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On the distribution of integers with missing digits under hereditary sum of digits function

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Abstract. The aim of this work is to estimate the cardinality of the set of integers with missing digits under an arithmetical constraint on their hereditary sum of digits. This proves in particular a theorem on the well-distribution in residue classes and on the equidistribution modulo 1 of the sequences $(w(n)\alpha)_{n\in\mathcal{W}_\mathcal{D}}$ and $(n\alpha)_{n\in\mathcal{W}_\mathcal{D},w(n)\equiv r \bmod m}$, where α is an irrational number, r and m are integers, w denotes the hereditary sum of digits function, and $\mathcal{W}_\mathcal{D}$ is the set of integers with missing digits.

1. Introduction

It is known that, given any integer $q \geqslant 2$, every positive integer n has a unique representation

$$n = \sum_{k=0}^{\nu} a_k q^k, \qquad \text{where } a_\nu \neq 0 \text{ and } a_k \in [\![0, q-1]\!] \text{ for all } k, \tag{1.1}$$

called q-ary expansion of n.

A base q being fixed and a subset $\mathcal{D} \subset [\![0, q-1]\!]$ such that $2 \leqslant |\mathcal{D}| \leqslant q-1$ (i.e., $\mathcal{D}$ expels at least one digit and keeps at least two) being chosen, we define the integers with missing digits expressed to base q relatively to $\mathcal{D}$ to be the integers whose q-ary expansion includes only the digits of $\mathcal{D}$. We denote by $\mathcal{W}_\mathcal{D}$ the set of such integers, and for $x \in \mathbb{R}$,

$$\mathcal{W}_\mathcal{D}(x) := \{n \in \mathcal{W}_\mathcal{D},\ n < x\}.$$

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A first estimate of the size of $\mathcal{W}_{\mathcal{D}}(x)$ is given when $x = q^\nu$ by

$$|\mathcal{W}_{\mathcal{D}}(q^\nu)| = \begin{cases} |\mathcal{D}|^\nu, & \text{if } 0 \in \mathcal{D}, \\ \sum_{k=1}^{\nu} |\mathcal{D}|^k, & \text{else,} \end{cases} \tag{1.2}$$

which, according to the bound $q^{\nu-1} \leqslant x < q^\nu$, gives

$$|\mathcal{W}_{\mathcal{D}}(x)| \asymp x^{\frac{\log|\mathcal{D}|}{\log q}}.$$

In particular, the set $\mathcal{W}_{\mathcal{D}}$ is of density zero, since $\dfrac{\log|\mathcal{D}|}{\log q} \in]0,1[$. As a matter of fact, the integers with missing digits do not only present a sparse sequence but also present a fractal structure. For instance, the integers with missing digits to base 3 associated to the set of digits $\{0,2\}$ have a distribution modeled on that of Cantor's set (after a rescaling). Their study is possible since their generating function is factorized completely, which enables to control its irregularities. The arithmetic properties of integers with missing digits have been intensively studied by many authors, namely Banks, Coquet, Dartyge, Erdős, Filaseta, Konyagin, Mauduit, Sárközy and Shparlinski (see [3], [6]–[9], [12]–[13]).

In their articles [8] and [9], Erdős, Mauduit and Sárközy studied the distribution of integers with missing digits in arithmetic progressions. If $q \geqslant 3$, $0 \in \mathcal{D}$ and $2 \leqslant |\mathcal{D}| \leqslant q-1$, then the main theorem of [8] states as follows:

Theorem A. *There exist positive constants $c_1 = c_1(q,|\mathcal{D}|)$, $c_2 = c_2(q,|\mathcal{D}|)$ and $c_3 = c_3(q,|\mathcal{D}|)$, such that writing $\mathcal{D} = \{d_1, d_2, \dots, d_{|\mathcal{D}|}\}$ where $d_1 = 0$ and $(d_2, \dots, d_{|\mathcal{D}|}) = 1$, $N \in \mathbb{N}$, $m' \in \mathbb{N}$, $m' \geqslant 2$, $((q-1)q, m') = 1$, $m' < \exp\left(c_1(\log N)^{\frac{1}{2}}\right)$ and $a \in \mathbb{Z}$ Then*

$$\left| |\{n \;;\; n \in \mathcal{W}_{\mathcal{D}}(N), n \equiv a \pmod{m'}\}| - \frac{1}{m'}|\mathcal{W}_{\mathcal{D}}(N)| \right| < c_2 \frac{1}{m'} |\mathcal{W}_{\mathcal{D}}(N)| \exp\left(-c_3 \frac{\log N}{\log m'}\right).$$

From Theorem A, it follows that the set $\mathcal{W}_{\mathcal{D}}(N)$ is uniformly equidistributed in the residue classes modulo m' once $m' < \exp\left(c_1(q,|\mathcal{D}|)(\log N)^{\frac{1}{2}}\right)$. This implies that for such m', $\mathcal{W}_{\mathcal{D}}(N)$ meets every residue class modulo m'.

It should be noted that Col improved the inequality [8, Lemma 2] providing a more refined result (see [4, Corollary 1]).

In the same direction, we could name an article of Konyagin [12, Theorem 1] that deals with the study of the distribution of $\mathcal{W}_{\mathcal{D}}(N)$ in the residue classes.

Theorem B. *Let $q \geqslant 3$, $\mathcal{D} \subset [\![0, q-1]\!]$ such that $0 \in \mathcal{D}$ and $2 \leqslant |\mathcal{D}| \leqslant q-1$, writing $\mathcal{D} = \{d_1, d_2, \ldots, d_{|\mathcal{D}|}\}$ where $d_1 = 0$ and $(d_2, \ldots, d_{|\mathcal{D}|}) = 1$, $N \in \mathbb{N}$, $\nu_0 \in \mathbb{N}, N_0 \geqslant 0$, $N \equiv N_0 \pmod{q^{\nu_0}}$, $K \in \mathbb{N}$, $\mathcal{M} \subset \mathbb{N}$, $(q, m) = 1$ for all $m \in \mathcal{M}$, $\mu_1, \ldots, \mu_{K+1}$ are integers, $\mu_1 \geqslant \mu_2 \cdots \geqslant \mu_K > \mu_{K+1} = 0$, $M_k = q^{\mu_k}$, and for any $k \in [\![1, K+1]\!]$ and any distinct elements $m_1, \ldots, m_k$ from $\mathcal{M}$, we have $(m_1, \ldots, m_k) \leqslant M_k$. Then*

$$\sum_{m \in \mathcal{M}} \max_{a (\mathrm{mod}\, m)} |m| \, |\{n \in \mathcal{W}_{\mathcal{D}}(N) : \ n \equiv a \pmod{m}\}| - |\mathcal{W}_{\mathcal{D}}(N)||$$

$$\leqslant |\mathcal{W}_{\mathcal{D}}(N_0)||\mathcal{M}|M_1 + \mathcal{W}_{\mathcal{D}}(N) \left(\sum_{k=1}^{K} \left(1 + q \left(1 - \frac{1}{(q-1)^5 (2q)^2} \right)^{\left\lfloor \frac{\nu_0}{2\mu_k} \right\rfloor} \right)^{2\mu_k} - K \right).$$

Theorem B induces a larger class of integers m for which the set $\mathcal{W}_{\mathcal{D}}(N)$ is uniformly equidistributed modulo m. Note that the major advantage of his formula, compared to Theorem A, lies in summing on the classes $m \in \mathcal{M}$, which allows to achieve an average result on the residue classes modulo m.

The interested reader may refer to [4], [9], [16]–[17] for additional results and details.

Given a positive integer n, we define its q-ary hereditary expansion, denoted $f_q(n)$, as follows:

$$f_q(n) = \sum_{k=0}^{\nu} a_k q^{f_q(k)},$$

where $a_0, \ldots, a_\nu$ are the integers defined in (1.1) (actually, we keep just the a_k that are nonzero). In other words, we expand n in base q as in (1.1), then we expand each power of q recursively till we get only the digits $0, \ldots, q$. So, in order to convert from q-ary expansion to q-ary hereditary expansion, we rewrite all of the exponents in q-ary expansion. Then rewrite any exponents inside the exponents, and continue in this way until every number appearing in the expression has been converted to q-ary expansion. Hence, the process continues with as many levels of exponentiation as required. For instance, the 3-ary expansion of 2018 is

$$2018 = 2 \cdot 3^0 + 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^5 + 2 \cdot 3^6,$$

and the 3-ary hereditary expansion of 2018 is

$$2018 = 2 \cdot 3^0 + 2 \cdot 3^{2 \cdot 3^0} + 2 \cdot 3^{1 \cdot 3^{1 \cdot 3^0}} + 2 \cdot 3^{1 \cdot 3^{1 \cdot 3^0} + 2 \cdot 3^0} + 2 \cdot 3^{2 \cdot 3^{1 \cdot 3^0}}.$$

The q-ary hereditary expansion was used to define Goodstein sequences and prove Goodstein's theorem, which is a statement about the natural numbers, proved by GOODSTEIN [10] in 1944, stating that every Goodstein sequence eventually terminates at 0. Indeed, the Goodstein sequence $G_m(n)$ of a positive integer n is a sequence of natural numbers whose first element $G_1(n)$ is n itself. To get $G_2(n)$, we write n in 2-ary hereditary notation, next we change all the 2's to 3's and then subtract 1 from the result. In general, the $(m+1)^{\text{th}}$ term $G_{m+1}(n)$ of the Goodstein sequence of n is obtained as follows: we take the $(m+1)$-ary hereditary representation of $G_m(n)$, replace each occurrence of the base $m+1$ with $m+2$ and then subtract one. Note that the next term depends both on the previous term and on the index m.

Goodstein's theorem states that if we continue this process, the result will be zero at some step. For example, the sequence $G_m(3)$ reaches zero at the sixth step (see [14, Table 1]). KIRBY and PARIS [11] showed that this is unprovable in Peano arithmetic. For more details about this topic, the reader can refer to [14] and [19].

The hereditary sum of digits function to base q, which we denote by w_q, assigns to each positive integer the sum of its q-ary hereditary digits. For example,

$$w_3(2018) = 2+2+2+2+1+1+2+1+1+2+2+2+1 = 21.$$

If there is no risk of confusion, we write simply w instead of w_q. It is easy to see that $w(0) = 0$, and if

$$n = \sum_{i=1}^{L} \ell_i q^{\nu_i}$$

with $\nu_1 > \cdots > \nu_L$ and $\ell_i \in [\![1, q-1]\!], \forall\, i \in [\![1, L]\!]$, then w satisfies

$$w(n) = \sum_{i=1}^{L} (\ell_i + w(\nu_i)).$$

In particular, if $k \geqslant 1$, a and b are integers such that $a \in [\![1, q-1]\!]$ and $0 \leqslant b < q^k$, then

$$w(aq^k + b) = a + w(k) + w(b).$$

In [20], SANNA gave optimal upper bounds for the exponential sum $\sum_{n<N} e(w(n)t)$, where t is a real number. In particular, his results imply that for each positive integer m, the sequence $(w(n))_{n\in\mathbb{N}_0}$ is uniformly distributed modulo m, and that for each irrational number α, the sequence $(\alpha w(n))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1.

1.1. Notation. Along this article, the following notations are adopted: we denote by $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of nonnegative integers, positive integers, integers and real numbers, respectively. Given a real number x, we denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x, $||x||$ the distance from x to the nearest integer (i.e., $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$), and we set $e(x) = e^{2i\pi x}$. The *gcd* of two integers a and b is denoted by (a, b), and if $a \leqslant b$, we denote the set $\{a, a+1, \ldots, b\}$ by $[\![a, b]\!]$. The number of elements of a set $\mathcal{A}$ is written as $|\mathcal{A}|$. We agree that, given a sequence of complex numbers $(a_j)_{j\in\mathbb{N}}$, $\sum_{j=m}^{n} a_j = 0$ and $\prod_{j=m}^{n} a_j = 1$ once $m > n$.

Given two arithmetic functions f and g; we write $f(n) = O(g(n))$ or $f(n) \ll g(n)$ if there exists a constant $c > 0$ such that $|f(n)| \leqslant c|g(n)|$ whenever n is sufficiently large. If $f(n) \ll g(n)$ and $g(n) \ll f(n)$, we write $f(n) \asymp g(n)$.

2. Statement of the results

Let a and r be integers, q, m and m' be integers $\geqslant 2$. Let $\mathcal{D}$ be a nonempty subset of $[\![0, q-1]\!]$ such that $|\mathcal{D}| \geqslant 2$. We denote by

$$\mathcal{D}^* := \mathcal{D} \setminus \{0\}$$

and for every integer $\ell \in [\![0, q]\!]$, we set

$$\mathcal{D}_\ell := [\![0, \ell-1]\!] \cap \mathcal{D} \quad \text{and} \quad \mathcal{D}_\ell^* = \mathcal{D}_\ell \setminus \{0\},$$

in particular, $\mathcal{D}_q = \mathcal{D}$ and $\mathcal{D}_0 = \varnothing$.

We set $\mathbb{1}_\mathcal{A}$ the characteristic function of the set $\mathcal{A}$, i.e.:

$$\begin{aligned} \mathbb{1}_\mathcal{A} : [\![0, q-1]\!] &\longrightarrow \{0, 1\} \\ x &\longmapsto \begin{cases} 1, & \text{if } x \in \mathcal{A}, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Moreover, we set

$$\mathcal{W}_\mathcal{D} := \left\{ n \in \mathbb{N},\ n = \sum_{k=0}^{\nu} a_k q^k,\ \text{where } a_k \in \mathcal{D} \text{ for every } k \in [\![0, \nu]\!] \text{ and } a_\nu \neq 0 \right\}$$

and for $N \geqslant 2$,

$$\begin{aligned} \mathcal{W}_\mathcal{D}(N) &:= \{n < N,\ n \in \mathcal{W}_\mathcal{D}\}, \\ \mathcal{W}_\mathcal{D}(N, a, m', r, m) &:= \{n \in \mathcal{W}_\mathcal{D}(N),\ n \equiv a \bmod m',\ w(n) \equiv r \bmod m\}. \end{aligned}$$

Our work is split as follows: In paragraph 3, we present some technical lemmas concerning the exponential sums and the cardinality of $\mathcal{W}_{\mathcal{D}}(N)$, which we shall need in order to settle our main theorem. In paragraph 4, we add to Theorem A some congruence constraint on the hereditary sum of digits in the set $\{n \in \mathcal{W}_{\mathcal{D}}(N)\ : n \equiv a \bmod m'\}$ so to obtain our main theorem:

Theorem 2.1. *Let $q \geqslant 3$, $\mathcal{D} \subset [\![0, q-1]\!]$ such that $2 \leqslant |\mathcal{D}| \leqslant q-1$, there exist positive constants $k_1 = k_1(q, |\mathcal{D}|)$, $k_2 = k_2(q, |\mathcal{D}|)$ and $k_3 = k_3(q, |\mathcal{D}|)$ such that, writing $\mathcal{D} = \{d_1, d_2, \ldots, d_{|\mathcal{D}|}\}$ with $d_1 = \min(\mathcal{D})$ and $(d_2, \ldots, d_{|\mathcal{D}|}) = 1$, $N \in \mathbb{N}, m$ and $m' \in \mathbb{N}$ such that $m, m' \geqslant 2, (q(q-1), m') = 1$,*

$$mm' < \exp(k_1(\log N)^{\frac{1}{2}}), \tag{2.1}$$

and $(a, r) \in \mathbb{Z}^2$. Then

$$\begin{aligned} &\left| |\mathcal{W}_{\mathcal{D}}(N, a, m', r, m)| - \frac{1}{mm'} |\mathcal{W}_{\mathcal{D}}(N)| \right| \\ &\qquad < k_2 \frac{1}{mm'} |\mathcal{W}_{\mathcal{D}}(N)| \exp\left(-k_3 \frac{\log N}{\log mm'} \right). \end{aligned} \tag{2.2}$$

Following this theorem, we prove that the set $\mathcal{W}_{\mathcal{D}}(N)$ is uniformly equidistributed in the residue classes modulo mm' admitting $mm' < \exp(k(q, \mathcal{D})(\log N)^{\frac{1}{2}})$. It follows that for such m and m', the set $\mathcal{W}_{\mathcal{D}}(N)$ meets every congruence class modulo mm'.

It is to note that Theorem 2.1 holds true even if $0 \notin \mathcal{D}$. Indeed, the formula differs between the cases $0 \in \mathcal{D}$ and $0 \notin \mathcal{D}$ (due to combinatorial reasons), but Theorem 2.1 remains true in the latter case (which is rarely considered in the literature, see [5], for instance).

As a consequence of this estimate, we care about a result concerning the equidistribution modulo 1 of the sequence $(w(n)\alpha)_{n \in \mathcal{W}_{\mathcal{D}}}$.

Corollary 2.2. *The sequence $(w(n)\alpha)_{n \in \mathcal{W}_{\mathcal{D}}}$ is equidistributed modulo 1 if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

In the same perspective, we could state the following corollary

Corollary 2.3. *Let $m \geqslant 2$ and $r \in \mathbb{Z}$. The sequence $(n\alpha)_{n \in \mathcal{W}_{\mathcal{D}}, w(n) \equiv r \bmod m}$ is equidistributed modulo 1 if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

3. Estimate of the sets $\mathcal{W}_{\mathcal{D}}(N)$ and $\mathcal{W}_{\mathcal{D}}(N, a, m', r, m)$

Now, we introduce the function T_N with complex values defined for every positive integer N, for every real numbers α, β and for every nonempty subset $\mathcal{D}$

of $[\![0, q-1]\!]$ by

$$T_N(\alpha, \beta) = T_N(\mathcal{D}, \alpha, \beta) := \sum_{n \in \mathcal{W}_{\mathcal{D}}(N)} e\left(n\alpha + w(n)\beta\right).$$

We also set

$$\Theta_{\mathcal{D}}(\alpha, \beta, j, \ell) = 1 + e(w(j)\beta) \sum_{d \in \mathcal{D}_\ell^*} e(d(q^j\alpha + \beta)) \quad \text{for } j \in \mathbb{N}_0 \text{ and } \ell \in [\![0, q]\!].$$

The function T_N enables to sieve the elements of $\mathcal{W}_{\mathcal{D}}\left(N, a, m', r, m\right)$. In fact, we have

Lemma 3.1. *Let q, m', m be integers $\geqslant 2$ and $(a, r) \in \mathbb{Z}^2$, and let N be a positive integer. Then, we have*

$$|\mathcal{W}_{\mathcal{D}}\left(N, a, m', r, m\right)| = \frac{1}{mm'} \sum_{t=0}^{m'-1} e\left(-\frac{ta}{m'}\right) \sum_{s=0}^{m-1} e\left(-\frac{sr}{m}\right) T_N\left(\frac{t}{m'}, \frac{s}{m}\right).$$

Proof. It is obvious according to the classic orthogonality relation that

$$\begin{aligned} |\mathcal{W}_{\mathcal{D}}\left(N, a, m', r, m\right)| &= \sum_{\substack{n \in \mathcal{W}_{\mathcal{D}}(N) \\ n \equiv a \bmod m' \\ w(n) \equiv r \bmod m}} 1 \\ &= \frac{1}{mm'} \sum_{n \in \mathcal{W}_{\mathcal{D}}(N)} \sum_{t=0}^{m'-1} e\left(t\, \frac{n-a}{m'}\right) \sum_{s=0}^{m-1} e\left(s\, \frac{w(n)-r}{m}\right) \\ &= \frac{1}{mm'} \sum_{t=0}^{m'-1} e\left(-\frac{ta}{m'}\right) \sum_{s=0}^{m-1} e\left(-\frac{sr}{m}\right) T_N\left(\frac{t}{m'}, \frac{s}{m}\right). \qquad \square \end{aligned}$$

We are going to state and prove some recursive relations enabling to express $|\mathcal{W}_{\mathcal{D}}|$ and T_N as previously done in [1]–[2] and [18], for instance. Note that this idea was already used in the work of Sanna [20] in order to establish an upper bound for the exponential sum $\sum_{n<N} e(\alpha w(n))$, where $N \in \mathbb{N}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

3.1. Study of the case $0 \in \mathcal{D}$. First, we state the following lemma that determines the cardinality of $\mathcal{W}_{\mathcal{D}}(N)$ for a given integer $N \geqslant 2$ whenever $0 \in \mathcal{D}$.

Lemma 3.2. *Let $\mathcal{D} \subset [\![0, q-1]\!]$ such that $0 \in \mathcal{D}$. For every integer $N = \ell_1 q^{\nu_1} + \cdots + \ell_L q^{\nu_L}$ with $\nu_1 > \cdots > \nu_L$ and $\ell_i \in [\![1, q-1]\!]$ $\forall\, i \in [\![1, L]\!]$, we have*

$$|\mathcal{W}_{\mathcal{D}}(N)| = |\mathcal{D}_{\ell_1}||\mathcal{D}|^{\nu_1} + \sum_{k=1}^{L-1} \left(\prod_{j=1}^{k} \mathbb{1}_{\mathcal{D}}(\ell_j)\right) |\mathcal{D}_{\ell_{k+1}}||\mathcal{D}|^{\nu_{k+1}}.$$

Proof. We shall write $N = \ell_1 q^{\nu_1} + N'$ with $\nu_1 = \left\lfloor \frac{\log N}{\log q} \right\rfloor$, $N' < q^{\nu_1}$ and $\ell_1 \in [\![1, q-1]\!]$. Evidently, we have

$$|\mathcal{W}_{\mathcal{D}}(N)| = |\mathcal{W}_{\mathcal{D}}(\ell_1 q^{\nu_1})| + \mathbb{1}_{\mathcal{D}}(\ell_1)|\mathcal{W}_{\mathcal{D}}(N')|.$$

Iterating the process, we are left with

$$|\mathcal{W}_{\mathcal{D}}(N)| = |\mathcal{W}_{\mathcal{D}}(\ell_1 q^{\nu_1})| + \sum_{k=1}^{L-1} \left(\prod_{j=1}^{k} \mathbb{1}_{\mathcal{D}}(\ell_j) \right) |\mathcal{W}_{\mathcal{D}}(\ell_{k+1} q^{\nu_{k+1}})|. \tag{3.1}$$

Let ν be a nonnegative integer, and let $\ell \in [\![1, q-1]\!]$. We separate the integers strictly smaller than ℓq^{ν} into two sets: the integers strictly smaller than q^{ν}, and the integers between q^{ν} and ℓq^{ν}. There follows the identity

$$|\mathcal{W}_{\mathcal{D}}(\ell q^{\nu})| = |\mathcal{W}_{\mathcal{D}}(q^{\nu})| + (|\mathcal{D}_{\ell}| - 1)|\mathcal{D}|^{\nu} = |\mathcal{D}_{\ell}||\mathcal{D}|^{\nu}. \tag{3.2}$$

Putting (3.2) in (3.1), we reach the required conclusion. □

Next, we are looking to simplify the expression of the function T_N whenever $0 \in \mathcal{D}$.

Lemma 3.3. *Let $\mathcal{D} \subset [\![0, q-1]\!]$ such that $0 \in \mathcal{D}$. Then, for all real numbers α, β, for every integer $N = \ell_1 q^{\nu_1} + \cdots + \ell_L q^{\nu_L}$ with $\nu_1 > \cdots > \nu_L$ and $\ell_i \in [\![1, q-1]\!]$ $\forall\, i \in [\![1, L]\!]$, it follows that*

$$T_N(\alpha, \beta) = \sum_{0 \leqslant k \leqslant L-1} \left(\prod_{1 \leqslant j \leqslant k} \mathbb{1}_{\mathcal{D}}(\ell_j) e(\ell_j q^{\nu_j} \alpha + (\ell_j + w(\nu_j))\beta) \right) \times \Theta_{\mathcal{D}}(\alpha, \beta, \nu_{k+1}, \ell_{k+1}) \prod_{j=0}^{\nu_{k+1}-1} \Theta_{\mathcal{D}}(\alpha, \beta, j, q).$$

Proof. We write $N = \ell_1 q^{\nu_1} + N'$ with $\nu_1 = \left\lfloor \frac{\log N}{\log q} \right\rfloor$, $N' < q^{\nu_1}$ and $\ell_1 \in [\![1, q-1]\!]$. Obviously,

$$T_N(\alpha, \beta) = T_{\ell_1 q^{\nu_1}}(\alpha, \beta) + \sum_{\substack{\ell_1 q^{\nu_1} \leqslant n < \ell_1 q^{\nu_1} + N' \\ n \in \mathcal{W}_{\mathcal{D}}}} e(n\alpha + w(n)\beta).$$

But, if $\ell_1 \notin \mathcal{D}$, then the second sum is zero and the expression is reduced to $T_{\ell_1 q^{\nu_1}}(\alpha, \beta)$.

Otherwise, we write

$$\sum_{\substack{\ell_1 q^{\nu_1} \leqslant n < \ell_1 q^{\nu_1} + N' \\ n \in \mathcal{W}_{\mathcal{D}}}} e(n\alpha + w(n)\beta) = \sum_{n \in \mathcal{W}_{\mathcal{D}}(N')} e((n + \ell_1 q^{\nu_1})\alpha + w(n + \ell_1 q^{\nu_1})\beta)$$
$$= e(\ell_1(q^{\nu_1}\alpha + \beta) + w(\nu_1)\beta) T_{N'}(\alpha, \beta).$$

So

$$T_N(\alpha, \beta) = T_{\ell_1 q^{\nu_1}}(\alpha, \beta) + \mathbb{1}_{\mathcal{D}}(\ell_1) e(\ell_1 q^{\nu_1}\alpha + (\ell_1 + w(\nu_1))\beta) T_{N'}(\alpha, \beta).$$

Reiterating, it follows that

$$T_N(\alpha, \beta) = T_{\ell_1 q^{\nu_1}}(\alpha, \beta) + \mathbb{1}_{\mathcal{D}}(\ell_1) e(\ell_1 q^{\nu_1}\alpha + (\ell_1 + w(\nu_1))\beta) T_{\ell_2 q^{\nu_2}}(\alpha, \beta) + \cdots$$
$$+ \mathbb{1}_{\mathcal{D}}(\ell_1) \cdots \mathbb{1}_{\mathcal{D}}(\ell_{L-1}) e\left(\sum_{j=1}^{L-1} \ell_j q^{\nu_j}\alpha + \sum_{j=1}^{L-1} (\ell_j + w(\nu_j))\beta\right) T_{\ell_L q^{\nu_L}}(\alpha, \beta). \quad (3.3)$$

Yet, for every nonnegative integer k and for every integer $\ell \in [\![2, q-1]\!]$, we have

$$T_{\ell q^k}(\alpha, \beta) = T_{q^k}(\alpha, \beta) + \sum_{\substack{1 \leqslant d < \ell \\ d \in \mathcal{D}}} \sum_{\substack{dq^k \leqslant n < (d+1)q^k \\ n \in \mathcal{W}_{\mathcal{D}}}} e(n\alpha + w(n)\beta)$$
$$= T_{q^k}(\alpha, \beta) + \sum_{d \in \mathcal{D}_\ell^*} \sum_{n \in \mathcal{W}_{\mathcal{D}}(q^k)} e((n + dq^k)\alpha + w(n + dq^k)\beta)$$
$$= \left[1 + e(w(k)\beta) \sum_{d \in \mathcal{D}_\ell^*} e(d(q^k\alpha + \beta))\right] T_{q^k}(\alpha, \beta). \quad (3.4)$$

Furthermore, this relation is trivially true for $\ell = 1$, and thus true for every $\ell \in [\![1, q-1]\!]$. Finally, we write:

$$T_{q^{k+1}}(\alpha, \beta) = 1 + \sum_{j=0}^{k} \sum_{\substack{1 \leqslant d < q \\ d \in \mathcal{D}}} \sum_{m \in \mathcal{W}_{\mathcal{D}}(q^j)} e((dq^j + m)\alpha + w(dq^j + m)\beta)$$
$$= 1 + \sum_{j=0}^{k} e(w(j)\beta) \sum_{d \in \mathcal{D}^*} e(d(q^j\alpha + \beta)) T_{q^j}(\alpha, \beta).$$

This implies the formula

$$T_{q^{k+1}}(\alpha, \beta) = \left[1 + e(w(k)\beta) \sum_{d \in \mathcal{D}^*} e(d(q^k\alpha + \beta))\right] T_{q^k}(\alpha, \beta),$$

which enables to affirm that

$$T_{q^k}(\alpha, \beta) = \prod_{j=0}^{k-1} \left[1 + e(w(j)\beta) \sum_{d \in \mathcal{D}^*} e(d(q^j\alpha + \beta))\right], \quad \text{for every } k \in \mathbb{N}. \quad (3.5)$$

Putting (3.5) in (3.4), which we insert in (3.3), we reach the required formula. □

3.2. Study of the case $0 \notin \mathcal{D}$. Now, we state a lemma enabling to determine the cardinality of $\mathcal{W}_{\mathcal{D}}(N)$ for $N \geqslant 2$ whenever $0 \notin \mathcal{D}$.

Lemma 3.4. *Let $\mathcal{D} \subset [\![1, q-1]\!]$. For every integer $N = \ell_\nu q^\nu + \cdots + \ell_1 q + \ell_0$ with $\ell_i \in [\![0, q-1]\!]$ $\forall\, i \in [\![0, \nu]\!]$ and $\ell_\nu \neq 0$, we have*

$$|\mathcal{W}_{\mathcal{D}}(N)| = |\mathcal{D}_{\ell_\nu}||\mathcal{D}|^\nu + \sum_{k=1}^{\nu} |\mathcal{D}|^k + \sum_{k=1}^{\nu} \left(\prod_{j=k}^{\nu} \mathbb{1}_{\mathcal{D}}(\ell_j) \right) |\mathcal{D}_{\ell_{k-1}}||\mathcal{D}|^{k-1}.$$

Proof. Let $N = \ell_\nu q^\nu + N'$ with $\nu = \left\lfloor \dfrac{\log N}{\log q} \right\rfloor$, $N' < q^\nu$ and $\ell_\nu \neq 0$. We write $N' = \ell_{\nu-1} q^{\nu-1} + \cdots + \ell_0$ and split the elements of $\mathcal{W}_{\mathcal{D}}(\ell_\nu q^\nu + N')$ into two sets: the integers strictly smaller than $\ell_\nu q^\nu$ ($|\mathcal{W}_{\mathcal{D}}(\ell_\nu q^\nu)|$ elements), and the integers between $\ell_\nu q^\nu$ and $\ell_\nu q^\nu + N'$. Then

- if $N' < q^{\nu-1}$, the second set is empty;
- if $q^{\nu-1} \leqslant N' < q^\nu$ and $\ell_\nu \notin \mathcal{D}$, the second set is empty again;
- if $q^{\nu-1} \leqslant N' < q^\nu$ and $\ell_\nu \in \mathcal{D}$, the second set contains $\big(|\mathcal{W}_{\mathcal{D}}(N')| - |\mathcal{W}_{\mathcal{D}}(q^{\nu-1})|\big)$ elements.

We set the function

$$\begin{aligned} \chi_\delta : [\![0, q-1]\!] &\longrightarrow \{0,1\} \\ \ell &\longmapsto \begin{cases} 1, & \text{if } \ell \geqslant \delta = \min(\mathcal{D}), \\ 0, & \text{else.} \end{cases} \end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned} |\mathcal{W}_{\mathcal{D}}(N)| &= |\mathcal{W}_{\mathcal{D}}(\ell_\nu q^\nu)| \\ &\quad + \mathbb{1}_{\mathcal{D}}(\ell_\nu)\chi_\delta(\ell_{\nu-1}) \big(|\mathcal{W}_{\mathcal{D}}(\ell_{\nu-1} q^{\nu-1} + \cdots + \ell_0)| - |\mathcal{W}_{\mathcal{D}}(q^{\nu-1})|\big). \end{aligned} \tag{3.7}$$

Let k be a nonnegative integer, and let $\ell \in [\![1, q-1]\!]$, we separate the elements of $\mathcal{W}_{\mathcal{D}}(\ell q^k)$ into two sets: the integers strictly smaller than q^k ($|\mathcal{W}_{\mathcal{D}}(q^k)|$ elements), and those between q^k and ℓq^k ($|\mathcal{D}_\ell||\mathcal{D}|^k$ elements). It follows that

$$|\mathcal{W}_{\mathcal{D}}(\ell q^k)| = |\mathcal{W}_{\mathcal{D}}(q^k)| + |\mathcal{D}_\ell||\mathcal{D}|^k. \tag{3.8}$$

Iterating the process, we insert (3.8) in (3.7), and applying the identities

$$\mathbb{1}_{\mathcal{D}}(\ell)\chi_\delta(\ell) = \mathbb{1}_{\mathcal{D}}(\ell) \quad \text{and} \quad \chi_\delta(\ell)|\mathcal{D}_\ell| = |\mathcal{D}_\ell|,$$

we get

$$|\mathcal{W}_{\mathcal{D}}(N)| = |\mathcal{W}_{\mathcal{D}}(q^{\nu})| + |\mathcal{D}_{\ell_\nu}||\mathcal{D}|^{\nu} + \sum_{k=1}^{\nu}\left(\prod_{j=k}^{\nu}\mathbb{1}_{\mathcal{D}}(\ell_j)\right)|\mathcal{D}_{\ell_{k-1}}||\mathcal{D}|^{k-1}. \tag{3.9}$$

But, following (1.2), we recall that

$$|\mathcal{W}_{\mathcal{D}}(q^{\nu})| = \sum_{k=1}^{\nu}|\mathcal{D}|^{k}. \tag{3.10}$$

Finally, it is sufficient to put (3.10) in (3.9) to get the desired conclusion. □

Last, we finish this paragraph by a lemma that simplifies the expression of the function T_N in the case $0 \notin \mathcal{D}$.

Lemma 3.5. *Let $\mathcal{D} \subset [\![1, q-1]\!]$. Then, for all real numbers α, β, for all integers ν and for every integer $N = \ell_\nu q^\nu + \cdots + \ell_1 q + \ell_0$ with $\ell_i \in [\![0, q-1]\!]$ $\forall\, i \in [\![0, \nu]\!]$ and $\ell_\nu \neq 0$, we have*

$$T_N(\alpha,\beta) = T_{\ell_\nu q^\nu}(\alpha,\beta) + e\left(\sum_{j=0}^{\nu} w(j)\beta\right)\sum_{1\leqslant k\leqslant \nu} T_{\ell_{k-1}}\left(q^{k-1}\alpha,\beta\right)$$
$$\prod_{j=0}^{k-2} T_q(q^j\alpha,\beta)\left(\prod_{k\leqslant j\leqslant \nu}\mathbb{1}_{\mathcal{D}}(\ell_j)e(\ell_j q^j\alpha + \ell_j\beta)\right).$$

Proof. Let $N = \ell_\nu q^\nu + N'$ with $\nu = \left\lfloor \dfrac{\log N}{\log q}\right\rfloor$, $N' < q^\nu$ and $\ell_\nu \neq 0$. We set $N' = \ell_{\nu-1}q^{\nu-1} + \cdots + \ell_0$, therefore

$$T_{\ell_\nu q^\nu + N'}(\alpha,\beta) = T_{\ell_\nu q^\nu}(\alpha,\beta) + \sum_{\substack{\ell_\nu q^\nu \leqslant n < \ell_\nu q^\nu + N' \\ n\in\mathcal{W}_{\mathcal{D}}}} e(n\alpha + w(n)\beta).$$

But, if $\ell_\nu \notin \mathcal{D}$ or if $N' < q^{\nu-1}$, the second sum is zero and the expression is reduced to $T_{\ell_\nu q^\nu}(\alpha,\beta)$.

Otherwise, we write

$$\sum_{\substack{\ell_\nu q^\nu \leqslant n < \ell_\nu q^\nu + N' \\ n\in\mathcal{W}_{\mathcal{D}}}} e(n\alpha + w(n)\beta) = \sum_{\substack{q^{\nu-1} \leqslant n < N' \\ n\in\mathcal{W}_{\mathcal{D}}}} e((n+\ell_\nu q^\nu)\alpha + w(n+\ell_\nu q^\nu)\beta)$$
$$= e(\ell_\nu(q^\nu\alpha+\beta)+w(\nu)\beta)\left(T_{N'}(\alpha,\beta)-T_{q^{\nu-1}}(\alpha,\beta)\right).$$

In conclusion,

$$T_N(\alpha,\beta)=\begin{cases} T_{\ell_\nu q^\nu}(\alpha,\beta), & \text{if } N'<q^{\nu-1},\\ T_{\ell_\nu q^\nu}(\alpha,\beta)+\mathbb{1}_{\mathcal{D}}(\ell_\nu)e(\ell_\nu q^\nu\alpha+(\ell_\nu+w(\nu))\beta) & \\ \quad\times(T_{N'}(\alpha,\beta)-T_{q^{\nu-1}}(\alpha,\beta)), & \text{if } q^{\nu-1}\leqslant N'<q^\nu. \end{cases}$$

Recall the function χ_δ already defined in (3.6), then

$$\begin{aligned} T_N(\alpha,\beta) &= T_{\ell_\nu q^\nu}(\alpha,\beta)+\mathbb{1}_{\mathcal{D}}(\ell_\nu)\chi_\delta(\ell_{\nu-1})\\ &\quad\times e(\ell_\nu q^\nu\alpha+(\ell_\nu+w(\nu))\beta)\left(T_{N'}(\alpha,\beta)-T_{q^{\nu-1}}(\alpha,\beta)\right). \end{aligned} \tag{3.11}$$

Next, for every nonnegative integer k and for every integer $\ell\in[\![2,q-1]\!]$, we write

$$\begin{aligned} T_{\ell q^k}(\alpha,\beta) &= T_{q^k}(\alpha,\beta)+\sum_{\substack{1\leqslant d<\ell\\ d\in\mathcal{D}}}\ \sum_{\substack{dq^k\leqslant n<(d+1)q^k\\ n\in\mathcal{W}_{\mathcal{D}}}} e(n\alpha+w(n)\beta)\\ &= T_{q^k}(\alpha,\beta)+\sum_{\substack{1\leqslant d<\ell\\ d\in\mathcal{D}}}\ \sum_{\substack{q^{k-1}\leqslant n<q^k\\ n\in\mathcal{W}_{\mathcal{D}}}} e((n+dq^k)\alpha+w(n+dq^k)\beta)\\ &= T_{q^k}(\alpha,\beta)+e(w(k)\beta)\sum_{\substack{1\leqslant d<\ell\\ d\in\mathcal{D}}} e(d(q^k\alpha+\beta))\sum_{\substack{q^{k-1}\leqslant n<q^k\\ n\in\mathcal{W}_{\mathcal{D}}}} e(n\alpha+w(n)\beta)\\ &= T_{q^k}(\alpha,\beta)+e(w(k)\beta)T_\ell(q^k\alpha,\beta)\left(T_{q^k}(\alpha,\beta)-T_{q^{k-1}}(\alpha,\beta)\right). \end{aligned} \tag{3.12}$$

Furthermore, this formula is evidently true for $\ell=1$, and so is true for every $\ell\in[\![1,q-1]\!]$. Then,

$$\begin{aligned} T_{q^{k+1}}(\alpha,\beta) &= \sum_{j=0}^{k}\ \sum_{\substack{q^j\leqslant n<q^{j+1}\\ n\in\mathcal{W}_{\mathcal{D}}}} e(n\alpha+w(n)\beta)\\ &= \sum_{j=0}^{k}\sum_{d\in\mathcal{D}}\ \sum_{\substack{q^{j-1}\leqslant m<q^j\\ m\in\mathcal{W}_{\mathcal{D}}}} e((dq^j+m)\alpha+w(dq^j+m)\beta)\\ &= \sum_{j=0}^{k} e(w(j)\beta)T_q(q^j\alpha,\beta)\left[T_{q^j}(\alpha,\beta)-T_{q^{j-1}}(\alpha,\beta)\right]. \end{aligned}$$

Subsequently, it follows that

$$T_{q^{k+1}}(\alpha,\beta)=T_{q^k}(\alpha,\beta)+e(w(k)\beta)T_q(q^k\alpha,\beta)\left[T_{q^k}(\alpha,\beta)-T_{q^{k-1}}(\alpha,\beta)\right].$$

Finally a strong induction on k enables to conclude that:

$$T_{q^k}(\alpha,\beta)=\sum_{h=0}^{k-1}\prod_{j=0}^{h}e(w(j)\beta)T_q(q^j\alpha,\beta). \tag{3.13}$$

We insert (3.13) in (3.12), and assuming $\nu>2$, we set $N''=\ell_{\nu-2}q^{\nu-2}+\cdots+\ell_0$ in (3.11) to obtain

$$\begin{aligned}T_{N'}(\alpha,\beta)-T_{q^{\nu-1}}(\alpha,\beta)&=e\left(\sum_{j=0}^{\nu-1}w(j)\beta\right)T_{\ell_{\nu-1}}(q^{\nu-1}\alpha,\beta)\prod_{j=0}^{\nu-2}T_q(q^j\alpha,\beta)\\&\quad+\mathbb{1}_{\mathcal{D}}(\ell_{\nu-1})\chi_\delta(\ell_{\nu-2})e(\ell_{\nu-1}q^{\nu-1}\alpha+(\ell_{\nu-1}\\&\quad+w(\nu-1))\beta)\left(T_{N''}(\alpha,\beta)-T_{q^{\nu-2}}(\alpha,\beta)\right).\end{aligned}$$

We reiterate the process and report to (3.11), taking in consideration the identities

$$\chi_\delta(\ell)T_\ell(q^\ell\alpha,\beta)=T_\ell(q^\ell\alpha,\beta)$$

and

$$\mathbb{1}_{\mathcal{D}}(\ell)\chi_\delta(\ell)=\mathbb{1}_{\mathcal{D}}(\ell),$$

true for every integer ℓ. We are left with the formula

$$\begin{aligned}T_N(\alpha,\beta)=T_{\ell_\nu q^\nu}(\alpha,\beta)+e\left(\sum_{j=0}^{\nu}w(j)\beta\right)\sum_{1\leqslant k\leqslant\nu}T_{\ell_{k-1}}\left(q^{k-1}\alpha,\beta\right)\\\prod_{j=0}^{k-2}T_q(q^j\alpha,\beta)\left(\prod_{k\leqslant j\leqslant\nu}\mathbb{1}_{\mathcal{D}}(\ell_j)e(\ell_jq^j\alpha+\ell_j\beta)\right).\end{aligned}$$

If $\nu=2$,

$$\begin{aligned}T_{N'}(\alpha,\beta)-T_q(\alpha,\beta)&=e((w(0)+w(1))\beta)T_{\ell_1}(q\alpha,\beta)T_q(\alpha,\beta)\\&\quad+\mathbb{1}_{\mathcal{D}}(\ell_1)\chi_\delta(\ell_0)e(\ell_1q\alpha+(\ell_1+w(1))\beta)T_{\ell_0}(\alpha,\beta).\end{aligned}$$

We put this in (3.11) to reach the previous formula (with $\nu=2$).

Hence, the lemma is proved. □

4. Proof of Theorem 2.1 – Analogon of Theorem A

4.1. Notations and lemmas. We start by introducing some notations and some lemmas used in [8] that will be useful to settle Theorem 2.1.

Lemma 4.1. *Let q and $\mathcal{D}$ be as defined in Theorem 2.1, and let $\alpha \in \mathbb{R}$, then there exist two integers i and j such that $2 \leqslant i < j \leqslant |\mathcal{D}|$ and*

$$\|(d_j - d_i)\alpha\| \geqslant \frac{1}{2(q-2)^2}\|\alpha\|.$$

Proof. The lemma can be proved following [8, Lemma 1] step by step modulo some elementary modifications, and it is easy to check that it works also when $0 \notin \mathcal{D}$. □

First, for $\alpha \in \mathbb{R}$ and $\mathcal{D} = \{d_1, \dots, d_{|\mathcal{D}|}\}$, we write

$$u(\alpha) = u_{\mathcal{D}}(\alpha) := \sum_{k=1}^{|\mathcal{D}|} e(d_k\alpha) \quad \text{and} \quad \mathcal{U}(\alpha) = \mathcal{U}_{\mathcal{D}}(\alpha) := \frac{u_{\mathcal{D}}(\alpha)}{|\mathcal{D}|}.$$

Next, we use an improved version of [8, Lemma 2].

Lemma 4.2. *Let q and $\mathcal{D}$ be as defined in Theorem 2.1, $\alpha \in \mathbb{R}$, then we have*

$$|\mathcal{U}(\alpha)| \leqslant 1 - \frac{64}{(q+1)^3}\|\alpha\|^2.$$

Proof. This is [4, Corollary 1]. □

Lemma 4.3. *If $q,\ m',\ t,\ \rho \in \mathbb{N}$ satisfy $q, m' \geqslant 2$, $t \in [\![1, m'-1]\!]$, $(q, m') = 1$ and $(q-1)\frac{t}{m'} \notin \mathbb{Z}$, then*

$$\rho \geqslant 2\frac{\log m'}{\log q} + 8 \tag{4.1}$$

and $\beta \in \mathbb{R}$, then

$$\sum_{j=0}^{\rho-1} \left\|\beta + q^j\frac{t}{m'}\right\|^2 \geqslant \frac{(q-1)^2}{20q^4}\frac{\rho}{\log m'}.$$

Proof. This is [15, Lemma 2'], which means a slight improvement of [17, Lemma 2]. □

4.2. Proof of Theorem 2.1. We go back to the generating function

$$T_N(\alpha,\beta)=\sum_{n\in\mathcal{W}_{\mathcal{D}}(N)} e(n\alpha+w(n)\beta),$$

in particular,

$$T_N(0,0)=|\mathcal{W}_{\mathcal{D}}(N)|, \tag{4.2}$$

then for $a,\ r\in\mathbb{Z}$, $m',\ m\in\mathbb{N}$, we get from Lemma 3.1

$$|\mathcal{W}_{\mathcal{D}}(N,a,m',r,m)|=\frac{1}{mm'}\sum_{t=0}^{m'-1} e\left(-\frac{ta}{m'}\right)\sum_{s=0}^{m-1} e\left(-\frac{rs}{m}\right)T_N\left(\frac{t}{m'},\frac{s}{m}\right). \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned}
&\left||\mathcal{W}_{\mathcal{D}}\left(N,a,m',r,m\right)|-\frac{1}{mm'}|\mathcal{W}_{\mathcal{D}}(N)|\right|\\
&=\left||\mathcal{W}_{\mathcal{D}}\left(N,a,m',r,m\right)|-\frac{1}{mm'}T_N(0,0)\right|\\
&\leqslant\frac{1}{mm'}\left(\sum_{s=1}^{m-1}\left|T_N\left(0,\frac{s}{m}\right)\right|+\sum_{(t,s)\in\Upsilon}\left|T_N\left(\frac{t}{m'},\frac{s}{m}\right)\right|\right),
\end{aligned} \tag{4.4}$$

where

$$\Upsilon:=[\![1,m'-1]\!]\times[\![0,m-1]\!],$$

so we are led to estimate $\left|T_N\left(0,\frac{s}{m}\right)\right|$ for $s\in[\![1,m-1]\!]$ and $\left|T_N\left(\frac{t}{m'},\frac{s}{m}\right)\right|$ for

$$(t,s)\in\Upsilon. \tag{4.5}$$

On the one hand, we write N as

$$N=\sum_{k=1}^{L}a_kq^{\nu_k},$$

where $\nu_1>\nu_2>\cdots>\nu_L$, $a_k\in[\![1,q-1]\!]$ for $k\in[\![1,L]\!]$ so that $q^{\nu_1}\leqslant N<q^{\nu_1+1}$, hence $\nu_1=\left\lfloor\frac{\log N}{\log q}\right\rfloor$.

Moreover, when $s\in[\![1,m-1]\!]$ and $0\in\mathcal{D}$, we can bound from above according to Lemma 3.3

$$\begin{aligned}
\left|T_N\left(0,\frac{s}{m}\right)\right|&\leqslant\sum_{k=1}^{L}\left|\Theta_{\mathcal{D}}\left(0,\frac{s}{m},\nu_k,\ell_k\right)\right|\prod_{j=0}^{\nu_k-1}\left|\Theta_{\mathcal{D}}\left(0,\frac{s}{m},j,q\right)\right|\\
&\leqslant(q-1)\sum_{k=0}^{\nu_1}\prod_{j=0}^{k-1}\left(1+\left|\sum_{d\in\mathcal{D}^*}e\left(d\frac{s}{m}\right)\right|\right).
\end{aligned} \tag{4.6}$$

Indeed, the second inequality holds true, because $\left|\Theta_{\mathcal{D}}\left(0,\frac{s}{m},\nu_k,\ell_k\right)\right| \leqslant (q-1)$ and the sum running over $[\![0,\nu_1]\!]$ (which is positive) includes $\nu_L,\ldots,\nu_1$.
But, Lemma 4.1 proves that $\forall\ s\in[\![1,m-1]\!]$ there exists some i and j such that $m\nmid s(d_j-d_i)$, so

$$\left|\sum_{d\in\mathcal{D}^*} e\left(d\frac{s}{m}\right)\right| < |\mathcal{D}|-1.$$

Thus, setting

$$\gamma := 1+\max_{1\leqslant s\leqslant m-1}\left|\sum_{d\in\mathcal{D}^*} e\left(d\frac{s}{m}\right)\right| \quad\text{and}\quad \omega := \frac{\log\gamma}{\log|\mathcal{D}|}<1,$$

it follows that $\forall\ s\in[\![1,m-1]\!]$,

$$1+\left|\sum_{d\in\mathcal{D}^*} e\left(d\frac{s}{m}\right)\right| \leqslant |\mathcal{D}|^{\omega},$$

which gives after substitution in (4.6)

$$\left|T_N\left(0,\frac{s}{m}\right)\right| \leqslant (q-1)\sum_{k=0}^{\nu_1}|\mathcal{D}|^{k\omega} \leqslant (q-1)\frac{|\mathcal{D}|^{\omega(\nu_1+1)}}{|\mathcal{D}|^{\omega}-1} \leqslant \frac{2^{\omega}(q-1)}{2^{\omega}-1}|\mathcal{W}_{\mathcal{D}}(N)|^{\omega}. \quad (4.7)$$

This upper bound follows from the fact that $|\mathcal{W}_{\mathcal{D}}(N)| \geqslant |\mathcal{D}|^{\nu_1}$ (thanks to Lemma 3.2) and that the function $x\mapsto\frac{x}{x-1}$ is decreasing and $|\mathcal{D}|\geqslant 2$.

An analogous upper bound holds true, from Lemmas 3.4 and 3.5, when $0\notin\mathcal{D}$. In fact, thanks to Lemma 3.5, we get (see the details in the proof of Corollary 2.2)

$$\left|T_N\left(0,\frac{s}{m}\right)\right| \leqslant q\frac{2^{\omega}}{2^{\omega}-1}|\mathcal{D}|^{\omega\nu} \leqslant \frac{q}{2^{\omega}-1}|\mathcal{W}_{\mathcal{D}}(N)|^{\omega},$$

where $\omega = \dfrac{\log\left|\sum_{d\in\mathcal{D}} e(d\alpha)\right|}{\log|\mathcal{D}|} < 1$. Here we use Lemma 3.4 to recall that $|\mathcal{W}_{\mathcal{D}}(N)| \geqslant 2|\mathcal{D}|^{\nu}$.
On the other hand, we use the ideas of [8], and we denote by $\mathcal{A}_\ell$, for $\ell\in[\![1,L]\!]$, the set of integers n written in the form

$$n=\sum_{i=1}^{\ell-1}a_iq^{\nu_i}+xq^{\nu_\ell}+\sum_{j=0}^{\nu_\ell-1}y_jq^j,$$

where

$$x \in \mathcal{D} \cap [\![0, a_\ell - 1]\!],\ y_j \in \mathcal{D}, \quad \text{for } j \in [\![0, \nu_\ell - 1]\!].$$

Thus, we clearly have

$$\mathcal{W}_{\mathcal{D}}(N) = \bigcup_{\ell=1}^{L} \mathcal{A}_\ell$$

with

$$\mathcal{A}_j \cap \mathcal{A}_\ell = \varnothing, \quad \text{for } 1 \leqslant j < \ell \leqslant L.$$

It follows that for all $\alpha,\ \beta \in \mathbb{R}$,

$$T_N(\alpha,\beta) = \sum_{n \in \mathcal{W}_{\mathcal{D}}(N)} e(n\alpha + w(n)\beta) = \sum_{\ell=1}^{L} \sum_{n \in \mathcal{A}_\ell} e(n\alpha + w(n)\beta) = \sum_{\ell=1}^{L} T_{N,\ell}(\alpha,\beta), \tag{4.8}$$

where, for $\ell \in [\![1, L]\!]$, we have

$$\begin{aligned}
T_{N,\ell}(\alpha,\beta) &= \sum_{x} \sum_{y_0} \cdots \sum_{y_{\nu_\ell - 1}} e\Bigg(\left[\sum_{i=1}^{\ell-1} a_i q^{\nu_i} + x q^{\nu_\ell} + \sum_{i=0}^{\nu_\ell - 1} y_i q^i \right] \alpha \\
&\quad + \left[\sum_{i=1}^{\ell-1} (a_i + w(\nu_i)) + x + \mathbb{1}_{\mathbb{N}}(x) w(\nu_\ell) + \sum_{i=0}^{\nu_\ell - 1} (y_i + \mathbb{1}_{\mathbb{N}}(y_i) w(i)) \right] \beta \Bigg) \\
&= e\left(\sum_{i=1}^{\ell-1} a_i q^{\nu_i} \alpha + \sum_{i=1}^{\ell-1} (a_i + w(\nu_i))\, \beta \right) \\
&\quad \times \left(\sum_{x \in \mathcal{D} \cap [\![0, a_\ell - 1]\!]} e(x(q^{\nu_\ell}\alpha + \beta) + \mathbb{1}_{\mathbb{N}}(x) w(\nu_\ell)\beta) \right) \\
&\quad \times \prod_{j=0}^{\nu_\ell - 1} \left(\sum_{y_j \in \mathcal{D}} e(y_j(q^j \alpha + \beta) + \mathbb{1}_{\mathbb{N}}(y_j) w(j)\beta) \right).
\end{aligned}$$

Then

$$|T_{N,\ell}(\alpha,\beta)| \leqslant |\mathcal{D}| \prod_{j=0}^{\nu_\ell - 1} |u_{\mathcal{D}}(q^j\alpha + \beta)| \leqslant q |\mathcal{D}|^{\nu_\ell} \prod_{j=0}^{\nu_\ell - 1} |\mathcal{U}_{\mathcal{D}}(q^j \alpha + \beta)|. \tag{4.9}$$

Subsequently, and since $|\mathcal{D}| \geqslant 2$, we get for $\alpha,\ \beta \in \mathbb{R}$

$$\begin{aligned}
&\sum_{\ell\ :\nu_\ell \leqslant \frac{1}{2}\frac{\log N}{\log q}} |T_{N,\ell}(\alpha,\beta)| \\
&\quad \leqslant \sum_{\ell\ :\nu_\ell \leqslant \frac{1}{2}\frac{\log N}{\log q}} q|\mathcal{D}|^{\nu_\ell} \prod_{j=0}^{\nu_\ell - 1} 1 < q \sum_{j=0}^{+\infty} |\mathcal{D}|^{\frac{\log N}{2\log q} - j} \\
&\quad \leqslant 2q|\mathcal{D}|^{\frac{\log N}{2\log q}} < 2q(|\mathcal{D}|^{\nu_1+1})^{\frac{1}{2}} < 2q^{\frac{3}{2}}(|\mathcal{D}|^{\nu_1})^{\frac{1}{2}} \leqslant 2q^{\frac{3}{2}}|\mathcal{W}_{\mathcal{D}}(N)|^{\frac{1}{2}}, \qquad (4.10)
\end{aligned}$$

as we have from Lemmas 3.2 and 3.4

$$|\mathcal{W}_{\mathcal{D}}(N)| \geqslant |\mathcal{D}|^{\nu_1}. \qquad (4.11)$$

In addition, whenever $\nu_\ell > \frac{1}{2}\frac{\log N}{\log q}$, then if N is sufficiently large, we get from (4.1)

$$\nu_\ell > \frac{1}{2}\frac{\log N}{\log q} > 2\frac{\log m'}{\log q} + 8,$$

so that condition (4.1) is true with ν_ℓ instead of ρ. Then using Lemmas 4.2 and 4.3, from (4.9) and the convexity inequality $1 - x \leqslant \exp(-x)$ (for $x \geqslant 0$), for $\ell \leqslant L$ such that $\nu_\ell > \frac{1}{2}\frac{\log N}{\log q}$, we have

$$\begin{aligned}
\left| T_{N,\ell}\left(\frac{t}{m'}, \frac{s}{m}\right) \right| &\leqslant q|\mathcal{D}|^{\nu_\ell} \prod_{j=0}^{\nu_\ell-1} \left(1 - \frac{64}{(q+1)^3}\left\| q^j\frac{t}{m'} + \frac{s}{m}\right\|^2\right) \\
&\leqslant q|\mathcal{D}|^{\nu_\ell} \exp\left(-\frac{64}{(q+1)^3}\sum_{j=0}^{\nu_\ell-1}\left\| q^j\frac{t}{m'} + \frac{s}{m}\right\|^2\right) \\
&\leqslant q|\mathcal{D}|^{\nu_\ell} \exp\left(-\frac{16(q-1)^2}{5q^4(q+1)^3}\frac{\nu_\ell}{\log m'}\right) \\
&\leqslant q|\mathcal{D}|^{\nu_\ell} \exp\left(-\frac{8(q-1)^2}{5q^4(q+1)^3\log q}\frac{\log N}{\log mm'}\right).
\end{aligned}$$

Then, from (2.1) and (4.11), for $t \in [\![1, m'-1]\!]$,

$$\begin{aligned}
&\sum_{\ell \leqslant L\ :\ \nu_\ell > \frac{1}{2}\frac{\log N}{\log q}} \left| T_{N,\ell}\left(\frac{t}{m'}, \frac{s}{m}\right)\right| \\
&\quad \leqslant q\exp\left(-\frac{\log N}{k_4\log mm'}\right)\sum_{j=0}^{+\infty}|\mathcal{D}|^{\nu_1 - j} \leqslant |\mathcal{D}|^{\nu_1}\exp\left(-\frac{\log N}{k_5\log mm'}\right) \\
&\quad \leqslant |\mathcal{W}_{\mathcal{D}}(N)|\exp\left(-\frac{\log N}{k_5\log mm'}\right), \qquad (4.12)
\end{aligned}$$

where $k_4,\ k_5$ depend at most on q and $|\mathcal{D}|$.

It follows from (4.8), (4.10) and (4.12) that for (t,s) satisfying (4.5), we get

$$\begin{aligned}\left|T_N\left(\frac{t}{m'},\frac{s}{m}\right)\right| &\\ &\leqslant 2q^{\frac{3}{2}}|\mathcal{W}_{\mathcal{D}}(N)|^{\frac{1}{2}}+|\mathcal{W}_{\mathcal{D}}(N)|\exp\left(-\frac{\log N}{k_5\log mm'}\right)\\ &=\frac{1}{mm'}|\mathcal{W}_{\mathcal{D}}(N)|\left(2mm'q^{\frac{3}{2}}|\mathcal{W}_{\mathcal{D}}(N)|^{-\frac{1}{2}}+mm'\exp\left(-\frac{\log N}{k_5\log mm'}\right)\right). \quad (4.13)\end{aligned}$$

Since $|\mathcal{D}|\geqslant 2$ and thanks to (4.11), there exist positive constants $k_6=k_6(q)$ and $k_7=k_7(q)$ such that

$$|\mathcal{W}_{\mathcal{D}}(N)|\geqslant|\mathcal{D}|^{\nu_1}\geqslant 2^{\left\lfloor\frac{(\log N)}{(\log q)}\right\rfloor}>k_6N^{k_7}. \quad (4.14)$$

Choosing k_1 in (2.1) sufficiently small, inequality (2.2) follows immediately from (2.1), (4.4), (4.7), (4.13) and (4.14), finishing the proof. □

5. Some applications to the equidistribution modulo 1

This paragraph is devoted to the applications of the formulae obtained in paragraph 3 to the problems of equidistribution modulo 1. The following results are direct consequences of the bounds of exponential sums T_N defined in paragraph 3.

Proof of Corollary 2.2. Indeed, if $\alpha\in\mathbb{Q}$, the sequence $(w(n)\alpha)_{n\in\mathcal{W}_{\mathcal{D}}}$ takes only a finite number of values modulo 1 and is clearly not equidistributed modulo 1.

From Weyl's criterion, Corollary 2.2 is hence equivalent to prove that for all $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ and for all $h\in\mathbb{Z}\setminus\{0\}$, we have

$$\sum_{n\in\mathcal{W}_{\mathcal{D}}(N)}e(w(n)h\alpha)=o(|\mathcal{W}_{\mathcal{D}}(N)|),$$

meaning that for all $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, we have

$$\sum_{n\in\mathcal{W}_{\mathcal{D}}(N)}e(w(n)\alpha)=o(|\mathcal{W}_{\mathcal{D}}(N)|).$$

Case 1. If $0\in\mathcal{D}$, we write $N=\ell_1q^{\nu_1}+\cdots+\ell_Lq^{\nu_L}$ with $\nu_1>\cdots>\nu_L$ and $\ell_i\in[\![1,q-1]\!]$ for every $i\in[\![1,L]\!]$. Thus

$$\sum_{n\in\mathcal{W}_{\mathcal{D}}(N)}e(w(n)\alpha)=T_N\left(0,\alpha\right).$$

It follows from (4.6) that

$$\left|\sum_{n\in\mathcal{W}_{\mathcal{D}}(N)} e(w(n)\alpha)\right| \leqslant (q-1)\sum_{k=0}^{\nu_1}\prod_{j=0}^{k-1}\left(1+\left|\sum_{d\in\mathcal{D}^*} e\,(d\alpha)\right|\right). \tag{5.1}$$

But, $\left|\sum_{d\in\mathcal{D}^*} e\,(d\alpha)\right| < |\mathcal{D}|-1$, so setting $\rho = \dfrac{\log\left(1+\left|\sum_{d\in\mathcal{D}^*} e\,(d\alpha)\right|\right)}{\log|\mathcal{D}|} < 1$, we get

$$1+\left|\sum_{d\in\mathcal{D}^*} e\,(d\alpha)\right| = |\mathcal{D}|^{\rho}.$$

Then substituting in (5.1), we get

$$\left|\sum_{n\in\mathcal{W}_{\mathcal{D}}(N)} e(w(n)\alpha)\right| \leqslant (q-1)\sum_{k=0}^{\nu_1}\prod_{j=0}^{k-1}|\mathcal{D}|^{\rho} \leqslant (q-1)\frac{|\mathcal{D}|^{\rho(\nu_1+1)}-1}{|\mathcal{D}|^{\rho}-1}. \tag{5.2}$$

But, Lemma 3.2 gives

$$|\mathcal{W}_{\mathcal{D}}(N)| \geqslant |\mathcal{D}|^{\nu_1},$$

which implies from (5.2)

$$\frac{\left|\sum_{n\in\mathcal{W}_{\mathcal{D}}(N)} e(w(n)\alpha)\right|}{|\mathcal{W}_{\mathcal{D}}(N)|} \leqslant (q-1)\frac{|\mathcal{D}|^{\rho}}{|\mathcal{D}|^{\rho}-1}|\mathcal{D}|^{(\rho-1)\left(\left\lfloor\frac{\log N}{\log q}\right\rfloor+1\right)},$$

and the result follows immediately by tending N towards $+\infty$.

Case 2. If $0\notin\mathcal{D}$, we write $N=\ell_\nu q^\nu+\cdots+\ell_1 q+\ell_0$ with $\ell_i\in[\![0,q-1]\!]$ for all $i\in[\![0,\nu]\!]$ and $\ell_\nu\neq 0$, then Lemmas 3.4 and 3.5 lead to the inequalities

$$|\mathcal{W}_{\mathcal{D}}(N)| \geqslant |\mathcal{D}|^{\nu},$$

and

$$\begin{aligned}
&\left|\sum_{n\in\mathcal{W}_{\mathcal{D}}(N)} e(w(n)\alpha)\right| \\
&\quad = |T_N(0,\alpha)| \leqslant |T_{\ell_\nu q^\nu}(0,\alpha)| + \sum_{1\leqslant k\leqslant\nu}\left|T_{\ell_{k-1}}(0,\alpha)\right|\prod_{j=0}^{k-2}|T_q(0,\alpha)| \\
&\quad \leqslant \sum_{h=0}^{\nu-1}\left|\sum_{d\in\mathcal{D}} e(d\alpha)\right|^{h+1} + (q-1)\sum_{1\leqslant k\leqslant\nu+1}\left|\sum_{d\in\mathcal{D}} e(d\alpha)\right|^{k-1} \leqslant q\frac{|\mathcal{D}|^{\rho'}}{|\mathcal{D}|^{\rho'}-1}|\mathcal{D}|^{\rho'\nu},
\end{aligned}$$

where $\rho' = \dfrac{\log\left|\sum_{d\in\mathcal{D}} e(d\alpha)\right|}{\log|\mathcal{D}|} < 1$, since α is an irrational number, so following Lemma 4.1, there exist two integers i and j such that $2 \leqslant i < j \leqslant |\mathcal{D}|$ and $(d_j - d_i)\,\alpha \notin \mathbb{Z}$.

Hence, Corollary 2.2 is proved. □

Proof of Corollary 2.3. Indeed, the case $\alpha \in \mathbb{Q}$ is rejected as done above, and from Weyl's criterion it remains to prove that for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we have

$$\sum_{n\in\mathcal{W}_{\mathcal{D}},w(n)\equiv r \mod m} e(n\alpha) = o\left(|\mathcal{W}_{\mathcal{D}}(N)|\right).$$

But, the left-side term is simply a combination of the functions $T_N\left(\alpha, \frac{h}{m}\right)$ (where h is a parameter going along the set $[\![0, m-1]\!]$) that could be bounded as done previously to reach the desired result. □

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