Publ. Math. Debrecen
94/3-4 (2019), 269–287
DOI: 10.5486/PMD.2019.8242

The inequality and its application of algebroid functions on annulus concerning some polynomials

By HONG YAN XU (Shangrao) and ZHAO JUN WU (Xianning)

Abstract. The main purpose of this paper is to investigate the value distribution of algebroid functions on annulus, and establish the second fundamental theorem of algebroid functions on annulus concerning some polynomials, which is an improvement of the previous results given by Tan. By applying this inequality, we obtain three results of algebroid functions on annulus concerning its derivatives and some polynomials.

1. Introduction

We first assume that the reader is familiar with the basic results and the standard notations of Nevanlinna's value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f), \dots$ (see Hayman [6], Yang [29], Yi and Yang [30]). As we know, research on the value distribution of meromorphic functions is very active in the field of complex analysis. It is well known that the Nevanlinna theory plays an important role in studying the value distribution of meromorphic functions. In the past one hundred years, there were many classic theorems and results in this respect, such as: the first and second main theorem, lemma on the logarithmic derivatives, the five values theorem, etc. It is also of

---

Mathematics Subject Classification: 32C20, 30D45.
Key words and phrases: algebroid function, annulus, polynomial.
This work was supported by the National Natural Science Foundation of China (11561033), the Natural Science Foundation of Jiangxi Province in China (20181BAB201001), and the Foundation of the Education Department of Jiangxi (GJJ170759, GJJ180734) of China. The second author was partially supported by the NSF of the Educational Department of Hubei Province (D20182801) and by Hubei Key Laboratory of Applied Mathematics (Hubei University).

interest to extend some classic theorems and results of value distribution of meromorphic functions in the whole complex plane to angular domains, unit disc, etc. For example, Shea and Sons [20] in 1986 studied the value distribution theory for meromorphic functions of slow growth in the disk, and Fang [4] in 1999 discussed the uniqueness of meromorphic functions sharing some values and sets in the unit disc, Valiron, Yang, Zhang, Wu, etc. had paid consideration attention to the singular direction of meromorphic functions by using the Nevanlinna characteristic function in the angular domain, and obtained some important results on the existence of some singular directions such as: Borel direction, Julia direction, Nevanlinna direction (see [25]–[26], [29], [33]), Zheng [31]–[32] in 2000s gave some interesting uniqueness theorems of meromorphic functions by using the Tujsi characteristic function in the angular domain. Besides, there were also a lot of papers focusing on the value distribution of meromorphic functions in the unit disc and angular domain, see [13]–[15], [18]–[19], [24], [28], [30], [33].

In fact, the whole complex plane, the unit disc and the angular domain can all be regarded as a single connected region, in other words, the theorems stated in the above references are only regarded as the uniqueness results in simply connected region. However, the annulus and the m-punctured complex plane in the whole complex plane can be called as the double connected domain and the several connected domain, respectively. Moreover, for meromorphic functions on the double connected domain and several connected region, there were only few papers about value distribution and uniqueness. Twelve years ago, Khrystiyanyn and Kondratyuk [9]–[10] in 2005 established the Nevanlinna theory for meromorphic functions on annuli (see also [11]–[12]), whereafter, Lund and Ye [16]–[17] in 2009 and 2010 studied meromorphic functions on annuli with the form $\{z : R_1 < |z| < R_2\}$, where $R_1 \geq 0$ and $R_2 \leq +\infty$. In 2009, Cao [2] investigated the uniqueness of meromorphic functions on annuli sharing some values, and established an analog of Nevanlinna's famous five-value theorem. Fernández [5] in 2010 further investigated the value distribution of meromorphic functions on annuli. Xu and Xuan [27] in 2012 studied the uniqueness of meromorphic functions sharing some values on annuli. In the same year, Chen and Wu [3] discussed the Borel exceptional values of meromorphic function and its derivative on annulus.

Let $H_k(z), \dots, H_0(z)$ be analytic functions in a single connected domain $\mathbb{S} \subseteq \mathbb{C}$ without common zeros, then a k-valued algebroid function $f(z)$ in $\mathbb{S} \subseteq \mathbb{C}$ can be determined by the irreducible equation (see [7], [21])

$$\Psi(z, f) = H_k(z)f^k + H_{k-1}(z)f^{k-1} + \cdots + H_0(z) = 0.$$

If $k = 1$, then $f(z)$ is a meromorphic function in $\mathbb{S}$. The notion of algebroid functions was firstly introduced by H. Poincaré, and G. Darboux pointed out that it is a very important class of functions (see [8]). As the extension of meromorphic functions, He, Sun, Gao, etc. investigated the value distribution of algebroid functions in $\mathbb{S}$, and obtained the first and second fundamental theorems, the lemma on the logarithmic derivatives, etc. of algebroid functions in some single connected domains — the whole complex $\mathbb{C}$, the unit disc $\mathbb{D}$ and the angular domain $\triangle$. Inspired by the idea in [9]–[10], Tan [22]–[23] first in 2015 and 2016 studied the value distribution and uniqueness of algebroid functions on annuli $\mathbb{A}$, and established some basic results such as the first and second fundamental theorems, and the Cartan theorem for algebroid functions on annuli $\mathbb{A}$. But there are few papers focusing on the value distribution of algebroid functions in some double connected domain and several connected regions.

In this paper, we will further study the value distribution of algebroid functions on annuli. The structure of this paper is as follows. In Section 2, we introduce the basic notations and fundamental theorems of algebroid functions on annuli. Section 3 gives our main theorems and corollaries including the second fundamental theorem for algebroid functions on annuli concerning finite many polynomials and related results for algebroid function concerning its derivation. Section 4 lists some required lemmas. Section 5 shows the proofs of our main results and corollaries.

2. Basic notions of algebroid function on annuli

From the Doubly Connected Mapping Theorem [1], it is easy to see that each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \le r < R \le +\infty$. If $0 < r < R < +\infty$, take the homothety $z \mapsto \frac{z}{\sqrt{rR}}$, then the annulus $\{z : r < |z| < R\}$ is reduced to $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R}{r}}$; if $r = 0$ and $R = +\infty$, then the annulus $\{z : r < |z| < R\}$ is $\{z : 0 < |z| < +\infty\}$. Thus, in two cases, every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

Similar to [7], [21], we will show the basic notions and theorems of algebroid functions on annulus $\mathbb{A}$ (see [22]–[23]) as follows. Let $A_k(z), \dots, A_0(z)$ be analytic functions on annulus $\mathbb{A} := \{z : \frac{1}{R_0} < |z| < R_0\}(1 < R_0 \le +\infty)$ without common zeros, then a k-valued algebroid function $W(z)$ on annulus $\mathbb{A}$ can be determined by the irreducible equation (see [22]–[23])

$$\psi(z, W) = A_k(z)W^k + A_{k-1}(z)W^{k-1} + \dots + A_0(z) = 0. \tag{1}$$

Let $W(z)$ be a k-valued algebroid function on annulus $\mathbb{A}$ and $1 < r < R_0 \leq +\infty$. We denote the notations

$$m(r, W) = \frac{1}{k}\sum_{j=1}^{k} m(r, w_j) = \frac{1}{k}\sum_{j=1}^{k} \frac{1}{2\pi} \log^+ |w_j(re^{i\theta})| d\theta,$$

$$N_1(r, W) = \frac{1}{k}\int_{\frac{1}{r}}^{1} \frac{n_1(t, W)}{t} dt, \qquad N_2(r, W) = \frac{1}{k}\int_{1}^{r} \frac{n_2(t, W)}{t} dt,$$

$$m_0(r, W) = m(r, W) + m\left(\frac{1}{r}, W\right) - 2m(1, W),$$

$$N_0(r, W) = N_1(r, W) + N_2(r, W),$$

and

$$N_{x_1}(r, W) = \frac{1}{k}\int_{\frac{1}{r}}^{1} \frac{n_{x_1}(t, W)}{t} dt, \qquad N_{x_2}(r, W) = \frac{1}{k}\int_{1}^{r} \frac{n_{x_2}(t, W)}{t} dt,$$

$$N_x(r, W) = N_{x_1}(r, W) + N_{x_2}(r, W),$$

where $w_j(z)(j = 1, 2, \ldots, k)$ is a one-valued branch of $W(z)$, $n_1(t, W)$ and $n_2(t, W)$ are the counting functions of poles of the function $W(z)$ in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$ (counting multiplicity), respectively, and $n_{x_1}(t, W)$ and $n_{x_2}(t, W)$ are the counting functions of branch points of the function $W(z)$ in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. $N_x(r, W)$ is the density index of branch point of $W(z)$ on annulus $\mathbb{A}$ (see [22]–[23]). The Nevanlinna characteristic of algebroid function W on annulus $\mathbb{A}$ is defined by

$$T_0(r, W) = m_0(r, W) + N_0(r, W).$$

Similarly, for $a \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, we have

$$N_0\left(r, \frac{1}{W-a}\right) = N_1\left(r, \frac{1}{W-a}\right) + N_2\left(r, \frac{1}{W-a}\right)$$
$$= \frac{1}{k}\int_{\frac{1}{r}}^{1} \frac{n_1(t, \frac{1}{W-a})}{t} dt + \frac{1}{k}\int_{1}^{r} \frac{n_2(t, \frac{1}{W-a})}{t} dt,$$

where $n_1(t, \frac{1}{W-a})$ and $n_2(t, \frac{1}{W-a})$ are the counting functions of poles of the function $\frac{1}{W(z)-a}$ in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$ (counting multiplicity), respectively. In addition, we use $\overline{n}_1(t, \frac{1}{W-a})$, $\overline{n}_2(t, \frac{1}{W-a})$ to denote the counting functions of distinct poles of the function $\frac{1}{W-a}$ in $\{z : t < |z| \leq 1\}$ and

$\{z : 1 < |z| \leq t\}$. Similarly, we have the notations $\overline{N}_1(r, W)$, $\overline{N}_2(r, W)$, $\overline{N}_0(r, W)$, and $\overline{N}_0(r, \frac{1}{W-a})$.

Let $W(z)$ be an algebroid function on annulus $\mathbb{A}$. If there are λ branches of $W(z)$ such that $W(z_0) = a, a(\neq \infty)$, then the fractional power series of $W(z)$ is

$$W(z) = a + b_\tau(z - z_0)^{\frac{\tau}{\lambda}} + b_{\tau+1}(z - z_0)^{\frac{\tau+1}{\lambda}} + \cdots, \tag{2}$$

and $n_0(r, a) = n_0(r, \frac{1}{W-a}) = \sum\limits_{W=a} \tau$, where $n_0(r, a)$ is the counting function of zeros of $W(z) - a$ on annulus $\mathbb{A}$ (counting multiplicity). If there are λ branches of $W(z)$ such that $W(z_0) = \infty$, then the fractional power series of $W(z)$ is

$$W(z) = b_{-\tau}(z - z_0)^{-\frac{\tau}{\lambda}} + b_{-\tau+1}(z - z_0)^{\frac{-\tau+1}{\lambda}} + \cdots, \tag{3}$$

and $n_0(r, \infty) = n_0(r, W) = \sum\limits_{W=\infty} \tau$, where $n_0(r, \infty)$ is the counting function of poles of $W(z) - a$ on the annulus $\mathbb{A}$ (counting multiplicity), $z = z_0$ is a branch point of $\lambda - 1$ degree of $W(z)$ on its Riemann surface $\widetilde{\mathscr{M}}$. Let $n_x(r, W)$ be the branch points of $W(z)$ on its Riemann surface on annulus $\mathbb{A}$, then $n_x(r, W) = \sum(\lambda - 1)$. In this paper, we suppose that zero is not a branch point of $W(z)$. Obviously, for $a \in \overline{\mathbb{C}}$, we have

$$n_0\left(r, \frac{1}{W-a}\right) = n_0\left(r, \frac{1}{\psi(z, a)}\right), \qquad N_0\left(r, \frac{1}{W-a}\right) = N_0\left(r, \frac{1}{\psi(z, a)}\right),$$

and especially, $N_0(r, \frac{1}{W}) = \frac{1}{k} N_0(r, \frac{1}{A_0})$ as $a = 0$, and $N_0(r, W) = \frac{1}{k} N_0(r, \frac{1}{A_k})$ as $a = \infty$. From the above definitions, we have some connections with the classical characteristics of algebroid functions in $\mathbb{C}$ as follows:

(a) $N_0(r, W) = N(r, W) + N(\frac{1}{r}, W) - 2N(1, W)$, for $r > 1$,

(b) $T_0(r, W) = T(r, W) + T(\frac{1}{r}, W) - 2T(1, W)$, for $r > 1$,

(c) $T(r, W) - 2T(1, W) \leq T_0(r, W) \leq T(r, W)$.

In fact, suppose $W(0) \neq \infty$, then we have $n_1(t, W) = n(1, W) - n(t, W), 0 < t < 1$ and $n_2(t, W) = n(t, W) - n(1, W), t > 1$. Thus

$$\begin{aligned} N_0(r, W) &= \int_{\frac{1}{r}}^{1} \frac{n(1, W) - n(t, W)}{t} dt + \int_{1}^{r} \frac{n(t, W) - n(1, W)}{t} dt \\ &= \int_{\frac{1}{r}}^{1} \frac{n(1, W)}{t} dt - \int_{\frac{1}{r}}^{1} \frac{n(t, W)}{t} dt + \int_{1}^{r} \frac{n(t, W)}{t} dt - \int_{1}^{r} \frac{n(1, W)}{t} dt \\ &= n(1, W) \log r - \int_{0}^{1} \frac{n(t, W)}{t} dt + \int_{0}^{\frac{1}{r}} \frac{n(t, W)}{t} d + \end{aligned}$$

$$+\int_0^r \frac{n(t,W)}{t}dt - \int_0^1 \frac{n(t,W)}{t}dt - n(1,W)\log r$$
$$= N(r,W) + N(\frac{1}{r},W) - 2N(1,W).$$

Similarly, we can prove the case $W(0) = \infty$. Because $T(r,W) = m(r,W) + N(r,W)$, from the above equality, then relation (b) follows immediately, which implies (c).

In addition, let $W(z), W_1(z), W_2(z)$ be k-valued algebroid functions on annulus $\mathbb{A}$. The following properties will be used in this paper (see [22]):

$$T_0(r,W) = T_0\left(r,\frac{1}{W}\right),$$

$$\max\left\{T_0(r,W_1\cdot W_2), T_0\left(r,\frac{W_1}{W_2}\right), T_0(r,W_1+W_2)\right\} \le T_0(r,W_1)+T_0(r,W_2)+O(1),$$

$$T_0\left(r,\frac{1}{W-a}\right) = T_0(r,W) + O(1), \quad \text{for every fixed } \ a\in\mathbb{C}.$$

3. Our main results

In 2016, the second fundamental theorem for algebroid functions on annulus $\mathbb{A}$ was first obtained by Tan [22]. Here we show this theorem as follows.

Theorem 3.1 (The second fundamental theorem for algebroid function on annuli [22, Lemma 3.5])**.** *Let $W(z)$ be a k-valued algebroid function which is determined by* (1) *on annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let $a_1, a_2, \ldots, a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Then*

$$(q-2k)T_0(r,W) < \sum_{j=1}^{q} N_0\left(r,\frac{1}{W-a_j}\right) - N_1(r,W) + S_0(r,W), \tag{4}$$

$N_1(r,W)$ is the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau-1$, and

$$S_0(r,W) = m_0\left(r,\frac{W'}{W}\right) + \sum_{j=1}^{q} m_0\left(r,\frac{W'}{W-a_j}\right) + O(1).$$

Remark 3.1. From [22], we know that (4) can be represented the following form

$$(q-2k)T_0(r,W) < \sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{W-a_j}\right) + S_0(r,W).$$

Remark 3.2. For the remainder $S_0(r,W)$ in (4), from [10, Theorem 1] and [22], we have

(i) in the case $R_0 = +\infty$,

$$S_0(r,W) = O(\log(rT_0(r,W)))$$

for $r \in (1,+\infty)$ outside a set of finite linear measure;

(ii) in the case $R_0 < +\infty$,

$$S_0(r,W) = O\left(\log\left(\frac{T_0(r,W)}{R_0 - r}\right)\right)$$

for $r \in (1,R_0)$ except for the set E of r such that $\int_E \frac{dr}{(R_0-r)} < +\infty$.

Definition 3.1. Let $W(z)$ be a k-valued algebroid function which is determined by (1) on annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Then the order of $W(z)$ is defined by

$$\rho(W) = \limsup_{r\to+\infty} \frac{\log^+ T_0(r,W)}{\log r}, \qquad \text{if } R_0 = +\infty,$$
$$\rho(W) = \limsup_{r\to R_0} \frac{\log^+ T_0(r,W)}{\log \frac{1}{R_0-r}}, \qquad \text{if } R_0 < +\infty.$$

Remark 3.3. From Definition 3.1 and Remark 3.2, we have

(i) in the case $R_0 = +\infty$, if $\rho(W) < +\infty$, then

$$S_0(r,W) = O(\log r) = o(T_0(r,W))$$

as $r \to +\infty$; if $\rho(W) = +\infty$, then

$$S_0(r,W) = O(\log(rT_0(r,W))) = o(T_0(r,W))$$

if $r \to +\infty$ outside a set of finite linear measure;

(ii) in the case $R_0 < +\infty$, if $\rho(W) \in [0, +\infty)$, then

$$S_0(r, W) = O\left(\log\left(\frac{T_0(r, W)}{R_0 - r}\right)\right) = O\left(\log\left(\frac{1}{R_0 - r}\right)\right),$$

as $r \to R_0-$ except for the set E of r such that $\int_E \frac{dr}{(R_0 - r)} < +\infty$, and if $\rho(W) = +\infty$, then

$$S_0(r, W) = O\left(\log\left(\frac{T_0(r, W)}{R_0 - r}\right)\right)$$

as $r \to R_0-$ except for the set E of r such that $\int_E \frac{dr}{(R_0 - r)} < +\infty$.

In this paper, we will further investigate the value distribution of algebroid functions on annulus $\mathbb{A}$, and establish the second fundamental theorem for algebroid functions concerning polynomials on the annulus as follows.

Theorem 3.2. *Let $W(z)$ be a k-valued algebroid function which is determined by* (1) *on annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, and $Q_j(z)(j = 1, 2, \ldots, q)$ be q distinct polynomials of degree $\leq d$ in z, then*

$$[q - 2k - (4k - 3)d]T_0(r, W) < \sum_{j=1}^{q} N_0\left(r, \frac{1}{W(z) - Q_j(z)}\right) + S_0(r, W),$$

where $S_0(r, W)$ is stated as in Remark 3.3.

When $k = 1$, we get the second fundamental theorem for meromorphic functions concerning polynomials on annulus

Corollary 3.1. *Let $f(z)$ be a meromorphic function on annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, and $Q_j(z)(j = 1, 2, \ldots, q)$ be q distinct polynomials of degree $\leq d$ in z, then*

$$[q - 2 - (4 - 3)d]T_0(r, f) < \sum_{j=1}^{q} N_0\left(r, \frac{1}{f(z) - Q_j(z)}\right) + S_0(r, f).$$

By applying Theorem 3.2, we can obtain the following theorems.

Theorem 3.3. *Let $W(z)$ be a k-valued algebroid function which is determined by* (1) *on annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, and $a_v(v = 1, 2, \ldots, p)$ and $b_j(j = 1, 2, \ldots, q)$ be $p + q$ distinct complex constants such that $a_v \neq b_j \neq 0$ for $v = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$ then*

$$[p+q-6(k-1)]T_0(r,W)$$
$$< (q+1)N_0\left(r,\frac{1}{W}\right)+2\overline{N}_0(r,W)+\sum_{v=1}^{p}N_0\left(r,\frac{1}{W-a_v}\right)$$
$$+\sum_{j=1}^{q}N_0\left(r,\frac{1}{W'-b_j}\right)-\left[N_0\left(r,\frac{1}{W''}\right)+qN_0\left(r,\frac{1}{W'}\right)\right]+S_0(r,W),$$

where $S_0(r,W)$ is stated as in Remark 3.3.

Theorem 3.4. *Let $W(z)$ be a k-valued algebroid function which is determined by (1) on annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$, and $Q_t(z)(t=1,2,\dots,q)$ and $G_l(\neq 0)(l=1,2,\dots,p)$ be $p+q$ distinct polynomials of degree $\le d$ in z, then*

$$[pq-(4k-3)(1+d)]T_0(r,W)$$
$$< p\sum_{t=1}^{q}N_0\left(r,\frac{1}{W(z)-Q_t(z)}\right)+\sum_{l=1}^{p}N_0\left(r,\frac{1}{W^{(d+1)}-G_l}\right)+S_0(r,W),$$

where $S_0(r,W)$ is stated as in Remark 3.3.

When $Q_t=a_t$, $G_l=b_l$ and $d=0$ in Theorem 3.4, we obtain the following corollary immediately:

Corollary 3.2. *Let $W(z)$ be a k-valued algebroid function which is determined by (1) on annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$, and $a_t(t=1,2,\dots,q)$ and $b_l(\neq 0)(l=1,2,\dots,p)$ be $p+q$ distinct numbers, then*

$$[pq-(6v-1)]T_0(r,W)$$
$$< \overline{N}_0(r,W)+p\sum_{t=1}^{q}N_0\left(r,\frac{1}{W(z)-a_t}\right)+\sum_{l=1}^{p}N_0\left(r,\frac{1}{W'-b_l}\right)$$
$$-\left[N_0\left(r,\frac{1}{W''}\right)+(p-1)N_0\left(r,\frac{1}{W'}\right)\right]+S_0(r,W),$$

where $S_0(r,W)$ is stated as in Remark 3.3.

From Theorem 3.2, we can get Theorem 3.1 immediately when $d=0$ and $Q_j(z)(j=1,2,\dots,q)$ are q distinct numbers. Moreover, we can also obtain the second fundamental theorem for meromorphic functions concerning polynomials on annulus. In addition, we can see that Theorem 3.2 is very useful in studying the value distribution of algebroid function on annulus concerning its derivatives and polynomials from Theorems 3.3 and 3.4. Our results are a generalization and improvement of the previous conclusions given by Tan, and Cao.

4. Some lemmas

To prove Theorem 3.2, we require some lemmas as follows.

Lemma 4.1 (see [22, Lemma 3.3])**.** *Let $W(z)$ be a k-valued algebroid function which is determined by* (1) *on annulus $\mathbb{A}$, then*

$$N_x(r,W) \le 2(k-1)T_0(r,W) + O(1).$$

Lemma 4.2. *Let $W(z)$ be a k-valued algebroid function which is determined by* (1) *on annulus $\mathbb{A}$, then*

$$N_0(r,W^{(j)}) \le N_0(r,W) + j\overline{N}_0(r,W) + (2j-1)N_x(r,W) + O(1).$$

Proof. Suppose $W(z_0) = a, (\ne \infty)$. Since $W(z)$ is a k-valued algebroid function on $\mathbb{A}$, it follows from (2) that

$$W^{(j)}(z) = (z-z_0)^{\frac{\tau - j\lambda}{\lambda}} \widehat{w}_j(z), \quad \widehat{w}_j(z_0) \ne 0, \infty.$$

Hence, z_0 is the pole of $W^{(j)}(z)$ with multiplicity $j\lambda - \tau$ if $j\lambda - \tau > 0$. On the other hand, if z_0 is the pole of $W(z)$, from (3) we have

$$W^{(j)}(z) = (z-z_0)^{-\frac{\tau + j\lambda}{\lambda}} \widehat{w}_j(z), \quad \widehat{w}_j(z_0) \ne 0, \infty.$$

Therefore,

$$n_0(r,W^{(j)}) = \sum_{W=\infty} (\tau + j\lambda) + \sum_{W\ne\infty} (j\lambda - \tau)^+,$$

where $(j\lambda - \tau)^+ = \max\{0, j\lambda - \tau\}$. Since $j\lambda - \tau \le j\lambda - 1 \le (2j-1)(\lambda - 1)$ as $\lambda > 1$, and $1 \le j \le 2j-1$, it yields

$$\begin{aligned} n_0(r,W^{(j)}) &\le \sum_{W=\infty} (\tau + j) + j \sum_{W\ne\infty} (\lambda - 1) + (2j-1) \sum_{W\ne\infty} (\lambda - 1) \\ &\le n_0(r,W) + j\overline{n}_0(r,W) + (2j-1)n_x(r,W), \end{aligned} \tag{5}$$

where $n_x(r,W) = n_{x_1}(r,W) + n_{x_2}(r,W)$. Thus, it follows from (5) that

$$N_0(r,W^{(j)}) \le N_0(r,W) + j\overline{N}_0(r,W) + (2j-1)N_x(r,W) + O(1).$$

Therefore, the proof of this lemma is completed. □

Remark 4.1. By Lemmas 4.1 and 4.2, we have

$$N_0(r,W^{(j)}) \le N_0(r,W) + j\overline{N}_0(r,W) + 2(k-1)(2j-1)T_0(r,W) + O(1).$$

Lemma 4.3. *Let $W(z)$ be a k-valued algebroid function which is determined by* (1) *on annulus $\mathbb{A}$, then for any positive integer j, we have*

$$m_0\left(r, \frac{W^{(j)}}{W}\right) = S_0(r, W),$$

where $S_0(r, W)$ is stated as in Remark 3.3.

Proof. By Remarks 3.2 and 3.3, it yields

$$\begin{aligned} T_0(r, W') &= m_0(r, W') + N_0(r, W') \\ &\le m_0(r, W) + m_0\left(r, \frac{W'}{W}\right) + N_0(r, W') + O(1) \\ &\le 2T_0(r, W) + S_0(r, W). \end{aligned}$$

Hence $S_0(r, W') = S_0(r, W)$. Similarly, we have $S_0(r, W^{(j)}) = S_0(r, W)$ for $j \in \mathbb{N}_+$. Thus, it follows that

$$\begin{aligned} m_0\left(r, \frac{W^{(j)}}{W}\right) &\le m_0\left(r, \frac{W^{(j)}}{W^{(j-1)}}\right) + \cdots + m_0\left(r, \frac{W''}{W'}\right) + m_0\left(r, \frac{W'}{W}\right) + O(1) \\ &= S_0(r, W). \end{aligned}$$

Therefore, this completes the proof of this lemma. □

Lemma 4.4. *Let $W(z)$ be a k-valued algebroid function which is determined by* (1) *on annulus $\mathbb{A}$ and is not an algebraic function, and $Q(z)$ be polynomials in z of degree $\le d$, then*

$$m_0\left(r, \frac{W^{(d+1)}(z)}{W(z) - Q(z)}\right) = S_0(r, W),$$

where $S_0(r, W)$ is stated as in Remark 3.2.

Proof. Let $V(z) = W(z) - Q(z)$. We first prove that $V(z)$ is a k-valued algebroid function on annulus $\mathbb{A}$. Substituting $W(z) = V(z) + Q(z)$ into (1), it leads to

$$A_k(z)(V - Q)^k + A_{k-1}(z)(V - Q)^{k-1} + \cdots + A_0(z) = 0, \tag{6}$$

and we can rewrite (6) as the following irreducible equation:

$$B_k(z)V^k + B_{k-1}(z)V^{k-1} + \cdots + B_0(z) = 0,$$

where

$$
\begin{aligned}
B_k(z) &= A_k(z),\\
B_{k-1}(z) &= A_{k-1}(z) + A_k(z)C_k^1 Q(z),\\
&\dots\\
B_{k-j}(z) &= A_{k-j}(z) + A_k(z)C_k^j Q(z)^j + \cdots + A_{k-j+1}(z)C_{k-j+1}^1 Q(z),\\
&\dots\\
B_0(z) &= A_0(z) + A_k(z)Q(z)^k + A_{k-1}(z)Q(z)^{k-1} + \cdots + A_1(z)Q(z).
\end{aligned}
$$

Since $A_k(z), \dots, A_0(z)$ are analytic functions on annulus $\mathbb{A}$ without common zeros, thus $B_k(z), \dots, B_0(z)$ are analytic functions on annulus $\mathbb{A}$ without common zeros. Hence $V(z)$ is a k-valued algebroid function. Since $W(z)$ is not an algebraic function and $Q(z)$ is a polynomial, then it follows that

$$T_0(r, V) = T_0(r, W - Q) \le T_0(r, W) + T_0(r, Q) + O(1),$$

that is,

$$S_0(r, V) = S_0(r, W).$$

Then it follows

$$m_0\left(r, \frac{W^{(d+1)}(z)}{W(z) - Q(z)}\right) = m_0\left(r, \frac{V^{(d+1)}}{V}\right) = S_0(r, V) = S_0(r, W).$$

Therefore, this completes the proof of this lemma. □

Lemma 4.5 (see [9, Theorem 1]). *Let f be a non-constant meromorphic function on annulus $\mathbb{A} = (\frac{1}{R_0}, R_0)(1 < R_0 \le +\infty)$, then*

$$
\begin{aligned}
N_0\left(r, \frac{1}{f}\right) - N_0(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta\\
- \frac{1}{\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta,
\end{aligned}
$$

where $1 \le r < R_0$.

5. Proofs of Theorems 3.2–3.4.

5.1. The Proof of Theorem 3.2. Since $W(z)$ is a k-valued algebroid function on annulus $\mathbb{A}$ and $Q_k(z)$ are polynomials, then $W^{(d+1)}(z)$ is also a k-valued algebroid function. Thus, assume that $W^{(d+1)}(z)$ satisfies the following equation:

$$\psi_0(z, W^{(d+1)}) \equiv E_k(z)(W^{(d+1)})^k + E_{k-1}(z)(W^{(d+1)})^{k-1} + \cdots + E_0(z) = 0,$$

where $E_j(z)(j=0,1,\ldots,k)$ are analytic on annulus $\mathbb{A}$, and $E_j(z)(j=0,1,\ldots,k)$ without common zeros. Further, let $\varphi_t(z)=W(z)-Q_t(z)$, $(t=1,2,\ldots,q)$, then $\varphi_t(z)$ $(t=1,2,\ldots,q)$ are also k-valued algebroid functions. Thus, we can assume that $\varphi_t(z),(t=1,2,\ldots,q)$ satisfy the following equations:

$$\psi_t(z,\varphi_t)\equiv D_{t,k}(z)\varphi_t^k+D_{t,k-1}(z)\varphi_t^{k-1}+\cdots+D_{t,0}(z)=0,$$

where $D_{t,j}(z)(t=1,2,\ldots,q;j=0,1,\ldots,k)$ are analytic on annulus $\mathbb{A}$, and for any fixed t, $D_{t,j}(z)(j=0,1,\ldots,k)$ without common zeros. In view of Lemma 4.4, it follows that $D_{1,k}(z)=D_{2,k}(z)=\cdots=D_{q,k}(z)=A_k$ and

$$n_0\left(r,\frac{1}{W-Q_t}\right)=n_0\left(r,\frac{1}{\varphi_t}\right)=n_0\left(r,\frac{1}{\psi_t(z,\varphi_t=0)}\right)=n_0\left(r,\frac{1}{D_{t,0}}\right),$$

$$N_0\left(r,\frac{1}{W-Q_t}\right)=N_0\left(r,\frac{1}{\varphi_t}\right)=\frac{1}{k}N_0\left(r,\frac{1}{\psi_t(z,\varphi_t=0)}\right).$$

Let $w_j=w_j(z)(j=1,2,\ldots,k)$ be k branches of $W(z)$, then the following equation

$$\prod_{j=1}^{k}\prod_{t=1}^{q}\frac{1}{w_j-Q_t}=\prod_{j=1}^{k}\left(\sum_{t=1}^{q}A_t\frac{1}{w_j-Q_t}\right)=\frac{\prod_{j=1}^{k}\left\{\sum_{t=1}^{q}A_t\frac{w_j^{(d+1)}}{w_j-Q_t}\right\}}{\prod_{j=1}^{k}w_j^{(d+1)}} \tag{7}$$

holds, at most except for finite poles of A_t, where

$$A_t(z)=[(Q_t-Q_1)\cdots(Q_t-Q_{t-1})(Q_t-Q_{t+1})\cdots(Q_t-Q_q)]^{-1},$$

that is, $A_t(z)$ is a rational function. Let $z=re^{i\theta}$, $z=\frac{1}{r}e^{i\theta}$ and $z=e^{i\theta}$, substitute (7), by Lemma 4.5 and similar to the argument as in [22], we can deduce

$$\begin{aligned}&qT_0(r,W)\\&\quad\le T_0(r,W^{(d+1)})+\sum_{t=1}^{q}N_0\left(r,\frac{1}{W-Q_t}\right)-N_0\left(r,\frac{1}{W^{(d+1)}}\right)+Q_0(r,W),\end{aligned} \tag{8}$$

where

$$Q_0(r,W)=\sum_{t=1}^{q}m_0\left(r,\frac{W^{(d+1)}}{W-Q_t}\right)+O(\log r).$$

By Lemma 4.2 and Remark 3.3, it follows that

$$\begin{aligned} T_0(r, W^{(d+1)}) &= m_0\left(r, W\frac{W^{(d+1)}}{W}\right) + N_0(r, W^{(d+1)}) \\ &\leq N_0(r, W^{(d+1)}) + T_0(r, W) - N_0(r, W) + m_0\left(r, \frac{W^{(d+1)}}{W}\right) \\ &\leq m_0\left(r, \frac{W^{(d+1)}}{W}\right) + T_0(r, W) + (d+1)\overline{N}_0(r, W) \\ &\quad + 2(2d+1)(k-1)T(r, W) + O(\log r). \end{aligned} \tag{9}$$

Substituting (9) into (8) and combining $\overline{N}_0(r, W) \leq T_0(r, W) + O(1)$, it leads to

$$\begin{aligned} qT_0(r, W) &< \sum_{t=1}^{q} N_0\left(r, \frac{1}{W - Q_t}\right) + (d+2)T_0(r, W) + m_0\left(r, \frac{W^{(d+1)}}{W}\right) \\ &\quad + 2(2d+1)(k-1)T_0(r, W) + Q_0(r, W) \\ &< \sum_{t=1}^{q} N_0\left(r, \frac{1}{W - Q_t}\right) + [(d+2) + 2(2d+1)(k-1)]T_0(r, W) + Q_1(r, W), \end{aligned}$$

where $Q_0 = 0$ and

$$Q_1(r, W) = \sum_{t=0}^{q} m_0\left(r, \frac{W^{(d+1)}}{W - Q_t}\right) + O(\log r).$$

Thus, it follows from Lemma 4.4 that

$$[q - 2k - (4k-3)d]T_0(r, W) < \sum_{j=1}^{q} N_0\left(r, \frac{1}{W(z) - Q_j(z)}\right) + S_0(r, W).$$

Thus, it means that this proves the conclusion of Theorem 3.2.

5.2. The Proof of Theorem 3.3. Applying Theorem 3.2 for $W(z), a_v$ $(v = 0, 1, \ldots, p)$ and $W'(z), b_j (j = 0, 1, \ldots, q)$, respectively, it follows that

$$pT_0(r, W) < N_0(r, W) + \sum_{v=0}^{p} N_0\left(r, \frac{1}{W - a_v}\right) - N_1(r) + S_0(r, W), \tag{10}$$

and

$$\begin{aligned} qT_0(r, W') &< \sum_{j=0}^{q} N_0\left(r, \frac{1}{W' - b}\right) + N_0(r, W'') \\ &\quad - N_0(r, W') - N_0\left(r, \frac{1}{W''}\right) + S_1(r, W'), \end{aligned} \tag{11}$$

where $a_0 = b_0 = 0$ and

$$S_1(r, W') = m_0\left(r, \frac{W''}{W'}\right) + \sum_{j=1}^{q} m_0\left(r, \frac{W''}{W' - b_j}\right) + O(1).$$

Thus, from (11) and in view of $T_0(r, W) = T_0(r, \frac{1}{W})$, it yields

$$\begin{aligned}
qT_0(r, W) &= qT_0\left(r, \frac{1}{W}\right) \\
&\le qT_0(r, W') + qN_0\left(r, \frac{1}{W}\right) - qN_0\left(r, \frac{1}{W'}\right) + qm_0\left(r, \frac{W'}{W}\right) + O(1) \\
&\le qN_0\left(r, \frac{1}{W}\right) + N_0(r, W'') + \sum_{j=1}^{q} N_0\left(r, \frac{1}{W' - b_j}\right) \\
&\quad - \left[(q-1)N_0\left(r, \frac{1}{W'}\right) + N_0(r, W') + N_0\left(r, \frac{1}{W'}\right)\right] + S_2(r, W'), \qquad (12)
\end{aligned}$$

where

$$S_2(r, W') = 2m_0\left(r, \frac{W''}{W'}\right) + \sum_{j=1}^{q} m_0\left(r, \frac{W''}{W' - b_j}\right) + qm_0\left(r, \frac{W'}{W}\right) + O(1).$$

By combining (10) with (12), it follows that

$$\begin{aligned}
&(p+q)T_0(r, W) \\
&< N_0(r, W'') + (q+1)N_0\left(r, \frac{1}{W}\right) + \sum_{v=1}^{p} N_0\left(r, \frac{1}{W - a_v}\right) + \sum_{j=1}^{q} N_0\left(r, \frac{1}{W' - b_j}\right) \\
&\quad - \left[qN_0\left(r, \frac{1}{W'}\right) + N_0(r, W) + N_0\left(r, \frac{1}{W''}\right)\right] + S_3(r, W) \\
&\le 2\overline{N}_0(r, W) + 3N_x(r, W) + (q+1)N_0\left(r, \frac{1}{W}\right) + \sum_{v=1}^{p} N_0\left(r, \frac{1}{W - a_v}\right) \\
&\quad + \sum_{j=1}^{q} N_0\left(r, \frac{1}{W' - b_j}\right) - \left[qN_0\left(r, \frac{1}{W'}\right) + N_0\left(r, \frac{1}{W''}\right)\right] + S_3(r, W),
\end{aligned}$$

where $S_3(r, W) = S_2(r, W') + S_1(r, W)$.

Thus, by Lemma 4.2, it means that the conclusions of Theorem 3.3 are proved easily.

5.3. The Proof of Theorem 3.4. From (8) and (9), it follows that

$$(q-1)T_0(r,W) < N_0(r,W) + \sum_{t=1}^{q} N_0\left(r, \frac{1}{W-Q_t}\right) - N_1(r) + H_1(r,W),$$

where

$$N_1(r) = 2N_0(r,W) - N_0(r, W^{(d+1)} + N_0\left(r, \frac{1}{W^{(d+1)}}\right)$$

and

$$H_1(r,W) = \sum_{t=0}^{q} m_0\left(r, \frac{W^{(d+1)}}{W-Q_t}\right), \qquad Q_0 = 0.$$

By applying (4) for $W^{(d+1)}$ and G_l, then, we conclude from Jensen's formula that

$$pT_0(r, W^{(d+1)}) < N_0(r, W^{(2d+2)}) + N_0\left(r, \frac{1}{W^{(d+1)}}\right) + \sum_{l=1}^{p} N_0\left(r, \frac{1}{W^{(d+1)} - G_l}\right) - \left[N_0(r, W^{(d+1)}) + N_0\left(r, \frac{1}{W^{(2d+2)}}\right)\right] + H_1(r, W^{(d+1)}), \quad (13)$$

where

$$H_1(r, W^{(d+1)}) = \sum_{l=1}^{p} m_0\left(r, \frac{W^{(2d+2)}}{W^{(d+1)} - G_l}\right) + 2m_0\left(r, \frac{W^{(2d+2)}}{W^{(d+1)}}\right) + O(1).$$

When (8) times p, it follows that

$$pqT_0(r,W) < p\sum_{t=1}^{q} N_0\left(r, \frac{1}{W-Q_t}\right) + pT_0(r, W^{(d+1)}) - pN_0\left(r, \frac{1}{W^{(d+1)}}\right) + pH_2(r,W), \quad (14)$$

where

$$H_2(r,W) = \sum_{t=1}^{q} m_0\left(r, \frac{W^{(d+1)}}{W-Q_t}\right) + O(\log r).$$

Substituting (13) into (14), and by Lemma 4.1, Lemma 4.2 and Remark 4.1, we have

$$pqT_0(r,W) < N_0(r, W^{(d+1)}) + p\sum_{t=1}^{q} N_0\left(r, \frac{1}{W-Q_t}\right) - \Big[N_0(r, W^{(d+1)}) + N_0\left(r, \frac{1}{W^{(2d+2)}}\right) + (p-1)N_0\left(r, \frac{1}{W^{(d+1)}}\right)\Big] + H_2(r,W),$$

and

$$
\begin{aligned}
&[pq-2(k-1)(2d+2)]T_0(r,W)\\
&\quad< (d+1)\overline{N}_0(r,W)+\sum_{l=1}^{p}N_0\left(r,\frac{1}{W^{(d+1)}-G_l}\right)+p\sum_{t=1}^{q}N_0\left(r,\frac{1}{W-Q_t}\right)\\
&\quad-\left[N_0\left(r,\frac{1}{W^{(2d+2)}}\right)+(p-1)N_0\left(r,\frac{1}{W^{(d+1)}}\right)\right]+H_3(r,W), \qquad (15)
\end{aligned}
$$

where $H_2(r,W^{(d+1)})+pH_1(r,W)=H_3(r,W)$. From the expression of $H_3(r,W)$ and (15), by Remark 3.3, and since $\overline{N}_0(r,W)\le T_0(r,W)+O(1)$, it follows that

$$
\begin{aligned}
&[pq-(4k-3)(1+d)]T_0(r,W)\\
&\quad< p\sum_{t=1}^{q}N_0\left(r,\frac{1}{W(z)-Q_t(z)}\right)+\sum_{l=1}^{p}N_0\left(r,\frac{1}{W^{(d+1)}-G_l}\right)+S_0(r,W).
\end{aligned}
$$

Therefore, this completes the proof of Theorem 3.4.

Acknowledgements. We thank the referee(s) for reading the manuscript very carefully and making a number of valuable and kind comments which improved the presentation.

References

[1] S. Axler, Harmonic functions from a complex analysis viewpoint, *Amer. Math. Monthly* **93** (1986), 246–258.

[2] T.-B. Cao, H.-X. Yi and H.-Y. Xu, On the multiple values and uniqueness of meromorphic functions on annuli, *Comput. Math. Appl.* **58** (2009), 1457–1465.

[3] Y. Chen and Z. Wu, Exceptional values of meromorphic functions and of their derivatives on annuli, *Ann. Polon. Math.* **105** (2012), 155–165.

[4] M. Fang, Uniqueness of admissible meromorphic functions in the unit disc, *Sci. China Ser. A* **42** (1999), 367–381.

[5] A. Fernández, On the value distribution of meromorphic functions in the punctured plane, *Mat. Stud.* **34** (2010), 136–144.

[6] W. K. Hayman, Meromorphic Functions, *Clarendon Press*, *Oxford*, 1964.

[7] Y. Z. He and X. Z. Xiao, Algebroid Functions and Ordinary Differential Equations in the Complex Domain, *Science Press*, *Beijing*, 1988.

[8] K.-L. Hiong, Modern research on some aspects of the theory of meromorphic functions, *Adv. Math. (China)* **6** (1963), 307–320 (in *Chinese*).

[9] A. Ya. Khrystiyanyn and A. A. Kondratyuk, On the Nevanlinna theory for meromorphic functions on annuli. I, *Mat. Stud.* **23** (2005), 19-30.

[10] A. Ya. Khrystiyanyn and A. A. Kondratyuk, On the Nevanlinna theory for meromorphic functions on annuli. II, *Mat. Stud.* **24** (2005), 57-68.

[11] A. A. Kondratyuk and I. Laine, Meromorphic functions in multiply connected domains, In: Fourier Series Methods in Complex Analysis, *University of Joensuu*, *Joensuu*, 2006.

[12] R. Korhonen, Nevanlinna theory in an annulus, In: Value Distribution Theory and Related Topics, *Kluwer Acad. Publ.*, *Boston, MA*, 2004, 167–179.

[13] X. M. Li and H. X. Yi, On uniqueness theorems of meromorphic functions concerning weighted sharing of three values, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), 1–16.

[14] L. W. Liao and C. C. Yang, Some new and old (unsolved) problems and conjectures on factorization theory, dynamics and functional equations of meromorphic functions, *J. Jiangxi Norm. Univ. Nat. Sci.* **41** (2017), 242–247.

[15] W. Lin, S. Mori and K. Tohge, Uniqueness theorems in an angular domain, *Tohoku Math. J. (2)* **58** (2006), 509–527.

[16] M. Lund and Z. Ye, Logarithmic derivatives in annuli, *J. Math. Anal. Appl.* **356** (2009), 441–452.

[17] M. Lund and Z. Ye, Nevanlinna theory of meromorphic functions on annuli, *Sci. China Math. Ser.* **53** (2010), 547–554.

[18] Z. Mao and H. Liu, Meromorphic functions in the unit disc that share values in an angular domain, *J. Math. Anal. Appl.* **359** (2009), 444–450.

[19] M. Ru, The recent progress in Nevanlinna theory, *J. Jiangxi Norm. Univ. Nat. Sci.* **42** (2018), 1–11.

[20] D. F. Shea and L. R. Sons, Value distribution theory for meromorphic functions of slow growth in the disk, *Houston J. Math.* **12** (1986), 249–266.

[21] D. C. Sun and Z. S. Gao, Value Distribution Theory of Algebroid Functions, *Science Press*, *Beijing*, 2014.

[22] Y. Tan, Several uniqueness theorems of algebroid functions on annuli, *Acta Math. Sci. Ser. B (Eng. Ed.)* **36** (2016), 295–316.

[23] Y. Tan and Q. C. Zhang, The fundamental theorems of algebroid functions on annuli, *Turkish J. Math.* **39** (2015), 293–312.

[24] F. Titzhoff, Slowly growing functions sharing values, *Fiz. Mat. Fak. Moksl. Semin. Darb.* **8** (2005), 143–164.

[25] G. Valiron, Directions de Borel des fonctions méromorphes, *Gauthier-Villars*, *Paris*, 1938.

[26] S. J. Wu, On the distribution of Borel directions of entire functions, *Chinese J. Contemp. Math.* **14** (1993), 203210.

[27] H.-Y. Xu and Z.-X. Xuan, The uniqueness of analytic functions on annuli sharing some values, *Abstr. Appl. Anal.* (2012), Art. ID 896596, 13 pp.

[28] H. Y. Xu, C. F. Yi and T. B. Cao, The uniqueness problem for meromorphic functions in the unit disc sharing values and a set in an angular domain, *Math. Scand.* **109** (2011), 240–252.

[29] L. Yang, Value Distribution Theory, *Springer-Verlag, Berlin; Science Press*, *Beijing*, 1993.

[30] H.-X. Yi and C.-C. Yang, Uniqueness Theory of Meromorphic Functions, *Kluwer Academic Publishers Group*, *Dordrecht*, 2003.

[31] J.-H. Zheng, On uniqueness of meromorphic functions with shared values in some angular domains, *Canad. Math. Bull.* **47** (2004), 152–160.

[32] J.-H. Zheng, On uniqueness of meromorphic functions with shared values in one angular domain, *Complex Var. Theory Appl.* **48** (2003), 777–785.

[33] J. Zheng, Value Distribution of Meromorphic Functions, *Tsinghua University Press, Beijing; Springer*, *Heidelberg*, 2010.

HONG YAN XU
SCHOOL OF MATHEMATICS
AND COMPUTER SCIENCE
SHANGRAO NORMAL UNIVERSITY
SHANGRAO JIANGXI 334001
P. R. CHINA
AND
DEPARTMENT OF INFORMATICS
AND ENGINEERING
JINGDEZHEN CERAMIC INSTITUTE
JINGDEZHEN JIANGXI 333403
P. R. CHINA

E-mail: xhyhhh@126.com

ZHAO JUN WU
SCHOOL OF MATHEMATICS
AND STATISTICS
HUBEI UNIVERSITY OF SCIENCE
AND TECHNOLOGY
XIANNING 437100
P. R. CHINA
AND
HUBEI KEY LABORATORY OF
APPLIED MATHEMATICS
FACULTY OF MATHEMATICS
AND STATISTICS
HUBEI UNIVERSITY
WUHAN 430062
P. R. CHINA

E-mail: wuzj52@hotmail.com

(Received January 17, 2018; revised October 10, 2018)