Publ. Math. Debrecen
94/3-4 (2019), 289–298
DOI: 10.5486/PMD.2019.8260

On weak quasicontractions in b-metric spaces

By ZORAN D. MITROVIĆ (Ho Chi Minh City) and NAWAB HUSSAIN (Jeddah)

Abstract. Recently, weak quasicontractions have been studied by Bessenyei [2]. The aim of the present paper is to establish fixed point results for weak quasicontractions involving comparison function in b-metric spaces. As applications of our theorems, we deduce certain well-known results as corollaries.

1. Introduction and preliminaries

Banach contraction mapping principle is a simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations.

Theorem 1.1. *Let (X, d) be a complete metric space. Let T be a contraction mapping on X, that is, one for which exists $\lambda \in [0, 1)$ satisfying*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point $x \in X$.

Because of its significance and simplicity, various authors have established numerous interesting extensions and generalizations of the Banach contraction principle (see, for example, the monographs of Rus [20], Kirk and Shazad [16]). In 2016, Bessenyei [2] rediscovered Theorems of Hegedűs and Szilágyi [9], and those of Walter [21], introduced weak quasicontraction involving comparison functions and also proved a theorem that generalizes the results obtained by Ćirić [7], Browder [4] and Matkowski [17].

Mathematics Subject Classification: 47H10.
Key words and phrases: weak quasicontraction, fixed points, comparison function, b-metric space.

The orbit and the double orbit induced by T are defined in the next way:

$$O(x) := \{T^n(x) | n \in \mathbb{N} \cup \{0\}\}; \qquad O(x,y) := O(x) \cup O(y),$$

where $T^{n+1} = T \circ T^n$ and $T^0 = id$.

We mention some properties of comparison functions $\varphi : [0,\infty) \to [0,\infty)$ which are being used in metric fixed point theory:

(P_1) φ is increasing.

(P_2) φ is upper semi-continuous.

(P_3) $\varphi(0) = 0$.

(P_4) $\varphi(t) < t$, for all $t > 0$.

(P_5) $\lim_{n\to\infty} \varphi^n(t) = 0$, for each $t > 0$.

Lemma 1.2 (Bessenyei, [2]). $(P_1) + (P_2) + (P_3) + (P_4) \Rightarrow (P_5)$.

Lemma 1.3 (Matkowski, [18]). $(P_1) + (P_5) \Rightarrow (P_4)$.

Definition 1.4 ([2]). Let (X,d) be a metric space. A mapping $T : X \to X$ is called a weak quasicontraction with comparison function φ (or briefly: *a weak φ-quasicontraction*) if it induces bounded orbits, and for all $x, y \in X$,

$$d(Tx,Ty) \le \varphi(\operatorname{diam} O(x,y)). \tag{1.2}$$

The main result of Bessenyei [2] is the below-mentioned fixed point theorem for weak quasicontractions defined on complete metric spaces.

Theorem 1.5 ([2]). *Let (X,d) be a complete metric space, and $T : X \to X$ a weak quasicontraction with comparison function φ that meets the conditions $(P_1), (P_2), (P_3)$ and (P_4). Then T has a unique fixed point. Moreover, the sequence of iterates at any point converges to this fixed point.*

Remark 1.6. We note that condition (1.1) implies $\operatorname{diam} O(x) \le \frac{d(x,Tx)}{1-\lambda}$, so if we put $\varphi(t) = \lambda t$, $t \ge 0$, in Theorem 1.5, we obtain the Banach fixed point theorem.

Bakhtin [1] and Czerwik [6] defined the notion of b-metric spaces and proved some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces. Successively, this notion has been reintroduced by Khamsi [14], Khamsi and Hussain [15], and Hussain *et al.* [11], [12] with the name of *metric-type space*.

Definition 1.7. Let X be a nonempty set, and let $s \ge 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A triplet (X, d, s) is called a b-metric space with coefficient s.

Note that the class of metric spaces is a proper subset of the class of b-metric spaces with coefficient $s \geq 1$. Fixed point theory in b-metric spaces was studied by many authors (see [8], [10], [13], [16], [19]). Note also that in a b-metric space, distance function d need not be continuous, i.e, there exists a b-metric space (X, d, s) and sequences$\{x_n\}, \{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, but $\lim_{n\to\infty} d(x_n, y_n) \neq d(x, y)$.

One of the main results of [6] Czerwik is the following;

Theorem 1.8 ([6, Theorem 1])**.** *Let (X, d, s) be a complete b-metric space, and suppose $T : X \to X$ satisfies*

$$d(T(x), T(y)) \leq \varphi(d(x, y)), \tag{1.3}$$

for each $x, y \in X$, where mapping $\varphi : [0, \infty) \to [0, \infty)$ satisfies conditions (P_1) and (P_5). Then T has a unique fixed point $x^* \in X$, and $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$.

Remark 1.9. Note that Theorem 1.8 is a direct consequence of the main result of [3]. The recent paper [5] gives excellent overview of possible generalizations of metric spaces.

Remark 1.10. We note that due to Lemma 1.3, condition (1.3) implies that T is a continuous mapping. Also, condition (1.3) implies that T induces bounded orbits (see [2]).

The fact that d is not continuous in the b-metric space leads to the introduction of strong b-metric space.

Definition 1.11 ([16, Definition 12.7]). Let X be a nonempty set, and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a strong b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq sd(x, y) + d(y, z)$.

A triplet (X, d, s), is called a strong b-metric space with coefficient s.

Remark 1.12. The distance function d in a strong b-metric space is continuous (see [16, Proposition 12.3]).

The aim of this paper is to obtain Theorem 1.5 in b-metric spaces using weak quasicontraction involving comparison function φ. As consequences, we derive certain known results as corollaries.

2. Main result

The proof of the next lemma is a straightforward adaptation of the reasoning from [2].

Lemma 2.1. *Let (X, d, s) be a complete b-metric space, and $T : X \to X$ a weak φ-quasicontraction where φ satisfies conditions (P_1) and (P_5). Then there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$.*

Proof. The boundedness of orbits implies the following:

$$\operatorname{diam} O(Tx, Ty) = \sup_{k,l\in\mathbb{N}} \{d(T^k x, T^l y), d(T^k x, T^l x), d(T^k y, T^l y)\}. \tag{2.1}$$

From condition (1.2), we obtain

$$d(T^k x, T^l y) \le \varphi(\operatorname{diam} O(T^{k-1}x, T^{l-1}y))) \le \varphi(\operatorname{diam} O(x, y)).$$

Similarly, we have

$$d(T^k x, T^l x) \le \varphi(\operatorname{diam} O(T^{k-1}x, T^{l-1}x)) \le \varphi(\operatorname{diam} O(x)) \le \varphi(\operatorname{diam} O(x, y)),$$

which implies

$$d(T^k y, T^l y) \le \varphi(\operatorname{diam} O(x, y)). \tag{2.2}$$

According to the foregoing, we conclude that

$$\operatorname{diam} O(Tx, Ty) \le \varphi(\operatorname{diam} O(x, y)). \tag{2.3}$$

From inequality (2.3), we have the following:

$$\operatorname{diam} O(T^2x, T^2y) \le \varphi(\operatorname{diam} O(Tx, Ty)) \le \varphi(\varphi(\operatorname{diam} O(x, y))) = \varphi^2(\operatorname{diam} O(x, y)).$$

Using induction, we obtain

$$\operatorname{diam} O(T^n x, T^n y) \le \varphi^n(\operatorname{diam} O(x, y)). \tag{2.4}$$

Choose $x \in X$, we show that $\{T^n x\}$ is a Cauchy sequence. The boundedness of orbits and condition (P_5) imply that for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(\operatorname{diam} O(x)) < \frac{\epsilon}{2s}$. Therefore, using inequality (2.4), we obtain, for all $n \geq n_0$,

$$d(T^{n_0}x, T^n x) \leq \varphi^{n_0}(\operatorname{diam} O(x, T^{n-n_0}x)) \leq \varphi^{n_0}(\operatorname{diam} O(x)) \leq \frac{\epsilon}{2s}.$$

Using inequality (3) in Definition 1.7, we conclude that for all $m, n \geq n_0$,

$$d(T^m x, T^n x) \leq s[d(T^m x, T^{n_0}x) + d(T^{n_0}x, T^n x)] \leq s\left[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}\right] = \epsilon.$$

So, $\{T^n x\}$ is a Cauchy sequence, and hence there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} T^n x = x^*$. □

Theorem 2.2. *Let (X, d, s) be a complete b-metric space, and $T : X \to X$ a weak φ-quasicontraction such that function φ satisfies conditions (P_1) and (P_5). Then $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$, and x^* is a unique fixed point of T, provided one of the following conditions is satisfied:*

(i) *T is continuous at x^*;*

(ii) *d is continuous.*

Proof. Lemma 2.1 implies that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$. Let us prove that x^* is a unique fixed point of T.

(i) Suppose that T is continuous at $x^* \in X$.

Then we have

$$x^* = \lim_{n\to\infty} T^{n+1}x = T \lim_{n\to\infty} T^{n+1}x = Tx^*.$$

(ii) Let d be continuous. If x^* is not a fixed point of T, then $\operatorname{diam} O(x^*) > 0$. The methods of [2] provide $n_0 \in \mathbb{N}$ such that $\operatorname{diam} O(x^*) = d(x^*, T^{n_0}x^*)$ holds. By Lemma 1.3, we obtain that

$$\varphi(\operatorname{diam} O(x^*)) < \operatorname{diam} O(x^*).$$

Since d is continuous, we obtain

$$\begin{aligned}\operatorname{diam} O(x^*) = d(x^*, T^{n_0}x^*) &= \lim_{n\to\infty} d(T^{n+n_0}x^*, T^{n_0}x^*)\\ &\leq \varphi^{n_0}(\operatorname{diam} O(T^n x^*, x^*)) \leq \varphi(\operatorname{diam} O(x^*)) < \operatorname{diam} O(x^*).\end{aligned}$$

Consequently, $\operatorname{diam} O(x^*) = 0$. Thus x^* is the fixed point of the mapping T.

For uniqueness, let y^* be another fixed point of T. Since,

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi(\operatorname{diam} O(x^*, y^*)) < \operatorname{diam} O(x^*, y^*) = d(x^*, y^*),$$

which implies T has exactly one fixed point. □

Remark 2.3. Since $d(x,y) \leq \operatorname{diam} O(x,y)$, for all $x, y \in X$ and φ, satisfies condition (P_4) (see Lemma 1.3), so from Theorem 2.2 we obtain Theorem 1.8.

Corollary 2.4. *Let (X, d, s) be a complete b-metric space, and let mapping $T : X \to X$ induce bounded orbits. Suppose that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \operatorname{diam} O(x,y) - \psi(\operatorname{diam} O(x,y)), \tag{2.5}$$

where function $\psi : [0,\infty) \to [0,\infty)$ satisfies the below conditions:

(a) *ψ is a decreasing,*

(b) *$id - \psi$ satisfy condition (P_5).*

Then there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$, and x^* is a unique fixed point of T, provided one of the following conditions is satisfied:

(i) *T is continuous at $x^* \in X$,*

(ii) *d is continuous.*

3. Some applications

In this section, we present certain consequences of Theorem 2.2 in b-metric spaces.

Lemma 3.1. *Let (X, d, s) be a complete b-metric space, and $T : X \to X$ a map such that for all $x, y \in X$ and some $\lambda \in [0, 1)$, we have*

$$d(Tx, Ty) \leq \lambda d(x, y). \tag{3.1}$$

Then T induces bounded orbits.

Proof. Since $\lim\limits_{n\to\infty} \lambda^n = 0$, there exists a natural number n_0 such that

$$0 < \lambda^{n_0} \cdot s^2 < 1. \tag{3.2}$$

Let $O_n(x) = \{x, Tx, \ldots, T^n x\}$. Then, we conclude that $\operatorname{diam} O_n(x) = d(T^k x, T^l x)$ for some $k, l \in \{1, 2, \ldots, n\}$, or $\operatorname{diam} O_n(x) = d(x, T^k x)$ for some $k \in \{1, 2, \ldots, n\}$. If $\operatorname{diam} O_n(x) = d(T^k x, T^l x)$, we obtain (where it is understood that $T^0 x = x$)

$$d(T^k x, T^l x) \leq \lambda d(T^{k-1} x, T^{l-1} x) < d(T^{k-1} x, T^{l-1} x) \leq \operatorname{diam} O_n(x).$$

Therefore, we conclude that $d(x, T^k x) = \operatorname{diam} O_n(x)$ for some $k \in \{1, 2, \ldots, n\}$. Applying inequality (3) in Definition 1.7, we obtain

$$\begin{aligned} d(x,T^kx) &\le s[d(x,T^{n_0}x)+d(T^{n_0}x,T^kx)] \\ &\le s[d(x,T^{n_0}x)+s(d(T^{n_0}x,T^{n_0+k}x)+d(T^{n_0+k}x,T^kx))] \\ &\le sd(x,T^{n_0}x)+s^2[\lambda^{n_0}d(x,T^kx)+\lambda^k d(T^{n_0}x,x)] \\ &\le (s+s^2)d(x,T^{n_0}x)+s^2\lambda^{n_0}d(x,T^kx)]. \end{aligned}$$

Therefore, we get

$$\operatorname{diam} O_n(x) \le \frac{s+s^2}{1-\lambda^{n_0}s^2} d(x,T^{n_0}x). \tag{3.3}$$

Since $\operatorname{diam} O(x) = \sup\{\operatorname{diam} O_n(x) : n \in \mathbb{N}\}$, we obtain that T induces bounds orbits. □

Theorem 3.2 (The Banach contraction principle in b-metric spaces, Dung [8, Theorem 2.1])**.** *Let (X,d,s) be a complete b-metric space, and let $T: X \to X$ be a continuous map such that for all $x,y \in X$ and some $\lambda \in [0,1)$,*

$$d(Tx,Ty) \le \lambda d(x,y). \tag{3.4}$$

Then T has a unique fixed point x^*, and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.

Proof. If we put $\varphi(t) = \lambda t$, then the assertion follows from Theorem 2.2 and Lemma 3.1. □

Theorem 3.3. *Let (X,d,s) be a complete b-metric space, and let $T: X \to X$ be a quasicontraction inducing bounded orbits, i.e, there exists $\lambda \in [0,1)$ such that*

$$d(Tx,Ty) \le \lambda \operatorname{diam} O(x,y), \tag{3.5}$$

for all $x,y \in X$. Then there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$, and T has a unique fixed point x^*, provided one of the following conditions are satisfied:

(i) *T is continuous at $x^* \in X$;*

(ii) *d is continuous.*

Proof. The proof follows from Theorem 2.2, if we put $\varphi(t) = \lambda t$. □

Remark 3.4. (1) If T is a quasicontraction on b-metric space (X,d,s) with $\lambda \in [0,\frac{1}{s})$, then similar to the proof of Lemma 3.1, there exists some $k \in \{1,2,\dots,n\}$, such that $d(x,T^kx) = \operatorname{diam} O_n(x)$. Since,

$$\operatorname{diam} O_n(x) \le s[d(x,Tx)+\operatorname{diam} O_{n-1}(Tx)] \le s[d(x,Tx)+\lambda \operatorname{diam} O_n(x)],$$

which implies

$$\operatorname{diam} O(x) \leq \frac{s}{1-\lambda s} d(x, Tx). \tag{3.6}$$

(2) If (X, d, s) is a strong b-metric space, then

$$\operatorname{diam} O_n(x) \leq sd(x, Tx) + \operatorname{diam} O_{n-1}(Tx) \leq sd(x, Tx) + \lambda \operatorname{diam} O_n(x),$$

which implies

$$\operatorname{diam} O(x) \leq \frac{s}{1-\lambda} d(x, Tx). \tag{3.7}$$

From Lemma 2.1, Theorem 3.3 and Remark 3.4, we obtain the following quasi-contraction principle in b-metric and strong b-metric spaces.

Corollary 3.5 (Version of the fixed point theorem of Ćirić in b-metric spaces)**.** *Let (X, d, s) be a complete b-metric space and let $T : X \to X$ be a map such that for all $x, y \in X$ and some $\lambda \in [0, 1/s)$,*

$$d(Tx, Ty) \leq \lambda \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)}{2s}, \frac{d(y, Tx)}{2s}\right\}.$$

Then T has a unique fixed point x^*.

Proof. From Lemma 2.1 and Remark 3.4, there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$. Let us show that x^* is a fixed point. We have the following:

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, T^{n+1}x) + d(T^{n+1}x, Tx^*)] \\ &\leq sd(x^*, T^{n+1}x) + s\lambda \max\Bigg\{d(T^n x, x^*), d(T^n x, T^{n+1}x), \\ &\qquad d(x^*, Tx^*), \frac{d(x^*, T^{n+1}x)}{2s}, \frac{d(T^n x, Tx^*)}{2s}\Bigg\}. \end{aligned}$$

Since,

$$\begin{aligned} \frac{d(x^*, T^{n+1}x)}{2s} &\leq \frac{d(x^*, T^n x) + d(T^n x, T^{n+1}x)}{2} \\ &\leq \max\{d(x^*, T^n x), d(T^n x, T^{n+1}x)\}, \end{aligned}$$

and

$$\frac{d(T^n x, Tx^*)}{2s} \leq \frac{d(T^n x, x^*) + d(x^*, Tx^*)}{2} \leq \max\{d(x^*, T^n x), d(x^*, Tx^*)\},$$

so we obtain

$$d(x^*, Tx^*) \leq sd(x^*, T^{n+1}x) + s\lambda \max\{d(T^n x, x^*), d(T^n x, T^{n+1}x), d(x^*, Tx^*)\}.$$

Since $\lim_{n\to\infty} T^n x = x^*$ and $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$, this shows that $(1 - \lambda s)d(x^*,\ Tx^*) = 0$, which implies that x^* is a fixed point of T. The uniqueness follows from the quasi-contractivity of T. □

Corollary 3.6 (Version of the fixed point theorem of Ćirić in strong b-metric spaces). *Let (X, d, s) be a complete strong b-metric space, and let $T : X \to X$ be a map such that for all $x, y \in X$ and some $\lambda \in [0, 1)$,*

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then T has a unique fixed point x^*.

Proof. The continuity of d follows directly from Theorem 3.3 and Remark 3.4. □

Remark 3.7.

(i) The conclusion of Corollary 3.6 does not hold in the setting of b-metric spaces for $\lambda \in [0, 1)$ (see [8, Example 2.6]).

(ii) Corollary 3.5 improves the result of Jovanović *et al.* ([13, Corollary 3.12]).

(ii) Corollary 3.6 improves the results of Dung (see [8, Corollaries 2.4 and 2.5]).

References

[1] I. A. Bakhtin, The contraction mapping principle in almost metric space, *Functional analysis* **30**, *Ul'yanovsk Gos. Ped. Inst.*, *Ul'yanovsk* (1989), 26–37 (in *Russian*).

[2] M. Bessenyei, The contraction principle in extended context, *Publ. Math. Debrecen* **89** (2016), 287–295.

[3] M. Bessenyei and Zs. Páles, A contraction principle in semimetric spaces, *J. Nonlinear Convex Anal.* **18** (2017), 515–524.

[4] F. E. Browder, On the convergence of successive approximations for nonlinear functional equations, *Indag. Math.* **30** (1968), 27–35.

[5] K. Chrząszcz, J. Jachymski and F. Turoboś, On characterizations and topology of regular semimetric spaces, *Publ. Math. Debrecen* **93** (2018), 87–105.

[6] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis* **1** (1993), 5–11.

[7] Lj. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* **45** (1974), 267–273.

[8] N. V. Dung and V. T. L. Hang, On relaxations of contraction constants and Caristi's theorem in b-metric spaces, *J. Fixed Point Theory Appl.* **18** (2016), 267–284.

[9] M. Hegedűs and T. Szilágyi, Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings, *Math. Japon.* **25** (1980), 147–157.

[10] N. Hussain and M. H. Shah, KKM mappings in cone b-metric spaces, *Comput. Math. Appl.* **62** (2011), 1677–1684.

[11] N. Hussain, D. Đorić, Z. Kadelburg and S. Radenović, Suzuki-type fixed point results in metric type spaces, *Fixed Point Theory Appl.* **2012:126** (2012), 12 pp.

[12] N. Hussain, R. Saadati and R. P. Agarwal, On the topology and wt-distance on metric type spaces, *Fixed Point Theory Appl.* **2014:88** (2014), 14 pp.

[13] M. Jovanović, Z. Kadelburg and S. Radenović, Common fixed point results in metric–type spaces, *Fixed Point Theory Appl.* (2010), Art. ID 978121, 15 pp.

[14] M. A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, *Fixed Point Theory and Appl.* (2010), Art. ID 315398, 7 pp.

[15] M. A. Khamsi and N. Hussain, KKM mappings in metric type spaces, *Nonlinear Anal.* **73** (2010), 3123–3129.

[16] W. Kirk and N. Shahzad, Fixed Point Theory in Distance Spaces, *Springer*, *Cham*, 2014.

[17] J. Matkowski, Integrable solutions of functional equations, *Dissertationes Math.* **127** (1975), 68 pp.

[18] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, *Proc. Amer. Math. Soc.* **62** (1977), 344–348.

[19] R. Miculescu and A. Mihail, New fixed point theorems for set-valued contractions in b-metric spaces, *J. Fixed Point Theory Appl.* **19** (2017), 2153–2163.

[20] I. A. Rus, Generalized Contractions and Applications, *Cluj University Press*, *Cluj-Napoca*, 2001.

[21] W. Walter, Remarks on a paper by F. Browder about contraction, *Nonlinear Anal.* **5** (1981), 21–25.

ZORAN D. MITROVIĆ
NONLINEAR ANALYSIS RESEARCH GROUP
TON DUC THANG UNIVERSITY
HO CHI MINH CITY
VIETNAM

AND

FACULTY OF MATHEMATICS AND STATISTICS
TON DUC THANG UNIVERSITY
HO CHI MINH CITY
VIETNAM

E-mail: zoran.mitrovic@tdtu.edu.vn

NAWAB HUSSAIN
DEPARTMENT OF MATHEMATICS
KING ABDULAZIZ UNIVERSITY
P. O. BOX 80203
JEDDAH 21589
SAUDI ARABIA

E-mail: nhusain@kau.edu.sa

(Received February 6, 2018; revised November 10, 2018)