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Cobordism of maps of locally orientable Witt spaces

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Abstract. The aim of this work is to present some remarks on cobordism of normally nonsingular maps between compact locally orientable Witt spaces. By using the Wu classes defined by Goresky and Pardon, we give a definition of Stiefel–Whitney numbers in this situation. Following Stong's method, we construct a map in the respective intersection homology groups and show that null-cobordism implies the vanishing of these Stiefel–Whitney numbers.

1. Introduction

R. Thom [21] was one of the first to define cobordism of smooth manifolds, and he determined the non-oriented cobordism algebra. Many authors worked on the subject, in particular R. E. Stong in his famous book [18] and in many papers. Let us quote also M. F. Atiyah [1], and P. E. Conner and E. E. Floyd [6].

Stong in [17] introduced and studied a notion of cobordism for maps $f : X \to Y$ of closed smooth manifolds. He defined Stiefel–Whitney numbers for such a map, presented a formula using cohomology groups with $\mathbb{Z}_2$ coefficients, and proved that two maps are cobordant if and only if they have the same

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characteristic numbers. These results concern smooth manifolds and use classical homology theories. They are recalled in Section 2.

The natural question arises: What can we say in the case of singular varieties? Among various works in this direction, we will quote those of G. Friedman [8], M. Goresky and W. Pardon [13], P. Siegel [16], D. Sullivan [19] and A. Szücs [20].

In [13, Introduction], Goresky and Pardon discussed four classes of singular spaces for which they constructed characteristic classes such that the respective characteristic numbers determine the cobordism groups: orientable $\bar{s}$-duality spaces, orientable locally square-free spaces, locally orientable Witt spaces, locally orientable spaces. They mention also other cobordism theories. Siegel in [16] described the class of $\mathbb{Q}$-Witt spaces and computed the cobordism groups of such spaces, showing that in non-trivial cases they are equal to the Witt group. Pardon [15] computed the cobordism groups of the "Poincaré duality spaces" defined by Goresky and Siegel in [14]. Friedman in [8] followed Siegel by computing the bordism groups of oriented K-Witt spaces for any coefficient field K as well as identifying the resulting generalized homology theories.

Characteristic classes of smooth manifolds are defined in usual cohomology groups on which there is a multiplicative structure given by the cup-product. On the other hand, characteristic classes of singular varieties do not exist in cohomology, they lie in homology where there is no product structure. In order to recover a product and define characteristic numbers, one can consider intersection homology. In general, characteristic classes cannot be lifted to intersection homology, and maps $f : X \to Y$ do not provide homomorphism of intersection homology (see [3]). Fortunately, the characteristic classes we consider lie in intersection homology and there are classes of maps, such as normally nonsingular maps or placid maps, which provide well-defined homomorphisms in intersection homology. But, unlike homology theories, intersection homology is not homotopy invariant and does not, in general, satisfy the universal coefficient theorem. In this case, as we see in [8], the Witt spaces provide an important class of examples defined by a relatively tractable condition concerning intersection homology.

Our aim in this work is to present some results on cobordism of maps, by considering pseudomanifolds X and Y which are compact locally orientable Witt spaces.

In this context, first we show that: *If a map $f : X \to Y$, where Y is a closed smooth manifold, is null-cobordant, then the Stiefel–Whitney numbers, defined using the Wu classes given by Goresky and Pardon in* [13]*, are zero.*

Then we consider the case when Y is not smooth, but the map is normally a nonsingular (or placid) map. While these conditions are rather restrictive, they are natural for the considered problem. We show that: *If a normally nonsingular (or placid) map $f : X \to Y$ is null-cobordant, then the Stiefel–Whitney numbers are zero.*

We remark that none of these results implies the other one, since the corresponding characteristic numbers are different.

Consequently, if two maps are cobordant under our hypothesis, then we have coincidence of the corresponding characteristic numbers.

2. Results of Stong and Atiyah on cobordism of maps

Let us recall results due to Stong in [17] and their relationship with the bordism groups of Atiyah.

A map of dimension (m, n) is a triple (f, M, N), where M and N are closed smooth manifolds of dimension m and n, respectively, and $f : M \to N$ is a continuous map.

Definition 2.1 ([17]). Two maps (f, M, N) and $(f^{'}, M^{'}, N^{'})$ of dimension (m, n) are cobordant if there exists a triple (F, V, W) where:

(1) V and W are compact smooth manifolds of dimensions $m + 1$ and $n + 1$, respectively, with boundaries $\partial V = M \sqcup M^{'}$ and $\partial W = N \sqcup N^{'}$, and

(2) $F : V \to W$ is a continuous map whose restrictions to M and $M^{'}$ are f and $f^{'}$, respectively.

The set of equivalence classes of maps of dimension (m, n) under this relation is denoted $\mathcal{N}(m, n)$. Following Conner and Floyd in [6], Stong shows how to reduce the calculation of the cobordism groups $\mathcal{N}(m, n)$ to a homotopy question involving the Thom spaces. He showed that the group $\mathcal{N}(m, n)$ is the n-th bordism group $\mathfrak{N}_n(\Omega^\infty MO(n - m + \infty))$ of the space $\Omega^\infty MO(n - m + \infty) = \lim_{q\to\infty} \Omega^q MO(n - m + q)$, where $MO(i)$ is the Thom space of the i-dimensional universal vector bundle and Ω stands for the loop space functor.

Atiyah introduced bordism groups $\mathcal{N}_m(N)$ which are equivalence classes of maps under the following relation:

Definition 2.2. Two maps $f : M \longrightarrow N$ and $f^{'} : M^{'} \longrightarrow N$, where M and $M^{'}$ are of dimension m, and N is of dimension n, are cobordant if there exist:

(1) A manifold V of dimension $m+1$ with boundary $\partial V = M \sqcup M^{'}$, and

(2) a continuous map $F : V \to N$ whose restrictions to M and $M^{'}$ are f and $f^{'}$, respectively.

In fact, this group is the group of homotopy classes of maps from the target manifold N to $\Omega^\infty MO(n-m+\infty)$, i.e., $\mathcal{N}_m(N) = [N, \Omega^\infty MO(n-m+\infty)]$.

Now the next propositions, showing the relation between the two definitions, are obvious.

Proposition 2.3. *Let (f, M, N) and $(f^{'}, M^{'}, N^{'})$ be maps of dimension (m, n) having the same target $N = N^{'}$. If two maps represent the same class in $\mathcal{N}_m(N)$, they determine the same class in $\mathcal{N}(m, n)$.*

Proposition 2.4. *If $f, g : M \to N$ are homotopic maps, then (f, M, N) is cobordant to (g, M, N).*

Stong in [17] defined the Stiefel–Whitney (S–W for short) numbers associated to the map (f, M, N), and proved that two maps of the same dimension (m, n) are cobordant if and only if they have the same S–W numbers. That is, he proved that the resulting cobordism groups $\mathcal{N}(m, n)$ are determined by the S–W numbers. In order to prove this result, Stong used a "Gysin" homomorphism $f_!$ defined below. Here, as well as throughout the whole paper, homology and cohomology groups are considered with $\mathbb{Z}_2$ coefficients. The universal coefficient theorem is used to identify $H^{i+n-m}(N)$ with $\mathrm{Hom}(H_{i+n-m}(N), \mathbb{Z}_2)$.

Definition 2.5 ([17, §6]). Let us consider a map $f : M \to N$, where M and N are manifolds of dimensions m and n, respectively. One defines the "Gysin" homomorphism by commutativity of the following diagram, where f_* is the map induced by f in homology, and PD_M and PD_N are Poincaré duality isomorphisms:

$$\begin{array}{ccc} H^i(M) & \xrightarrow{f_!} & H^{i+n-m}(N) \\ {\scriptstyle PD_M}\downarrow\cong & & \cong\downarrow{\scriptstyle PD_N} \\ H_{m-i}(M) & \xrightarrow{f_*} & H_{m-i}(N). \end{array}$$

The main reason why we restrict ourselves to the study of normally nonsingular maps (see Definition 3.3) is that for them the Atiyah–Hirzebruch definition of the Gysin map can be repeated word by word (see Remark 6.9). Moreover, its analogues can be defined in the intersection (co)homologies as well.

Remark 2.6. The Gysin homomorphism $f_! : H^i(M) \to H^{i+n-m}(N)$ is the map such that for any $\alpha \in H^i(M)$ and $\beta \in H_{i+n-m}(N)$, one has $\langle f_!(\alpha), \beta\rangle = \langle f^*(\tilde{\beta}) \cup \alpha, [M]\rangle \in \mathbb{Z}_2$, where $\tilde{\beta} = PD_N^{-1}(\beta) \in H^{m-i}(N)$ and $[M]$ is the fundamental class of M.

We will use an analogous description of the map $f_!$, due to Atiyah and Hirzebruch [2]: Let us consider $h : M \to \mathbb{S}^s$ an embedding of M in some s-dimensional sphere $\mathbb{S}^s$, and T a tubular neighborhood of $(f \times h)(M)$ in $N \times \mathbb{S}^s$, then $f_!$ is the composition of the maps:

$$H^i(M) \xrightarrow{\varphi} H^{i+s+n-m}(T/\partial T) \xrightarrow{c^*} H^{i+s+n-m}(N \times \mathbb{S}^s) \to H^{i+n-m}(N),$$

where φ denotes the Thom isomorphism, $c : N \times \mathbb{S}^s \to T/\partial T$ is the collapsing map, and the last map is the projection (denoted as isomorphism in Stong [17, p. 255])

$$H^{i+s+n-m}(N \times \mathbb{S}^s) \cong H^{i+n-m}(N) \oplus H^s(\mathbb{S}^s) \to H^{i+n-m}(N).$$

3. Intersection homology

3.1. Pseudomanifolds.

Definition 3.1 ([13, §2.1]). An m-dimensional pseudomanifold without boundary is a purely m-dimensional piecewise linear (P.L. for short) polyhedron which admits a triangulation such that each $(m-1)$ simplex is a face of exactly two m-simplices.

A pseudomanifold admits a piecewise linear stratification [4, §I.1.4], which is a filtration by closed subspaces $\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_{m-2} \subset X_m = X$, with the singular part $\Sigma(X)$ of X being (included in) the element X_{m-2} of the filtration and such that for each point $x \in X_i - X_{i-1}$ there is a neighborhood U and a homeomorphism preserving the P.L. stratification between U and $\mathbb{R}^i \times \mathrm{c}(L)$, where $\mathrm{c}(L)$ denotes the (open) cone on the link L of the stratum $X_i - X_{i-1}$. Note that the difference $X_i - X_{i-1}$ itself is a stratified pseudomanifold (see [4, §I.1.1]). If $X_i - X_{i-1}$ is not empty, it is a (non-necessarily connected) manifold of dimension i, and is called the i-dimensional stratum of the stratification.

Definition 3.2 ([13, §2.3]). An m-pseudomanifold X with boundary ∂X is an m-dimensional compact P.L. space such that $X - \partial X$ is a pseudomanifold, and ∂X is a compact $(m-1)$-dimensional P.L. subspace of X which has a collar neighborhood U in X, i.e., there is a P.L. homeomorphism $\varphi : U \cong \partial X \times [0, 1)$ such that the restriction $\varphi|_{\partial X}$ is the identity map.

In the following, we will consider mainly two classes of maps: normally nonsingular maps and placid maps. Here we recall these definitions.

Definition 3.3 ([7], [12, §5.3.1]). A map $f : X \to Y$ between pseudomanifolds is normally nonsingular if there exists a diagram

$$\begin{array}{ccc} N_f & \overset{i}{\hookrightarrow} & Y \times \mathbb{R}^k \\ {\scriptstyle \pi}\downarrow \uparrow{\scriptstyle s} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

where $\pi : N_f \longrightarrow X$ is a vector bundle with zero-section s, i is an open embedding, p is the first projection, and one has $f = p \circ i \circ s$. The bundle N_f is called the normal bundle.

Remark 3.4. According to Fulton–MacPherson [7], this definition says that *geometrically the singularities of* X *at any point* x *are no better or worse than the singularities of* Y *at* $f(x)$. In particular, if the target space Y is smooth, then the domain is smooth after crossing with some $\mathbb{R}^k$, so it is a homology manifold.

Definition 3.5. A map $f : X \to Y$ between two pseudomanifolds is called placid if there exists a stratification of Y such that for each stratum S in Y, we have

$$\operatorname{codim}_X f^{-1}(S) \geq \operatorname{codim}_Y(S).$$

3.2. Intersection Homology and Cohomology. Reference for this section is Goresky–MacPherson's original paper [11].

The notion of perversity is fundamental for the definition of intersection homology and cohomology. A perversity $\bar{p}$ is a multi-index sequence of integers $(p(0), p(1), \ldots)$ such that $p(0) = p(1) = p(2) = 0$ and $p(c) \leq p(c+1) \leq p(c) + 1$ for $c \geq 2$. Any perversity $\bar{p}$ lies between the zero perversity $\bar{0} = (0, 0, 0, \ldots)$ and the total perversity $\bar{t} = (0, 1, 2, 3, \ldots)$. In particular, we will use the lower middle perversity, denoted by $\overline{m}$, and the upper middle perversity, denoted by $\overline{n}$, such that

$$\overline{m}(c) = \left[\frac{c-2}{2}\right] \quad \text{and} \quad \overline{n}(c) = \left[\frac{c-1}{2}\right], \quad \text{for } c \geq 2.$$

Let $\overline{p}$ be a perversity, the complementary perversity $\overline{q}$ is defined by

$$q(c) + p(c) = t(c) = c - 2, \quad \text{for all } c \geq 2.$$

Let $\overline{p}$ and $\overline{r}$ be two perversities, if $p(c) \leq r(c)$ for every $c \geq 2$, one will write $\overline{p} \leq \overline{r}$.

Let X be an m-dimensional pseudomanifold and $\overline{p}$ a perversity. The intersection homology groups with $\mathbb{Z}_2$ coefficients, denoted by $IH_i^{\overline{p}}(X)$, are the homology groups of the chain complex

$$IC_i^{\overline{p}}(X)=\left\{\xi \in C_i(X) \mid \begin{array}{lcl} \dim(|\xi| \cap X_{m-c}) & \leq & i-c+p(c) \quad \text{and} \\ \dim(|\partial\xi| \cap X_{m-c}) & \leq & i-1-c+p(c), \ \forall\ c \geq 2 \end{array}\right\},$$

where $C_i(X)$ denotes the group of compact i-dimensional P.L. chains ξ of X with $\mathbb{Z}_2$ coefficients, and $|\xi|$ denotes the support of ξ.

The intersection cohomology groups with $\mathbb{Z}_2$ coefficients, denoted by $IH_{\overline{p}}^{m-i}(X)$, are defined as the groups of the cochain complex (see also [13])

$$IC_{\overline{p}}^{m-i}(X)=\left\{\gamma \in C^{m-i}(X) \middle| \begin{array}{lcl} \dim(|\gamma| \cap X_{m-c}) & \leq & i-c+p(c) \quad \text{and} \\ \dim(|\partial\gamma| \cap X_{m-c}) & \leq & i-1-c+p(c), \ \forall c \geq 2. \end{array}\right\},$$

where $C^{m-i}(X)$ denotes the abelian group, with $\mathbb{Z}_2$ coefficients, of all $(m-i)$-dimensional P.L. cochains of X with closed supports in X.

The main properties of intersection homology that we will use are the following:

Properties 3.6. *Let X be a compact m-dimensional pseudomanifold, then, for any perversity $\bar{p}$, the Poincaré duality map PD_X given by the cap-product with the fundamental class of X naturally factorizes in the following way* [11]*:*

$$\begin{array}{ccc} H^{m-i}(X) & \xrightarrow{\quad PD_X \quad} & H_i(X) \\ & {\scriptstyle \alpha_X} \searrow \quad \nearrow {\scriptstyle \omega_X} & \\ & IH_i^{\bar{p}}(X), & \end{array} \tag{3.7}$$

where α_X is induced by the cap-product by the fundamental class $[X]$, and ω_X is induced by the inclusion $IC_i^{\overline{p}}(X) \hookrightarrow C_i(X)$.

For perversities $\bar{p}$ and $\bar{r}$ such that $\bar{p}+\bar{r} \leq \bar{t}$, the intersection product

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{r}}(X) \to IH_{(i+j)-m}^{\bar{p}+\bar{r}}(X)$$

is well defined.

If X is compact, the Poincaré duality homomorphism

$$IH_{\bar{p}}^{m-i}(X;\mathbb{Z}_2) \to IH_i^{\bar{p}}(X;\mathbb{Z}_2) \tag{3.8}$$

is an isomorphism.

4. Locally orientable Witt spaces and Wu classes

In this section, we use definitions and notations of M. Goresky [10] and M.Goresky and W. Pardon [13]. First of all, let us fix notations in the smooth case.

Let M be an m-dimensional manifold. We will denote by $w^i(M) \in H^i(M)$ the i-th Stiefel–Whitney cohomology class (i-th S–W cohomology class) of the tangent bundle TM. The $(m-i)$-th Stiefel–Whitney homology class ($(m-i)$-th S–W homology class) of TM denoted by $w_{m-i}(M)$ in $H_{m-i}(M)$ is the image of $w^i(M)$ by Poincaré duality homomorphism.

4.1. $\mathbb{Z}_2$-Witt spaces and Wu classes. In the singular case, the mod 2 *Steenrod square operations* have been defined in intersection cohomology by M. Goresky in [10] (see also[13, §4]), as operations

$$Sq^i\colon IH^j_{\bar c}(X;\mathbb{Z}_2) \to IH^{i+j}_{2\bar c}(X;\mathbb{Z}_2)$$

provided $2\bar c \le \bar t$. Via the above Poincaré duality, one has similar operations in intersection homology (with compact supports).

Definition 4.1 ([13, §5.1]). Let X be an m-dimensional pseudomanifold. Suppose $\bar c$ is a perversity such that $2\bar c \le \bar t$. Let $\bar b = \bar t - \bar c$ be the complementary perversity. For any i with $0 \le i \le [m/2]$, the Steenrod square operation

$$Sq^i\colon IH^{\bar c}_i(X) \to IH^{2\bar c}_0(X) \to \mathbb{Z}_2$$

is given by multiplication with the i^{th}-intersection cohomology Wu class of X:

$$v^i(X) = v^i_{\bar b}(X) \in IH^i_{\bar b}(X).$$

One defines $v^i(X) = 0$ for $i > [m/2]$.

Definition 4.2 ([9, §5.1],[16]). A stratified pseudomanifold X is a $\mathbb{Z}_2$-*Witt space* if, for each stratum of odd codimension $2k+1$, one has that the link of the stratum L satisfies $IH^k_{\bar m}(L) = 0$ with the lower middle perversity $\bar m$.

There is a large class of examples of $\mathbb{Z}_2$-Witt spaces, such as manifolds, complex varieties and suspensions of odd-dimensional $\mathbb{Z}_2$-Witt spaces.

If X is a $\mathbb{Z}_2$-Witt space, then the middle intersection homology group is self-dual, i.e., satisfies the Poincaré duality over $\mathbb{Z}_2$. Also the natural homomorphism

$$IH^i_{\bar m}(X) \to IH^i_{\bar n}(X)$$

is an isomorphism.

Let X be a $\mathbb{Z}_2$-Witt space, then for any i with $0 \leq i \leq [m/2]$, the Wu classes $v^i_{\bar{m}}(X)$ lift canonically to $IH^i_{\bar{m}}(X) = IH^i_{\bar{n}}(X)$ (see [13] §10). From now on, we shall denote, for short, the Wu classes $v^i_{\bar{m}}(X)$ by $v^i(X)$.

We denote by $v_{m-i}(X) \in IH^{\bar{n}}_{m-i}(X)$ the $(m-i)$-th homology Wu class of X in intersection homology, the dual to the Wu class $v^i(X)$ (denoted by $Iv^i \in IH^i_{\bar{m}}(X)$ in [10, §5.2]).

Let $j : X \to V$ be a normally nonsingular inclusion [7], [9], [11] with normal bundle ν. Then j induces homomorphisms

$$j_* : IH^{\bar{p}}_i(X;\mathbb{Z}_2) \to IH^{\bar{p}}_i(V;\mathbb{Z}_2), \qquad j^* : IH^k_{\bar{p}}(V;\mathbb{Z}_2) \to IH^k_{\bar{p}}(X;\mathbb{Z}_2),$$

and one has the following Proposition ([13, Proposition 4.2])

Proposition 4.3. *For any $\eta \in IH^k_{\bar{p}}(V;\mathbb{Z}_2)$, with $2\bar{p} \leq \bar{t}$, we have*

$$Sq^i j^*(\eta) = j^* Sq^i(\eta),$$

which means the commutativity of the following diagram:

$$\begin{array}{ccc} IH^k_{\bar{p}}(V) & \xrightarrow{j^*} & IH^k_{\bar{p}}(X) \\ {\scriptstyle Sq^i}\downarrow & & \downarrow{\scriptstyle Sq^i} \\ IH^{i+k}_{2\bar{p}}(V) & \xrightarrow{j^*} & IH^{i+k}_{2\bar{p}}(X). \end{array}$$

Notice that in the case X denotes the boundary of an $(m+1)$-pseudomanifold V, then, the normal bundle of X in V is trivial (see Definition 3.2).

Corollary 4.4. *Let V be an $(m+1)$-pseudomanifold with boundary X, and $j : X \hookrightarrow V$ the induced inclusion. For any i with $0 \leq i \leq [m/2]$, one has $j^*(v^i(V)) = v^i(X)$.*

4.2. Locally orientable Witt spaces.

Definition 4.5 ([13, §8.1]). A stratified pseudomanifold X is *locally orientable* if the link of each stratum is an orientable pseudomanifold.

Definition 4.6 ([13, §10.2]). A stratified pseudomanifold X is a *locally orientable Witt space* if it is both locally orientable and a $\mathbb{Z}_2$*-Witt space.*

Lemma 4.7 ([13, §10.2]). *If X is a locally orientable Witt space, then $Sq^1 Sq^{2i} = Sq^{2i+1}$ as homomorphisms*

$$IH^{\bar{m}}_j(X,\mathbb{Z}_2) \to IH^{\bar{m}}_{j-2i-1}(X,\mathbb{Z}_2).$$

In our proofs, we will use the following property of locally orientable Witt spaces: For a locally orientable Witt space, the Wu classes which are defined as middle intersection homology classes, can be multiplied to construct the characteristic numbers

$$\varepsilon(v_i(X) \bullet v_j(X)) = \langle v^{m-i}(X) \cup v^{m-j}(X), [X] \rangle \in \mathbb{Z}_2,$$

where $i+j=m$, the map $\varepsilon : H_0(X,\mathbb{Z}_2) \to \mathbb{Z}_2$ denotes the augmentation, and the following diagram commutes (here $\bullet$ denotes intersection of cycles):

$$\begin{array}{ccccc} IH_i^{\bar m}(X) \times IH_j^{\bar m}(X) & \xrightarrow{\bullet} & IH_0^{\bar t}(X) & \xrightarrow{\varepsilon} & \mathbb{Z}_2 \\ \cong\times\cong \uparrow & & \cong \uparrow & & \\ IH^{m-i}_{\bar m}(X) \times IH^{m-j}_{\bar m}(X) & \xrightarrow{\cup} & IH^m_{\bar 0}(X). & & \end{array}$$

We notice that "absolute" cobordism of locally orientable Witt spaces is computed in [13, §10.5].

5. Cobordism of maps between pseudomanifolds

Definition 5.1. Let $f : X \to Y$ be a map between pseudomanifolds of dimensions m and n, respectively. The triple (f, X, Y) is null-cobordant if there exist:

(1) pseudomanifolds V and W with dimensions $m+1$ and $n+1$, respectively, such that $\partial V = X$ and $\partial W = Y$;

(2) a map $F : V \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} U_X & \xrightarrow{F_{|U_X}} & U_Y \\ \cong \downarrow \phi & & \psi \downarrow \cong \\ \partial V \times [0,1) & \xrightarrow{f \times Id} & \partial W \times [0,1), \end{array}$$

where U_X and U_Y are the collar neighborhoods of X and Y in V and W, respectively, and ϕ and ψ are P.L. diffeomorphisms such that $\phi(x) = (x,0)$, $x \in \partial V$ and $\psi(y) = (y,0)$, $y \in \partial W$;

(3) $F_{|\partial V} = f : \partial V \to \partial W$.

Given maps $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$, one may define the triple $(f \sqcup g, X_1 \sqcup X_2, Y_1 \sqcup Y_2)$ by the map $f \sqcup g$ from the disjoint union $X_1 \sqcup X_2$ of X_1 and X_2 into the disjoint union $Y_1 \sqcup Y_2$ of Y_1 and Y_2 such that $(f \sqcup g)|_{X_1} = f$ and $(f \sqcup g)|_{X_2} = g$.

Definition 5.2. Two maps $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are cobordant if the triple

$$(f \sqcup g, X_1 \sqcup X_2, Y_1 \sqcup Y_2)$$

is null-cobordant.

When fixing the target space Y, we denote the group of cobordism of maps $f : X \to Y$ between pseudomanifolds by $S\mathcal{N}_*(Y)$, the operation being given by the disjoint union.

Remark 5.3. In the smooth case, there is an isomorphism between $\mathcal{N}_*(Y)$ and $H_*(Y, \mathbb{Z}_2) \otimes \mathcal{N}_*$ where $\mathcal{N}_*(Y)$ is the group of non-orientable bordism, and $\mathcal{N}_*$ is the Thom bordism group [6].

A natural question is to ask if there exists an isomorphism between $S\mathcal{N}_*(Y)$ and $IH_*(Y, \mathbb{Z}_2) \otimes \mathcal{N}_*{}^{\mathbb{Z}_2-\mathrm{Witt}}$ where $\mathcal{N}_*{}^{\mathbb{Z}_2-\mathrm{Witt}}$ denotes the $\mathbb{Z}_2$-Witt bordism group for the unoriented case. Friedman and Siegel defined the $\mathbb{Z}_2$-Witt bordism group $\Omega_*{}^{\mathbb{Z}_2-\mathrm{Witt}}$ for the oriented case; they do not consider the case of maps between $\mathbb{Z}_2$-Witt spaces. The existence of an isomorphism between $S\mathcal{N}_*(Y)$ and $IH_*(Y, \mathbb{Z}_2) \otimes \mathcal{N}_*{}^{\mathbb{Z}_2-\mathrm{Witt}}$ is still an open conjecture.

6. Main results

In this section, we consider different cases of compact pseudomanifolds X and Y and maps $f : X \to Y$, and we prove that if (f, X, Y) is null-cobordant, then characteristic numbers associated in each case vanish. First, we consider the case X is a locally orientable Witt space of pure dimension m, and Y an n-dimensional smooth manifold. Then we consider the case where X and Y are locally orientable Witt spaces and $f : X \to Y$ a normally nonsingular (or placid) map. Remark that in these two cases we deal with different characteristic numbers.

6.1. Case of a map $f : X \longrightarrow Y$ with Y a smooth manifold.

Let $f : X \longrightarrow Y$ be a map with X a compact locally orientable Witt space of pure dimension m, and Y a closed n-dimensional smooth manifold. Then, we can define the map $f_! : IH_i^{\bar p}(X) \to IH_i^{\bar p}(Y)$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} H_i(X) & \xrightarrow{f_*} & H_i(Y) \\ \uparrow{\scriptstyle \omega_X} & & {\scriptstyle \omega_Y}\uparrow{\scriptstyle \simeq} \\ IH_i^{\bar p}(X) & \xrightarrow{f_!} & IH_i^{\bar p}(Y), \end{array}$$

i.e. $f_! = (\omega_Y)^{-1} \circ f_* \circ \omega_X$, where the map ω_Y is an isomorphism since Y is smooth.

We denote by $\widetilde{f_!}$ the composition map $\widetilde{f_!} = \alpha_Y^{-1} \circ f_!$, i.e. the composition map

$$IH_i^{\bar p}(X) \xrightarrow{\omega_X} H_i(X) \xrightarrow{f_*} H_i(Y) \xrightarrow{PD_Y^{-1}} H^{n-i}(Y)$$

where the last arrow denotes the inverse Poincaré duality isomorphism.

Definition 6.1. For any partition $\ell = \ell_1 + \cdots + \ell_s$ of a non-negative integer ℓ and r numbers $u_1, \ldots, u_r$ satisfying $u_i \le [m/2]$ for all i and

$$(\ell_1 + \ell_2 + \cdots + \ell_s) + u_1 + \cdots + u_r + r(n - m) = n, \qquad (6.2)$$

let us denote $w^\ell(Y) = w^{\ell_1}(Y) \cup \cdots \cup w^{\ell_s}(Y)$, where $w^{\ell_i}(Y)$ denotes the ℓ_i-th Stiefel–Whitney cohomology class of Y, which corresponds to $w_{n-\ell}(Y) = w_{n-\ell_1}(Y) \bullet \cdots \bullet w_{n-\ell_s}(Y)$, where $w_{n-\ell_i}(Y)$ denotes the $(n-\ell_i)$-th Stiefel–Whitney homology class of Y.

In the following, in order to avoid heavy notations, we will identify intersection homology classes $v_i(X) \in IH_i^{\bar m}(X)$ with intersection cohomology classes $v_{\bar m}^{m-i}(X) \in IH_{\bar m}^{m-i}(X) \cong IH_i^{\bar m}(X)$.

The S–W numbers of any triple (f, X, Y) corresponding to the numbers $\ell_1, \ldots, \ell_s, u_1, \ldots, u_r$ are defined by

$$\langle w^\ell(Y) \cup \widetilde{f_!}(v_{m-u_1}(X)) \cup \cdots \cup \widetilde{f_!}(v_{m-u_r}(X)), [Y] \rangle. \qquad (6.3)$$

Theorem 6.4. *Let $f : X \longrightarrow Y$ be a map, where X is compact locally orientable Witt space of pure dimension m, and Y a closed n-dimensional smooth manifold. If (f, X, Y) is null-cobordant, with $(f, X, Y) = \partial(F, V, W)$, and W is a smooth manifold, then for any partition ℓ and r numbers $u_1, \ldots, u_r$ satisfying $u_i \le [m/2]$ for all i and (6.2), the S–W numbers*

$$\langle w^\ell(Y) \cup \widetilde{f_!}(v_{m-u_1}(X)) \cup \cdots \cup \widetilde{f_!}(v_{m-u_r}(X)), [Y] \rangle$$

are zero.

Proof. With the notations of the theorem, let us define the map

$$\widetilde{F_!} : IH^{\bar{m}}_{i+1}(V) \xrightarrow{\omega_V} H_{i+1}(V) \xrightarrow{F_*} H_{i+1}(W) \xrightarrow{i_*} H_{i+1}(W,Y) \xrightarrow{LD_W} H^{n-i}(W),$$

where i_* is the morphism induced by inclusion $(W,\emptyset) \to (W,Y)$ in the long exact sequence of the pair, and LD_W is the Poincaré–Lefschetz duality isomorphism.

One has $\langle w^\ell(Y) \cup \widetilde{f_!}(v_{m-u_1}(X)) \cup \cdots \cup \widetilde{f_!}(v_{m-u_r}(X)), [Y]\rangle =$ $\langle j^* w^\ell(W) \cup j^*\widetilde{F_!}(v_{m-u_1}(V)) \cup \cdots \cup j^*\widetilde{F_!}(v_{m-u_r}(V)), \partial[W]\rangle$, by Corollary 4.4 and commutativity of the following diagram:

$$\begin{array}{ccc} IH^{\bar{m}}_i(X) & \xrightarrow{\widetilde{f_!}} & H^{n-i}(Y) \\ \uparrow{\scriptstyle j^*_X} & & \uparrow{\scriptstyle j^*} \\ IH^{\bar{m}}_{i+1}(V) & \xrightarrow{\widetilde{F_!}} & H^{n-i}(W). \end{array}$$

So, we obtain:

$$\begin{aligned} &\left\langle j^*\left(w^\ell(W) \cup \widetilde{F_!}(v_{m-u_1}(V)) \cup \cdots \cup \widetilde{F_!}(v_{m-u_r}(V))\right), \partial[W]\right\rangle \\ &\quad = \left\langle \delta j^*\left(w^\ell(W) \cup \widetilde{F_!}(v_{m-u_1}(V)) \cup \cdots \cup \widetilde{F_!}(v_{m-u_r}(V))\right), [W,\partial W]\right\rangle = 0, \end{aligned}$$

where

$$H^k(W) \xrightarrow{j^*} H^k(Y) \xrightarrow{\delta} H^{k+1}(W,\partial W)$$

is part of a long exact sequence, so $\delta j^* = 0$. □

Remark 6.5. If $f : X \to Y$ is a normally nonsingular map with Y a manifold, then $\omega_X : IH^{\overline{p}}_i(X) \to H_i(X)$ is an isomorphism, see Remark 3.4, and the map $\widetilde{f_!}$ coincides with the composition

$$IH^{\overline{p}}_i(X) \cong H_i(X) \to H_i(f(X)) \cong H^{n-i}(Y, Y \setminus f(X)) \xrightarrow{\beta} H^{n-i}(Y),$$

where the second isomorphism is Alexander duality and β is the connecting map in the long exact sequence of a pair.

6.2. Maps $f : X \longrightarrow Y$ with X and Y locally orientable Witt spaces.

Let $f : X \to Y$ be a map of pseudomanifolds, Then, in general neither the existence nor the unicity of induced map holds in intersection homology, i.e. of a Gysin map $f_!$ making the following diagram (6.6) commutative. However, if f is proper and normally nonsingular, then there is a well-defined Gysin map

$$f_! : IH^{\bar{m}}_i(X) \to IH^{\bar{m}}_i(Y)$$

such that the following diagram commutes [12, §5.4.3]:

$$\begin{array}{ccc} H_i(X) & \xrightarrow{f_*} & H_i(Y) \\ \uparrow{\scriptstyle \omega_X} & & \uparrow{\scriptstyle \omega_Y} \\ IH_i^{\bar{p}}(X) & \xrightarrow{f_!} & IH_i^{\bar{p}}(Y). \end{array} \tag{6.6}$$

The same result holds for placid maps as well (see [12] and [3, Proposition 3.2]).

We remark that the definition of the map $f_!$ is made at the level of sheaves in [3]. More precisely, the intersection homology $IH_i^{\bar{m}}(X)$ is the hypercohomology $\mathbb{H}^{m-i}(X;\mathcal{IC}_X^\bullet)$, where $\mathcal{IC}_X^\bullet$ is the so-called intersection sheaf such that for any open subset $U \subset X$, the complex of sections $\Gamma(U,\mathcal{IC}_X^\bullet)$ is the complex of intersection chains $IC_{m-\bullet}^{\overline{m}}(U)$, see [4, Corollary 5.2] and [12, §2.1].

Note that the index notation we use here is the one in [11] or [3], see [12, §2.3] for the correspondence of indices between this one and those of [4] and [12].

Let $f : X \to Y$ be a map either normally nonsingular (see [12]), or placid, or a closed embedding of a 1-codimensional complex variety (see [3, §2.4]), the existence of a unique map $f_! : IH_\bullet^{\bar{m}}(X) \to IH_\bullet^{\bar{m}}(Y)$ such that the diagram 6.6 commutes is proved at the level of intersection sheaves

$$\mathcal{IC}_X^\bullet \to f^!\mathcal{IC}_Y^\bullet[n-m]$$

(see [3]), or equivalently, by adjunction $f_!\mathcal{IC}_X^\bullet \to \mathcal{IC}_Y^\bullet$.

Lemma 6.7. *Let $f : X \longrightarrow Y$ be a normally nonsingular map, or a placid map, with X and Y compact and $(f, X, Y) = \partial(F, V, W)$. Then there exists a map $F_!$ such that the following diagram commutes:*

$$\begin{array}{ccc} IH_u^{\bar{m}}(X) & \xleftarrow{j_X^*} & IH_{u+1}^{\bar{m}}(V) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle F_!} \\ IH_u^{\bar{m}}(Y) & \xleftarrow{j_Y^*} & IH_{u+1}^{\bar{m}}(W). \end{array}$$

Proof. We apply [4, Proposition 10.7] (see also [12]) to the following diagram

$$\begin{array}{ccc} X & \xhookrightarrow{j_X} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ Y & \xhookrightarrow{j_Y} & W, \end{array}$$

where $j_X : X \longrightarrow V$ and $j_Y : Y \longrightarrow W$ are inclusion maps.

Therefore, one has equality of sheaves on Y:

$$j_Y^* F_! \mathcal{A} = f_! j_X^* \mathcal{A}$$

for any sheaf $\mathcal{A}$ on V. That provides a commutative diagram of complexes of sheaves on Y (perverse intersection sheaves for the middle perversity $\bar{m}$):

$$\begin{array}{ccc} f_! \mathcal{I}C_X^\bullet & \longleftarrow & f_! j_X^* (\mathcal{I}C_V^\bullet) = j_Y^* F_! (\mathcal{I}C_V^\bullet) \\ \downarrow & & \downarrow \\ \mathcal{I}C_Y^\bullet & \longleftarrow & j_Y^* (\mathcal{I}C_W^\bullet)\,. \end{array}$$

Taking hypercohomology $I\!H^{n-u}(Y;\bullet)$ of the previous diagram, one obtains for any u:

$$\begin{array}{ccc} I\!H^{n-u}(X;\mathcal{I}C_X^\bullet) & \xleftarrow{j_X^*} & I\!H^{n-u}(V;\mathcal{I}C_V^\bullet) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle F_!} \\ I\!H^{n-u}(Y;\mathcal{I}C_Y^\bullet) & \xleftarrow{j_Y^*} & I\!H^{n-u}(W;\mathcal{I}C_W^\bullet), \end{array}$$

where, by unicity and the construction in [3], the maps coincide with the ones previously defined. The Lemma follows. □

Theorem 6.8. *Let $f : X \longrightarrow Y$ be a normally nonsingular (or placid) map, with X and Y compact locally orientable Witt spaces of pure dimension m and n, respectively. If (f, X, Y) is null-cobordant, then for any u with $0 \leq u \leq n$, the following S–W numbers vanish:*

$$\langle v_{n-u}(Y) \bullet f_!(v_u(X)), [Y] \rangle = 0.$$

Proof. If $m > n$, the result is trivial. If $m \leq n$ and u is out of the interval $[m - [m/2], [n/2]]$ the result is trivial. Let us consider u in the interval. The diagram of Lemma 6.7 can be written in the cohomology setting

$$\begin{array}{ccc} IH_{\bar{n}}^{m-u}(X) & \xrightarrow{f_!} & IH_{\bar{n}}^{n-u}(Y) \\ \uparrow{\scriptstyle j_X^*} & & \uparrow{\scriptstyle j_Y^*} \\ IH_{\bar{n}}^{m-u}(V) & \xrightarrow{F_!} & IH_{\bar{n}}^{n-u}(W), \end{array}$$

where $\bar{m}+\bar{n} = \bar{t}$, and we use the same notation for corresponding maps j_X^* and j_Y^*.

Let us consider the intersection homology Wu class $v_{n-u}(Y) \in IH^{\bar{n}}_{n-u}(Y)$, written $v^u(Y)$ in $IH^u_{\bar{m}}(Y)$ in the intersection cohomology setting, by Corollary 4.4, $v^u(Y) = j^*_Y v^u(W)$, where $v^u(W) \in IH^u_{\bar{m}}(W)$ is the intersection cohomology Wu class corresponding to the intersection homology Wu class $v_{n+1-u}(W) \in IH^{\bar{n}}_{n+1-u}(W)$ of W.

In the same way, let us denote by $v^{m-u}(X) \in IH^{m-u}_{\bar{n}}(X)$ the intersection cohomology Wu class corresponding to the intersection homology Wu class $v_u(X) \in IH^{\bar{m}}_u(X)$, then Corollary 4.4 implies that $v^{m-u}(X) = j^*_X(v^{m-u}(V))$, where $v^{m-u}(V) \in IH^{m-u}_{\bar{n}}(V)$ is the intersection cohomology Wu class corresponding to the intersection homology class $v_{u+1}(V) \in IH^{\bar{m}}_{u+1}(V)$.

One has the intersection product $v_{n-u}(Y) \bullet f_!(v_u(X)) \in IH^{\bar{t}}_0(Y)$ corresponding to the product $v^u(Y) \cup f_!(v^{m-u}(X)) \in IH^n_{\bar{0}}(Y)$ and $f_!(v^{m-u}(X)) = f_! j^*_X(v^{m-u}(V)) \in IH^{n-u}_{\bar{n}}(Y)$.

Therefore,

$$\begin{aligned} \langle v^u(Y) \cup f_!(v^{m-u}(X)), [Y]\rangle &= \langle j^*_Y v^u(W) \cup f_! j^*_X(v^{m-u}(V)), [Y]\rangle \\ &= \langle j^*_Y v^u(W) \cup j^*_Y F_!(v^{m-u}(V)), [Y]\rangle \\ &= \langle j^*_Y \left(v^u(W) \cup F_!(v^{m-u}(V))\right), [\partial W]\rangle \\ &= \langle \delta j^*_Y \left(v^u(W) \cup F_!(v^{m-u}(V))\right), [W, \partial W]\rangle = 0, \end{aligned}$$

where the first equality is a consequence of [10, Theorem 5.3] of Goresky, the second one is from Lemma 6.7, and the last equality is obtained in an analogous way like the last equality given in the proof of Theorem 6.4. □

Remark 6.9. Let us consider the case where X is an m-dimensional compact smooth manifold, and Y is a locally orientable Witt space of pure dimension n. Then one can show that the map $f_!$ exists and is well-defined without sheaf theory, in the following way.

Since f is a normally nonsingular map, one may consider the normal bundle N_f over X and an open embedding $i : N_f \to Y \times \mathbb{R}^s$. Let T be a tubular neighborhood of $(f \times h)(X)$ in $Y \times \mathbb{R}^s$, where $h : X \to \mathbb{R}^s$ is defined in such a way that the following diagram commutes:

$$\begin{array}{ccc} N_f & \xrightarrow{\ i\ } & Y \times \mathbb{R}^s. \\ \uparrow{\scriptstyle \sigma} & \nearrow{\scriptstyle f\times h} & \\ X & & \end{array}$$

By compactness of X, the image of i lies in $Y \times \mathbb{S}^s$ for some s-dimensional sphere $\mathbb{S}^s$, and then the diagram reduces to

$$\begin{array}{ccc} N_f & \xrightarrow{i} & Y \times \mathbb{S}^s. \\ \uparrow \sigma & \nearrow_{f\times h} & \\ X & & \end{array}$$

Following Remark 2.6, there exists a map ϕ which is the composition of the maps:

$$H^i(X) \xrightarrow{\varphi} H^{i+s+n-m}(T/\partial T) \xrightarrow{c^*} H^{i+s+n-m}(Y \times \mathbb{S}^s) \to H^{i+n-m}(Y),$$

here φ denotes the Thom homomorphism and $c : Y \times \mathbb{S}^s \to T/\partial T$ is the collapsing map. The last homomorphism is given by the $K\ddot{u}nneth\ formula$ for a product of a smooth manifold with a Z_2-Witt space [5].

Since X is a smooth manifold, $\alpha_X : H^i(X) \to IH^{\bar{p}}_{m-i}(X)$ is an isomorphism, then one defines the map $f_!$ by commutativity of the following diagram, i.e. $f_! = \alpha_Y \circ \phi \circ \alpha_X^{-1}$:

$$\begin{array}{ccc} H^i(X) & \xrightarrow{\phi} & H^{n-(m-i)}(Y) \\ \alpha_X \downarrow \cong & & \downarrow \alpha_Y \\ IH^{\bar{m}}_{m-i}(X) & \xrightarrow{f_!} & IH^{\bar{m}}_{m-i}(Y). \end{array}$$

From these results, it is possible to conclude that in each case above, the S–W numbers are cobordism invariants in the following sense.

Corollary 6.10. *Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two cobordant maps with X_1 and X_2 compact locally orientable Witt spaces of pure dimension m, and Y be a closed n-dimensional smooth manifold. Then for any partition $\ell = \ell_1 + \cdots + \ell_s$ and r numbers $u_1, \ldots, u_r$ satisfying $u_i \leq [m/2]$ for all i and equation* (6.2)*, we have the equality of the corresponding S–W numbers:*

$$\begin{aligned} &\langle w^\ell(Y) \cup \widetilde{f_!}(v_{m-u_1}(X)) \cup \cdots \cup \widetilde{f_!}(v_{m-u_r}(X)), [Y]\rangle \\ &\quad = \langle w^\ell(Y) \cup \widetilde{g_!}(v_{m-u_1}(X)) \cup \cdots \cup \widetilde{g_!}(v_{m-u_r}(X)), [Y]\rangle. \end{aligned}$$

Corollary 6.11. *Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two cobordant normally nonsingular (or placid) maps, with X_1 and X_2 (of pure dimension m) and Y (of pure dimension n) compact locally orientable Witt spaces. Then for any u with $0 \leq u \leq n$, we have the equality of the corresponding S–W numbers:*

$$\langle v_{n-u}(Y) \bullet f_!(v_u(X)), [Y]\rangle = \langle v_{n-u}(Y) \bullet g_!(v_u(X)), [Y]\rangle.$$

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