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A generalization of the Gelfand–Kolmogoroff theorem

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Abstract. We introduce the notion of a ring isomorphism in norm. For such maps, we obtain an extension of the Gelfand–Kolmogoroff theorem by showing that a ring isomorphism in norm between spaces of continuous functions on compact spaces is a weighted composition operator.

1. Introduction

Let X be a compact Hausdorff space, and let $C(X)$ denote the Banach algebra of all continuous real-valued functions on X equipped with the supremum norm. The classical Banach–Stone theorem asserts that the linear metric structure of $C(X)$ determines the topology of X (see [1], [7]). More precisely, every surjective isometry T between $C(X)$ and $C(Y)$ must be of the form $Tf = h \cdot (f \circ \tau)$ (that is, it is a *weighted composition operator*), where $\tau : Y \to X$ is a homeomorphism and $h \in C(Y)$ satisfies $|h| = 1$. In 1966, Holsztyński [5] showed that any non-surjetive isometry between spaces of continuous functions is also a weighted composition operator.

The Banach–Stone theorem has found a large number of extensions, generalizations and variants in many different contexts (see, for example, [2]). Among

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them, one line of research which started with the Banach–Stone theorem established a link between the algebraic properties. In this direction, the following well-known result was obtained by GELFAND and KOLMOGOROFF [3] in 1939.

Theorem 1.1 (Gelfand–Kolmogoroff)**.** *Suppose that X and Y are compact Hausdorff spaces. If $C(X)$ and $C(Y)$ are isomorphic as rings, then X and Y are homeomorphic and every ring isomorphism $T : C(X) \to C(Y)$ is of the form $Tf = f \circ \tau$, where $\tau : Y \to X$ is a homeomorphism.*

Indeed, this result can be obtained by the Banach–Stone theorem, since ring isomorphism implies isometry (see, for example, GILLMAN and JERISON [4, 1J.6]). The main purpose of this short note is to generalize this theorem by considering some class of more general maps.

2. Main results

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras. If $T : \mathcal{A} \to \mathcal{B}$ is a bijection so that

$$\|T(f \cdot g)\| = \|Tf \cdot Tg\| \tag{1}$$

and

$$\|T(f + g)\| = \|Tf + Tg\| \tag{2}$$

for all $f, g \in \mathcal{A}$, then we call T a *ring isomorphism in norm*. We say that T is unital if T preserves the identity element. A natural question is whether every unital ring isomorphism in norm from $\mathcal{A}$ onto $\mathcal{B}$ is indeed a ring isomorphism.

The next theorem provides a positive answer for the class of $C(X)$-spaces and strengthens the Gelfand–Kolmogoroff theorem.

Theorem 2.1. *Suppose that X and Y are compact Hausdorff spaces. If there exists a ring isomorhpism in norm $T : C(X) \to C(Y)$, then there is a continuous function $h \in C(Y)$ with $|h(y)| = 1$ ($y \in Y$) and a homeomorphism $\tau : Y \to X$ such that*

$$(Tf)(y) = h(y) \cdot f(\tau(y)) \ (y \in Y, f \in C(X)) . \tag{3}$$

PROOF. The proof will be divided into two parts. In the first part, we show that it is true if $T1 = 1$. Then, making use of this result, we show in the second part that the result holds for a general ring isomorphism in norm T.

Part I. Assume that $T1 = 1$. Since T is a bijection satisfying (2), [8, Corollary 1] shows that T is additive, and hence we have $T(\alpha f) = \alpha(Tf)$ for all $\alpha \in \mathbb{Q}$.

In what follows, we show that if $f(x) \geq 0$ for all $x \in X$, then

$$(Tf)(y) \geq 0 \quad \text{for all} \quad y \in Y. \tag{4}$$

Since

$$\left\| T\left(\left(\frac{f}{n^2}\right)^{\frac{1}{2}}\right)\right\| = \frac{\|T(f^{\frac{1}{2}})\|}{n}, \tag{5}$$

we can choose n large enough such that

$$\left\| T\left(\left(\frac{f}{n^2}\right)^{\frac{1}{2}}\right)\right\| \leq 1. \tag{6}$$

Then

$$\begin{aligned}
\left\| T\left(1 - \frac{f}{n}\right)\right\| &= \left\| T\left(\left(1 - \left(\frac{f}{n}\right)^{\frac{1}{2}}\right)\left(1 + \left(\frac{f}{n}\right)^{\frac{1}{2}}\right)\right)\right\| \\
&= \left\| T\left(1 - \left(\frac{f}{n}\right)^{\frac{1}{2}}\right) \cdot T\left(1 + \left(\frac{f}{n}\right)^{\frac{1}{2}}\right)\right\| \\
&= \left\| \left(1 - T\left(\left(\frac{f}{n}\right)^{\frac{1}{2}}\right)\right)\left(1 + T\left(\left(\frac{f}{n}\right)^{\frac{1}{2}}\right)\right)\right\| \\
&= \left\| 1 - \left(T\left(\left(\frac{f}{n}\right)^{\frac{1}{2}}\right)\right)^2\right\| \leq 1.
\end{aligned} \tag{7}$$

Therefore, $\|1 - T(f/n)\| \leq 1$, which implies $(T(f/n))(y) \geq 0$ for all $y \in Y$, and hence (4) holds.

Fix $f \in C(X)$. Put $f_+ = \max\{f, 0\}$, and $f_- = \max\{-f, 0\}$. Clearly, $f = f_+ - f_-$. By (1), we get $\|Tf_+ \cdot Tf_-\| = 0$, and then

$$\operatorname{supp}(Tf_+) \cap \operatorname{supp}(Tf_-) = \emptyset. \tag{8}$$

It follows that

$$\|Tf\| = \|Tf_+ - Tf_-\| = \max\{\|Tf_+\|, \|Tf_-\|\} = \|Tf_+ + Tf_-\| = \|T(|f|)\|. \tag{9}$$

We next show that

$$\|Tf\| \leq \|f\| \tag{10}$$

for all $f \in C(X)$. Let λ be any rational number so that $\|f\| \leq \lambda$. Then (4) implies $T(\lambda - |f|) \geq 0$, and hence $0 \leq T(|f|) \leq \lambda$. Therefore, $\|T(|f|)\| \leq \lambda$, and then (9) shows $\|Tf\| \leq \lambda$. Since λ is an arbitrary rational number satisfying $\|f\| \leq \lambda$, we obtain (10).

Now the additivity of T and (10) imply that T is continuous and linear. Since T is a continuous linear bijection, by the open mapping theorem, T is an isomorphism. Therefore, there is $\beta > 0$ such that

$$\beta\|f\| \leq \|Tf\| \leq \|f\| \tag{11}$$

for all $f \in C(X)$.

We claim that β in (11) can be chosen to be 1. Suppose not. Then there exists $f \in C(X)$ so that $\|f\| = 1$ and $\|Tf\| < \|f\|$. For any positive integer n, by (1),

$$\begin{aligned} \left\|T\left(f^{2^n}\right)\right\| &= \left\|T\left(f^{2^{n-1}}\right) \cdot T\left(f^{2^{n-1}}\right)\right\| = \left\|T\left(f^{2^{n-1}}\right)\right\|^2 \\ &= \left\|T\left(f^{2^{n-2}}\right)\right\|^4 = \ldots = \|Tf\|^{2^n}. \end{aligned} \tag{12}$$

Since $\|Tf\| < \|f\| = 1$, choose n large enough so that $\|Tf\|^{2^n} < \beta$. (12) shows that $\left\|T\left(f^{2^n}\right)\right\| < \beta$. On the other hand, it is clear that $\left\|f^{2^n}\right\| = 1$, since $\|f\| = 1$. Then (11) shows that $\left\|T\left(f^{2^n}\right)\right\| \geq \beta\left\|f^{2^n}\right\| = \beta$, which leads to a contradiction.

Now (11) becomes $\|Tf\| = \|f\|$ for all $f \in C(X)$. Since T is linear, we have proved that T is a surjective linear isometry with $T1 = 1$. By the Banach–Stone theorem, there is a homeomorphism $\tau : Y \to X$ such that

$$(Tf)(y) = f(\tau(y)) \tag{13}$$

for all $f \in C(X)$ and $y \in Y$.

Part II. We claim that $|(T1)(y)| = 1$ for all $y \in Y$. Assume not. There is an open subset U of Y and $\varepsilon > 0$ such that $|(T1)(y)| \geq 1 + \varepsilon$ for all $y \in U$ or $|(T1)(y)| \geq 1 - \varepsilon$ for all $y \in U$. Without loss of generality, we assume that

$$|(T1)(y)| \geq 1 - \varepsilon \tag{14}$$

for all $y \in U$. By Urysohn's lemma, there is a nonzero function $h \in C(Y)$ such that $\operatorname{supp}(h) \subset U$. Let f be an element in X such that $Tf = h$ (note that T is surjective). Then

$$\|h\| = \|Tf\| = \|T(1 \cdot f)\| = \|T1 \cdot Tf\| = \|T1 \cdot h\| \leq (1-\varepsilon)\|h\|. \tag{15}$$

We get a contradiction.

Clearly, $T/T1$ also satisfies (1) and (2), since $|(T1)(y)| = 1$. Replacing T by $T/T1$ in Part I, we get that there exists a homeomorphism $\tau : Y \to X$ such that

$$(Tf)(y) = (T1)(y) \cdot (f(\tau(y))) \tag{16}$$

for all $f \in C(X)$ and $y \in Y$. Letting $h = T1$ finishes the proof. □

Remark 2.2. In the proof of the above theorem, we see that if T is a unital ring isomorphism in norm, then T has the form $Tf = f \circ \tau$, and hence T is a ring isomorphism (indeed, an algebraic isomorphism).

Remark 2.3. A direct consequence of Theorem 2.1 is that the existence of a ring isomorphism in norm between $C(X)$ and $C(Y)$ implies a homeomorphism between X and Y. The referee told us this consequence can also be obtained by the main result of [6].

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