Commutativity of Cho and normal Jacobi operators on real hypersurfaces in the complex quadric

By JUAN DE DIOS PÉREZ (Granada) and YOUNG JIN SUH (Daegu)

Abstract. On a real hypersurface in the complex quadric we can consider the Levi-Civita connection and, for any non-zero real constant k, the k-th generalized Tanaka–Webster connection. We prove the non-existence of real hypersurfaces in the complex quadric for which the covariant derivatives associated to both connections coincide when they act on the normal Jacobi operator of the real hypersurface.

1. Introduction

The complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ is a compact Hermitian symmetric space of rank 2. It is also a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ (see [4]). The space Q^m is equipped with two geometric structures: a Kaehler structure J and a parallel circle subbundle $\mathfrak{A}$ of the endomorphism bundle $\text{End}(TQ^m)$, which consists of all the real structures on the tangent space of Q^m. For any $A \in \mathfrak{A}$ the following relations hold: $A^2 = I$ and $AJ = -JA$. A nonzero tangent vector W at a point of Q^m is called *singular* if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for Q^m: $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic vectors.

Real hypersurfaces M are immersed submanifolds of real co-dimension 1 in a Hermitian manifold such as a complex space form, or the complex two-plane Grassmannian $SU_{m+2}/S(U_2 U_m)$, or the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 U_m)$. Since Q^m is a compact Hermitian symmetric space with

rank 2, it is interesting to study real hypersurfaces M in Q^m. The Kaehler structure J of Q^m induces on M an almost contact metric structure (ϕ, ξ, η, g), where ϕ is the structure tensor field, ξ is the Reeb vector field, η is a 1-form and g is the induced Riemannian metric of Q^m.

The study of real hypersurfaces M in Q^m is initiated by BERNDT and SUH in [1]. In this paper the geometric properties of real hypersurfaces M in complex quadric Q^m, which are tubes of radius r, $0 < r < \pi/2$, around the totally geodesic $\mathbb{C}P^k$ in Q^m, when $m = 2k$, or tubes of radius r, $0 < r < \pi/2\sqrt{2}$, around the totally geodesic Q^{m-1} in Q^m, are presented. The condition of isometric Reeb flow is equivalent to the commuting condition of the shape operator S with the structure tensor ϕ of M. The classification of such real hypersurfaces in Q^m is obtained in [2].

Given a Riemannian manifold $(\tilde{M}, \tilde{g})$, Jacobi fields along geodesics satisfy a differential equation which results in the notion of Jacobi operator. That is, if $\tilde{R}$ is the Riemannian curvature tensor of $\tilde{M}$ and X is a tangent vector field on $\tilde{M}$, then the Jacobi operator with respect to X at a point $p \in \tilde{M}$ is given by

$$(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p),$$

and is a self adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$, i.e., $\tilde{R}_X \in \mathrm{End}(T_p\tilde{M})$. In the case of real hypersurfaces M in Q^m, we can consider the normal Jacobi operator $\bar{R}_N$, where $\bar{R}$ is the Riemannian curvature tensor of Q^m and N is the unit normal vector field on the real hypersurface M.

As M has an almost contact metric structure, for any non-zero real constant k, we can define the so called k-th generalized Tanaka–Webster connection $\hat{\nabla}^{(k)}$ on M by

$$\hat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\phi SX, Y)\xi - \eta(Y)\phi SX - k\eta(X)\phi Y$$

for any X, Y tangent to M, where ∇ is the Levi-Civita connection on M, S denotes the shape operator on M associated to N (see [3]). ϕ, η and ξ will be defined in Section 3. Let us call $F_X^{(k)} Y = g(\phi SX, Y)\xi - \eta(Y)\phi SX - k\eta(X)\phi Y$, for any X, Y tangent to M. $F_X^{(k)}$ is called the k-th Cho operator on M associated to X. Notice that if $X \in \mathcal{C}$, the maximal holomorphic distribution on M, given by all the vector fields orthogonal to ξ, the associated Cho operator does not depend on k and we will denote it simply by F_X. Then, given a symmetric tensor field L of type (1,1) on M, $\nabla_X L = \hat{\nabla}_X^{(k)} L$ for a tangent vector field X on M if and only if $F_X^{(k)} L = LF_X^{(k)}$, that is, the eigenspaces of L are preserved by $F_X^{(k)}$.

In this paper we study real hypersurfaces M in Q^m such that the covariant derivatives associated to the Levi-Civita and the k-th generalized Tanaka–Webster

connections coincide when we apply them to the normal Jacobi operator $\bar{R}_N$, that is

$$\nabla \bar{R}_N = \hat{\nabla}^{(k)} \bar{R}_N \tag{1.1}$$

for some non-zero real k. We will prove the following

Theorem 1.1. *There do not exist real hypersurfaces M in Q^m, $m \geq 3$, such that $\nabla \bar{R}_N = \hat{\nabla}^{(k)} \bar{R}_N$, for any non-zero real k.*

2. The space Q^m

The complex projective space $\mathbb{C}P^{m+1}$ is considered as the Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. The symbol $o = [0, \ldots, 0, 1]$ in $\mathbb{C}P^{m+1}$ is the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The action of the special orthogonal group $SO_{m+2} \subset SU_{m+2}$ on $\mathbb{C}P^{m+1}$is of cohomogeneity one. The totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$ is a singular orbit of the action of SO_{m+2} containing the point o. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. It is a homogeneous model, which interprets geometrically the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. Thus, the complex quadric Q^m is considered as a Hermitian space of rank 2. For $m = 1$ the complex quadric Q^1 is isometric to a sphere S^2 of constant curvature. For $m = 2$ the complex quadric Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. Therefore, we assume the dimension of complex quadric Q^m to be greater than or equal to 3.

Moreover, the complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ defined by the homogeneous quadric equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where z_i, $i = 1, \ldots, m+2$, are homogeneous coordinates on $\mathbb{C}P^{m+1}$. The Kaehler structure of the complex projective space $\mathbb{C}P^{m+1}$ induces canonically a Kaehler structure (J, g) on Q^m, where g is a Riemannian metric with maximal holomorphic sectional curvature 4 induced by the Fubini Study metric of $\mathbb{C}P^{m+1}$.

A point $[z]$ in $\mathbb{C}P^{m+1}$ is the complex span of z, i.e., $[z] = \{\lambda z | \lambda \in \mathbb{C}\}$, where z is a nonzero vector of $\mathbb{C}^{m+2}$. Take the Riemannian fibration $\pi : S^{2m+3} \subset \mathbb{C}^{m+2} \longrightarrow \mathbb{C}P^{m+1}$ given by $z \mapsto [z]$. Then $\mathbb{C}^{m+2} \ominus [z]$ is the horizontal space of π at $z \in S^{2m+3}$.

The shape operator $A_{\bar{z}}$ of Q^m with respect to the unit normal vector $\bar{z}$ is given by

$$A_{\bar{z}} \pi_*|_z w = \pi_*|_z \bar{w},$$

for all $w \in T_{[z]}Q^m$. The shape operator $A_{\bar{z}}$ is a complex conjugation restricted to $T_{[z]}Q^m$. The complex vector space $T_{[z]}Q^m$ is decomposed into

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (+1)-eigenspace of $A_{\bar{z}}$, i.e., $A_{\bar{z}}X = X$ and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$, i.e., $A_{\bar{z}}JX = -JX$ for any $X \in V(A_{\bar{z}})$. Geometrically, it means that $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, which is an antilinear involution. The set of all shape operators $A_{\lambda\bar{z}}$ defines a parallel circle subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$, which consists of all the real structures on the tangent space of Q^m. For any $A \in \mathfrak{A}$ the following relations hold:

$$A^2 = I \quad \text{and} \quad AJ = -JA.$$

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ yields that the Riemannian curvature tensor R of Q^m is given by

$$\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ$$
$$+ g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY,$$

where J is the complex structure, g is the Riemannian metric and A is a real structure in $\mathfrak{A}$.

A nonzero tangent vector $W \in T_{[z]}Q^m$ is called *singular* if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for Q^m:

(1) $\mathfrak{A}$-*principal*. In this case, there exists a real structure $A \in \mathfrak{A}$ such that $W \in V(A)$.

(2) $\mathfrak{A}$-*isotropic*. In this case, there exists a real structure $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$.

For every unit vector field $W \in T_{[z]}Q^m$, there is a complex conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY,$$

for some $t \in [0, \pi/4]$. The singular vectors correspond to the values $t = 0$ and $t = \pi/4$.

3. Real hypersurfaces in Q^m

Let M be a real hypersurface in Q^m and N a unit normal vector field of M. For any vector X tangent to M, we write

$$JX = \phi X + \eta(X)N, \tag{3.1}$$

where ϕX denotes the tangential component of JX and $\eta(X)N$ the normal component. The above relation defines on M a skew-symmetric tensor field of type (1,1) ϕ, named the structure tensor. The structure vector field ξ is defined by $\xi = -JN$ and is called the Reeb vector field. The 1-form η is given by $\eta(X) = g(X,\xi)$ for any vector field X tangent to M. So, on M an almost contact metric structure (ϕ,ξ,η,g) is defined. The elements of the almost contact structure satisfy the following relations:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3.2}$$

for all tangent vectors X, Y to M. Relation (3.2) implies

$$\phi\xi = 0.$$

The tangent bundle TM of M splits orthogonally into

$$TM = \mathcal{C} \oplus \mathcal{F},$$

where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J.

At each point $[z] \in M$, we choose a real structure $A \in \mathfrak{A}_{[z]}$ such that

$$N_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2, \qquad AN_{[z]} = \cos(t)Z_1 - \sin(t)JZ_2, \tag{3.3}$$

where Z_1, Z_2 are orthonormal vectors in $V(A)$ and $0 \le t \le \frac{\pi}{4}$. Moreover, the above relations due to $\xi = -JN$ imply

$$\xi_{[z]} = -\cos(t)JZ_1 + \sin(t)Z_2, \qquad A\xi_{[z]} = \cos(t)JZ_1 + \sin(t)Z_2. \tag{3.4}$$

So we have $g(AN_{[z]}, \xi_{[z]}) = 0$.

Let $X \in T_{[z]}M$, then AX is decomposed into

$$AX = BX + \rho(X)N, \tag{3.5}$$

where BX is the tangential component, $\rho(X)N$ is the normal component with $\rho(X) = g(AX, N) = g(X, AN)$.

Moreover, we define the maximal $\mathfrak{A}_{[z]}$-invariant subspace of $T_{[z]}M$ to be given by

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M | AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\}.$$

In [8], the following lemma concerning the normal vector N of M is included:

Lemma 3.1. *Let M be a real hypersurface in Q^m. Then for each $[z] \in M$, we have:*

(i) *If $N_{[z]}$ is $\mathfrak{A}$-principal, then $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$.*

(ii) *If $N_{[z]}$ is not $\mathfrak{A}$-principal, then $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]} \ominus \mathbb{C}(JX + Y)$.*

The shape operator of a real hypersurface M in Q^m is denoted by S. The real hypersurface is called *Hopf hypersurface* if the Reeb vector field is an eigenvector of the shape operator, i.e.,

$$S\xi = \alpha\xi, \tag{3.6}$$

where $\alpha = g(S\xi, \xi)$ is the Reeb function. The Codazzi equation of M due to (3.5) is given by

$$
\begin{aligned}
g((\nabla_X S)Y &- (\nabla_Y S)X, Z) \\
&= \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) + \rho(X)g(BY, Z) \\
&\quad - \rho(Y)g(BX, Z) - \eta(BX)g(BY, \phi Z) - \eta(BX)\rho(Y)\eta(Z) \\
&\quad + \eta(BY)g(BX, \phi Z) + \eta(BY)\rho(X)\eta(Z),
\end{aligned} \tag{3.7}
$$

and the curvature tensor $R(X, Y)Z$ of M is given by

$$
\begin{aligned}
R(X, Y)Z = {}&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
&+ g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX \\
&- g(JAX, Z)JAY + g(SY, Z)SX - g(SX, Z)SY,
\end{aligned} \tag{3.8}
$$

for any X, Y, Z tangent to M.

4. Proof of Theorem 1.1

The normal Jacobi operator of a real hypersurface in Q^m is calculated by the Gauss equation for $Y = Z = N$ and, because of (3.3), is given by

$$\bar{R}_N(X) = X + 3\eta(X)\xi + \cos(2t)AX - g(AX, N)AN - g(AX, \xi)A\xi, \tag{4.1}$$

for any $X \in TM$.

Let M be a real hypersurface in Q^m whose normal Jacobi operator satisfies relation (1.1). This is equivalent to having $F_X^{(k)} \bar{R}_N Y = \bar{R}_N F_X^{(k)} Y$, for any X, Y tangent to M. In particular, if $X = \xi$, we obtain

$$g(\phi S\xi, \bar{R}_N Y)\xi - g(\bar{R}_N \xi, Y)\phi S\xi - k\phi \bar{R}_N Y$$
$$= g(\phi S\xi, Y)\bar{R}_N \xi - \eta(Y)\bar{R}_N \phi S\xi - k\bar{R}_N \phi Y \qquad (4.2)$$

for any Y tangent to M. If $X \in \mathcal{C}$. we have

$$g(\phi SX, \bar{R}_N Y)\xi - g(\bar{R}_N \xi, Y)\phi SX = g(\phi SX, Y)\bar{R}_N \xi - \eta(Y)\bar{R}_N \phi SX \qquad (4.3)$$

for any $X \in \mathcal{C}$, Y tangent to M.

All the following calculations take place at an arbitrary point $[z] \in M$, but we omit the subscript $_{[z]}$ from the vector fields and other objects for the sake of brevity.

Let us suppose that M is Hopf at $[z]$, i.e., that $S\xi = \alpha\xi$ holds. From (4.2) we get $k\phi\bar{R}_N Y = k\bar{R}_N \phi Y$ for any Y tangent to M. As $k \neq 0$, $\phi\bar{R}_N = \bar{R}_N \phi$ and N must be $\mathfrak{A}$-isotropic, see [5]. In this case, $g(AN, N) = g(A\xi, \xi) = g(AN, \xi) = 0$ and $\bar{R}_N \xi = 4\xi$. From (4.3) we get $g(\bar{R}_N \xi, \xi)\phi SX = \bar{R}_N \phi SX$ for any $X \in \mathcal{C}$. That is, $3\phi SX = -g(\phi SX, AN)AN - g(\phi SX, A\xi)A\xi$. Let us suppose there exists a unit $Y \in \mathcal{C}$ such that $SY = \lambda Y$, $\lambda \neq 0$. Then we have $3\phi Y = -g(\phi Y, AN)AN - g(\phi Y, A\xi)A\xi$. Its scalar product with ϕY yields $3 = -g(\phi Y, AN)^2 - g(\phi Y, A\xi)^2$, which is impossible. Therefore $SX = 0$ for any $X \in \mathcal{C}$.

Then the equation of Codazzi for a unit $X \in \mathcal{C}$, $Y = \phi X$, $Z = \xi$ yields $0 = -2 + g(X, AN)g(A\phi X, \xi) - g(\phi X, AN)g(AX, \xi) + g(X, A\xi)g(JA\phi X, \xi) - g(\phi X, A\xi)g(JAX, \xi) = -2 + 2g(X, AN)^2 + 2g(X, A\xi)^2$ for any $X \in \mathcal{C}$. This yields $X = g(X, AN)AN + g(X, A\xi)A\xi$ for any unit $X \in \mathcal{C}$, which gives $\dim(\mathcal{C})=2$, and this is impossible because $m \geq 3$.

Thus M must be non-Hopf at $[z]$. We write $S\xi = \alpha\xi + \beta U$, with a unit vector $U \in \mathcal{C}$, and a non-zero real number β. Then (4.2) becomes

$$\beta g(\phi U, \bar{R}_N X)\xi - \beta g(\bar{R}_N \xi, X)\phi U - k\phi\bar{R}_N X$$
$$= \beta g(\phi U, X)\bar{R}_N \xi - \beta\eta(X)\bar{R}_N \phi U - k\bar{R}_N \phi X \qquad (4.4)$$

for any X tangent to M. Taking $X = \xi$ in (4.4), we obtain $\beta g(\phi U, \bar{R}_N \xi)\xi - \beta g(\bar{R}_N \xi, \xi)\phi U - k\phi\bar{R}_N \xi = -\beta\bar{R}_N \phi U$. Its scalar product with ξ, bearing in mind that $\beta \neq 0$, yields $g(\bar{R}_N \xi, \phi U) = 0 = 2\cos(2t)g(\phi U, A\xi)$. Then, if $\cos(2t) = 0$, $t = \frac{\pi}{4}$ and N must be $\mathfrak{A}$-isotropic.

From now on, we will suppose $\cos(2t) \neq 0$. Thus $g(\phi U, A\xi) = 0 = -g(U, JA\xi) = g(U, AN)$. That is,

$$g(\bar{R}_N \xi, \phi U) = g(\phi U, A\xi) = g(U, AN) = 0. \tag{4.5}$$

The scalar product of (4.4) and ϕU implies

$$-\beta g(\bar{R}_N \xi, X) - kg(\bar{R}_N X, U) = -\beta \eta(X) g(\bar{R}_N \phi U, \phi U) - kg(\bar{R}_N \phi X, \phi U) \tag{4.6}$$

for any X tangent to M. Taking $X = \phi U$ we obtain, bearing in mind (4.5), $g(\bar{R}_N U, \phi U) = 0$. This yields $\cos(2t) g(AU, \phi U) = 0$. As we suppose $\cos(2t) \neq 0$, we obtain

$$g(\bar{R}_N U, \phi U) = g(AU, \phi U) = 0. \tag{4.7}$$

Moreover, if we take $X \in \mathcal{C}_U = \{X \in \mathcal{C} | g(X, U) = g(X, \phi U) = 0\}$ in (4.6), we get

$$-\beta g(\bar{R}_N \xi, X) = kg((\phi \bar{R}_N - \bar{R}_N \phi)\phi U, X) \tag{4.8}$$

for any $X \in \mathcal{C}_U$.

The scalar product of (4.4) and U, bearing in mind (4.7), gives $g(\phi \bar{R}_N X, U) = g(\bar{R}_N \phi X, U)$, for any $X \in \mathcal{C}_U$. That is, $g((\phi \bar{R}_N - \bar{R}_N \phi)U, X) = 0$, for any $X \in \mathcal{C}_U$. Taking ϕX instead of X, we have $g(\phi \bar{R}_N U - \bar{R}_N \phi U, \phi X) = 0 = g(\bar{R}_N U, X) - g(\bar{R}_N \phi U, \phi X) = -g((\bar{R}_N \phi)\phi U, X) + g((\phi \bar{R}_N)\phi U, X)$. That is, $g((\phi \bar{R}_N - \bar{R}_N \phi)\phi U, X) = 0$, for any $X \in \mathcal{C}_U$. From (4.8) we obtain

$$g(\bar{R}_N \xi, X) = 0 \tag{4.9}$$

for any $X \in \mathcal{C}_U$. This yields

$$g(A\xi, X) = g(AN, X) = 0 \tag{4.10}$$

for any $X \in \mathcal{C}_U$.

Take the scalar product of (4.4) and $Y \in \mathcal{C}_U$. From (4.9) we obtain $g((\phi \bar{R}_N - \bar{R}_N \phi)Y, X) = 0$, for any $X \in \mathcal{C}, Y \in \mathcal{C}_U$. As $g((\phi \bar{R}_N - \bar{R}_N \phi)Y, \xi) = 0$ we conclude

$$(\phi \bar{R}_N - \bar{R}_N \phi)Y = 0 \tag{4.11}$$

for any $Y \in \mathcal{C}_U$. From (4.10) we have

$$\bar{R}_N Y = Y + \cos(2t)AY \quad \text{and} \quad \bar{R}_N \phi Y = \phi Y + \cos(2t)A\phi Y$$

for any $Y \in \mathcal{C}_U$. Therefore (4.11) yields $\cos(2t)\phi AY = \cos(2t)A\phi Y$, and, as $\cos(2t) \neq 0$, $\phi AY = A\phi Y$, for any $Y \in \mathcal{C}_U$. Thus $\phi AY = JAY = A\phi Y = AJY = -JAY$ yields $JAY = 0$ for any $Y \in \mathcal{C}_U$. This gives $AY = 0$, which is impossible and we have proved that N must be $\mathfrak{A}$-isotropic.

Then $g(AN, N) = g(A\xi, \xi) = 0$ and $\bar{R}_N \xi = 4\xi$. Taking $X = \xi$ in (4.4), we have $-4\beta\phi U = -\beta\bar{R}_N\phi U$. Its scalar product with ϕU gives $3 = -g(\phi U, AN)^2 - g(\phi U, A\xi)^2$, a contradiction, finishing the proof.

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JUAN DE DIOS PÉREZ
DEPARTAMENTO DE GEOMETRÍA
Y TOPOLOGÍA
UNIVERSIDAD DE GRANADA
18071 GRANADA
SPAIN
E-mail: jdperez@ugr.es

YOUNG JIN SUH
DEPARTMENT OF MATHEMATICS
AND RESEARCH INSTITUTE
OF REAL AND COMPLEX MANIFOLDS
COLLEGE OF NATURAL SCIENCES
KYUNGPOOK NATIONAL UNIVERSITY
DAEGU 41566
REPUBLIC OF KOREA
E-mail: yjsuh@knu.ac.kr