 is a codimension-1 submanifold of $T_{x_0}^\circ M \times \mathbb{R}$, and the restriction of $\hat{\Delta}$ to $T_{x_0}^\circ M \times \mathbb{R}$ is tangent to σ_0. I denote by $\mathcal{L}(x, y, z)$ the leaf of the integrable distribution $\mathcal{D}$ through $(x, y, z) \in T^\circ M \times \mathbb{R}$. Set

$$\sigma = \bigcup_{(y,z)\in\sigma_0} \mathcal{L}(x_0, y, z) = \bigcup_{y\in T_{x_0}^\circ M} \mathcal{L}(x_0, y, P_0(y)).$$

Then σ is a codimension-1 submanifold of $T^\circ M \times \mathbb{R}$, defined in a neighbourhood of $T_{x_0}^\circ M \times \mathbb{R}$, transverse to the fibration $T^\circ M \times \mathbb{R} \to T^\circ M$, to which both $\mathcal{D}$ and $\hat{\Delta}$ are tangent. It is the image of the required section. □

Corollary 2. *If M, with $\dim M > 2$, is connected and simply connected, then for any isotropic spray Γ on $T^\circ M$, there is a unique globally-defined function on $T^\circ M$, positively-homogeneous of degree 1, with specified values on $T^\circ_{x_0} M$ for any $x_0 \in M$, such that $\Gamma - 2P\Delta$ is R-flat.*

PROOF. Since $\mathcal{D}$ is horizontal with respect to the projection $T^\circ M \times \mathbb{R} \to M$, which is to say that for each $(x, y, z) \in T^\circ M \times \mathbb{R}$, $\mathcal{D}_{(x,y,z)}$ is isomorphic to $T_x M$, every leaf $\mathcal{L}(x, y, z)$ of the integrable distribution $\mathcal{D}$ is a covering manifold of M. But by assumption, M is simply connected, so $\mathcal{L}(x, y, z)$ is actually diffeomorphic to M. This means that $\sigma = \cup_{y \in T^\circ_{x_0} M} \mathcal{L}(x_0, y, P_0(y))$ defines a global section of $T^\circ M \times \mathbb{R} \to T^\circ M$. $\qquad\square$

What about the case $\dim M = 2$? In the first place, from Proposition 2, we know that every spray is isotropic, and therefore $W^h_{ij} = 0$ automatically. But of course it does not follow automatically that every spray is locally projectively R-flat. The proof of Theorem 2 made use of the fact that when $\dim M > 2$, the projective Cotton tensor of an isotropic spray vanishes. When $\dim M = 2$, this condition has to be imposed.

It follows from Proposition 11 that if $W^h_{ij} = 0$, then r_{ij} is projectively invariant; and since no restriction was made on dimension, this holds in particular if $\dim M = 2$. So the vanishing of the Cotton tensor is then a projectively invariant condition. Moreover, if we assume that the Cotton tensor vanishes, the proof of Theorem 2 proceeds as before. We can conclude that the projective Cotton tensor plays the role of the Weyl projective curvature when $\dim M = 2$.

Theorem 3. *When $\dim M = 2$, a spray is projectively R-flat if and only if its projective Cotton tensor vanishes.*

7. Finsler spaces

The following property of the type $(1, 1)$ Riemann curvature of the Berwald connection of the geodesic spray of a Finsler space is well-known: it is self-adjoint with respect to the fundamental tensor, which is to say that if $R_{ij} = g_{ik} R^k_j$, then $R_{ij} = R_{ji}$. (This is to be distinguished from the Ricci tensor.)

Proposition 14. *The Riemann curvature of a Finsler space is isotropic if and only if there is a function κ on $T^\circ M$, positively-homogeneous of degree 0, such that*

$$R_{ij} = \kappa\big((g_{kl}y^k y^l)g_{ij} - y_i y_j\big),$$

where $y_i = g_{ik}y^k$.

PROOF. Recall from Proposition 1 that one version of the condition for the Riemann curvature to be isotropic is that there is a function ρ, positively-homogeneous of degree 2, and a covector τ, positively-homogeneous of degree 1, such that $R^i_j = \rho\delta^i_j + \tau_j y^i$; or equivalently, in the Finsler case,

$$R_{ij} = \rho g_{ij} + \tau_j y_i.$$

So evidently, if R_{ij} takes the given form, then the Riemann curvature is isotropic.

For the converse, suppose that $R_{ij} = \rho g_{ij} + \tau_j y_i$. Recall from Proposition 1 that $\tau_i y^i = -\rho$. On the other hand, since $R_{ij} = R_{ji}$ we have $\tau_j y_i = \tau_i y_j$, whence

$$\tau_j g_{kl} y^k y^l = -\rho y_j.$$

So R_{ij} takes the given form, with $\kappa = \rho/(g_{kl}y^k y^l)$. $\square$

For $x \in M$, $y, v \in T_x M$, $y, v \neq 0$, the flag curvature $K_{(x,y)}(v)$ is defined as

$$K_{(x,y)}(v) = \frac{R_{ij}v^i v^j}{(g_{kl}y^k y^l)(g_{pq}v^p v^q) - (g_{rs}y^r v^s)^2}.$$

A Finsler space for which the flag curvature is independent of the flag, in other words for which $K_{(x,y)}(v) = \kappa(x, y)$ for some function κ on $T^\circ M$ (which must be positively-homogeneous of degree 0), is said to have scalar flag curvature. It should come as no surprise in view of the previous proposition that for a space to have scalar flag curvature its Riemann curvature must be isotropic. (The flag curvature is scalar if it is the same in all directions of the flag, which perhaps finally explains the use of the term 'isotropic' in the expression 'isotropic Riemann curvature'.)

Theorem 4. *Assume that* $\dim M > 2$. *A Finsler space has scalar flag curvature if and only if its Riemann curvature is isotropic, that is, if and only if its Weyl tensor vanishes. A Finsler space of scalar flag curvature is projectively R-flat. If two Finsler spaces over the same base manifold have projectively equivalent geodesic sprays and one is of scalar flag curvature, so is the other.*

The last assertion is the analogue in Finsler geometry of Beltrami's Theorem in Riemannian geometry: a metric projectively equivalent to a metric of constant curvature is itself a metric of constant curvature. (An R-flat Finsler space is of course of zero scalar flag curvature.)

PROOF. If the Riemann curvature is isotropic, and takes the form given in the proposition above, then the flag curvature is κ. If, on the other hand, the space is of scalar flag curvature κ, then

$$R_{ij}v^iv^j = \kappa\big((g_{pq}y^py^q)(g_{rs}v^rv^s) - (g_{pq}y^pv^q)^2\big) = \kappa\big((g_{pq}y^py^q)g_{ij} - y_iy_j\big)v^iv^j,$$

for every v^i; so since R_{ij} is symmetric,

$$R_{ij} = \kappa((g_{pq}y^py^q)g_{ij} - y_iy_j)$$

and the Riemann curvature is isotropic. The remaining results follow from known properties of sprays with isotropic Riemann curvature. □

In this context, I should mention the following local results (see [4], [8]): for $\dim M > 2$, every R-flat spray is Finsler metrizable (that is to say, there is a Finsler function of which it is the canonical geodesic spray), and therefore every isotropic spray is projectively metrizable (that is to say, is projectively equivalent to the canonical geodesic spray of some Finsler function); and every spray over a 2-dimensional base is projectively metrizable.

Finally, I must acknowledge that BERWALD analysed fully the case of a 2-dimensional Finsler space in his paper [3] of 1941. I shall explain below how his results are related to mine.

In a 2-dimensional Finsler space,

$$R_{ij} = \kappa((g_{pq}y^py^q)g_{ij} - y_iy_j) = \kappa F^2(g_{ij} - l_il_j), \qquad l_i = \frac{1}{F}y_i.$$

It is conventional to work in terms of an orthonormal basis of vector fields $\{l, m\}$, with $l^i = y^i/F$, and the corresponding covector fields whose components are $l_i = g_{ij}l^j = F_{,i} = y_i/F$ as above, $m_i = g_{ij}m^j$. This approach dates back at least to Berwald's paper [3], though the notation used here is closer to that of MATSUMOTO [13]. We have $g_{ij} = l_il_j + m_im_j$, and so

$$R_{ij} = \kappa F^2 m_im_j.$$

One useful technique is to express tensors as linear combinations of tensor products of the basis vectors and covectors. Consider, for example, R_{ij}^h. It is skew in i and j, and satisfies $l_hR_{ij}^h = 0$ (since $H_i(F) = H_j(F) = 0$, and therefore $[H_i, H_j](F) = R_{ij}^hF_{,h} = 0$): it must therefore take the form

$$R_{ij}^h = km^h(l_im_j - l_jm_i)$$

for some scalar k. But $R_j^h = y^i R_{ij}^h = F l^i R_{ij}^h$, whence

$$R_{ij} = k F m_i m_j = \kappa F^2 m_i m_j,$$

and therefore

$$R_{ij}^h = \kappa F m^h (l_i m_j - l_j m_i).$$

This identifies κ with the curvature of the 2-dimensional Finsler space as defined in [3], [13] – which is hardly surprising, but one has to be a bit careful about factors F, etc.

To compute the Cotton tensor, we first need r_i, which is given by

$$r_i = -\tfrac{1}{3}(R_i + (R_k y^k)_{,i}).$$

We have

$$R_i = R_{ij}^j = \kappa F l_i,$$

so that

$$r_i = -\tfrac{1}{3}(\kappa F l_i + (\kappa F^2)_{,i}) = -\tfrac{1}{3}(3\kappa F l_i + \kappa_{,i} F^2).$$

The Cotton tensor is $r_{i;j} - r_{j;i}$. It has but one component which does not vanish automatically, which may be taken to be $(r_{i;j} - r_{j;i}) l^i m^j$. One can take advantage of the fact that $l^i_{;j} = 0$ to write the first term as $(r_i l^i)_{;j} m^j = -F m^j H_j(\kappa)$. It so happens that $m_{;i}^j l^i = 0$, as I now show, which allows one to carry out a similar manoeuvre on the second term.

Lemma 4.
$$m_{;i}^j l^i = 0.$$

PROOF. We can express the vector $m_{;i}^j l^i$ as a linear combination of l^j and m^j. But since $m_{;i}^j l^i l_j = (m^j l_j)_{;i} l^i = 0$, there is no l^j component: $m_{;i}^j l^i = \mu m^j$ for some scalar μ. Now

$$g^{ij}{}_{;k} l^k = 0 = m_{;k}^i l^k m^j + m^i m_{;k}^j l^k = 2\mu m^i m^j,$$

so $\mu = 0$. $\qquad\square$

It follows that

$$r_{j;i} l^i m^j = (r_j m^j)_{;i} l^i = -\tfrac{1}{3} F l^i H_i(F m^j \kappa_{,j}) = -\tfrac{1}{3} F l^i H_i(F m^j V_j(\kappa)).$$

We have the following expression for the Cotton tensor of a 2-dimensional Finsler space.

Proposition 15.

$$(r_{i;j} - r_{j;i})l^i m^j = \tfrac{1}{3}F(l^i H_i(Fm^j V_j(\kappa)) - 3m^j H_j(\kappa)).$$

The expression $l^i H_i(Fm^j V_j(\kappa)) - 3m^j H_j(\kappa)$ agrees with one for a quantity playing a similar role which occurs in Berwald's paper [3]. Berwald marks the effects of the various operators appearing in the formula above by attaching certain subscripts to their arguments, as follows. For any function Φ which is positively-homogeneous of degree 0,

$$l^i H_i(\Phi) = \Phi_s, \qquad m^i H_i(\Phi) = \Phi_b, \qquad Fm^i V_i(\Phi) = \Phi_\vartheta.$$

In Berwald's notation, the expression given in the proposition above is (apart from unimportant factors)

$$\kappa_{\vartheta s} - 3\kappa_b.$$

Berwald does indeed show that the vanishing of this quantity is the necessary and sufficient condition for there to be a projective transformation which makes the Riemann curvature zero.

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