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Restricted summability of the multi-dimensional Cesàro means of Walsh–Kaczmarz–Fourier series

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Abstract. The properties of the maximal operator of the (C,α)-means ($\alpha = (\alpha_1,\dots,\alpha_d)$) of the multi-dimensional Walsh–Kaczmarz–Fourier series are discussed, where the set of indices is inside a cone-like set. We prove that the maximal operator is bounded from dyadic Hardy space H_p^γ to Lebesgue space L_p for $p_0 < p$ ($p_0 = \max\{1/(1+\alpha_k) : k = 1,\dots,d\}$) and is of weak type $(1,1)$. As a corollary, we get a theorem of Simon on the a.e. convergence of cone-restricted two-dimensional Fejér means of integrable functions. In the endpoint case $p = p_0$, we show that the maximal operator $\sigma_L^{\kappa,\alpha,*}$ is not bounded from the dyadic Hardy space $H_{p_0}^\gamma$ to the Lebesgue space L_{p_0}.

1. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis (for more details, see [1] and [16]). Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2$ has two elements 0 and 1, the group operation is the modulo 2 addition. The topology is given by that every subset is open. The Haar measure on $\mathbb{Z}_2$ is given by the assumption that the measure of a singleton is 1/2, that is, $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let G be the complete direct product of countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are sequences of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with coordinates

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$x_k \in \{0,1\}$ $(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) is the product measure, and the topology is the product topology. The compact Abelian group G is called the Walsh group. A base for the neighbourhoods of G can be given by

$$I_0(x) := G,$$
$$I_n(x) := I_n(x_0,\dots,x_{n-1}) := \{y \in G : y = (x_0,\dots,x_{n-1},y_n,y_{n+1},\dots)\},$$

$(x \in G, n \in \mathbb{N})$, where $I_n(x)$ are called dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G, and for simplicity, we write $I_n := I_n(0)$ $(n \in \mathbb{N})$. Set $e_n := (0,\dots,0,1,0,\dots) \in G$, the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$.

Let r_k denote the k-th Rademacher function, it is defined by

$$r_k(x) := (-1)^{x_k} \quad (k \in \mathbb{N}, x \in G).$$

The Walsh–Paley system is defined as the product system of Rademacher functions. Now, we give more details. If $n \in \mathbb{N}$, then n can be expressed in the form $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0,1\}$ $(i \in \mathbb{N})$. Define the order of a natural number n by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is $2^{|n|} \le n < 2^{|n|+1}$.

The Walsh–Paley functions are

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbb{P}).$$

The Walsh–Kaczmarz functions, defined by $\kappa_0 = 1$, are

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}$$

for $n \ge 1$. The set of Walsh–Kaczmarz functions and the set of Walsh–Paley functions are equal in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \le n < 2^{k+1}\} = \{w_n : 2^k \le n < 2^{k+1}\}$$

for all $k \in \mathbb{P}$. Moreover, $\kappa_0 = w_0$.

V. A. Skvortsov (see [21]) gave a relation between the Walsh–Kaczmarz functions and the Walsh–Paley functions. Namely, Skvortsov defined a transformation $\tau_A : G \to G$

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_1, x_0, x_A, x_{A+1}, \dots)$$

for $A \in \mathbb{N}$. The transformation τ_A satisfies $\tau_A(\tau_A(x)) = x$ for all $x \in G$, and it is a measure-preserving transformation [21]. By the definition of τ_A, we have the following connection:

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G).$$

Let us set $0 < \alpha$, and let

$$A_j^\alpha := \binom{j+\alpha}{j} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+j)}{j!} \quad (j \in \mathbb{N}; \alpha \neq -1, -2, \ldots).$$

It is known that $A_j^\alpha \sim O(j^\alpha)$ $(j \in \mathbb{N})$ (see ZYGMUND [32]). The one-dimensional Dirichlet kernels and Cesàro kernels are defined by

$$D_n^\psi := \sum_{k=0}^{n-1} \psi_k, \qquad K_n^{\psi,\alpha}(x) := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k^\psi(x),$$

for $\psi_n = w_n$ $(n \in \mathbb{P})$ or $\psi_n = \kappa_n$ $(n \in \mathbb{P})$, $D_0^\psi := 0$.

Choosing $\alpha = 1$ we defined the n-th Fejér mean, as a special case. For Walsh–Paley–Fejér kernel functions we have $K_n(x) \to 0$, while $n \to \infty$ for every $x \neq 0$. However, they can take negative values, which is a different situation from the trigonometric case. On the other hand, for Walsh–Kaczmarz–Fejér kernel functions we have $K_n(x) \to \infty$, while $n \to \infty$ at every dyadic rational (and $K_n(x)$ can take negative values, as well). The last fact shows that the behavior of the Walsh–Kaczmarz system is worse than the behavior of the Walsh–Paley system in this special sense. See later inequality (2), as well.

It is well-known that the 2^n-th Dirichlet kernels have a closed form (see, e.g., [16]):

$$D_{2^n}^w(x) = D_{2^n}^\kappa(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n, \\ 2^n, & \text{if } x \in I_n. \end{cases} \tag{1}$$

The Kronecker product $\left(\psi_n : n \in \mathbb{N}^d\right)$ of d Walsh–(Kaczmarz) systems is said to be the d-dimensional (multi-dimensional) Walsh–(Kaczmarz) system. That is,

$$\psi_n(x) = \psi_{n_1}\left(x^1\right) \cdots \psi_{n_d}\left(x^d\right),$$

where $n = (n_1, \ldots, n_d)$ and $x = (x^1, \ldots, x^d)$.

If $f \in L^1\left(G^d\right)$, then the number $\widehat{f}^\psi(n) := \int_{G^d} f \psi_n$ $\left(n \in \mathbb{N}^d\right)$ is said to be the n-th Walsh–(Kaczmarz)–Fourier coefficient of f. We can extend this definition to martingales in the usual way (see WEISZ [25], [26]).

The d-dimensional (C,α) $(\alpha=(\alpha_1,\dots,\alpha_d))$ or Cesàro mean of a martingale is defined by

$$\sigma_n^{\psi,\alpha}f(x):=\frac{1}{\prod_{i=1}^d A_{n_i}^{\alpha_i}}\sum_{j=1}^{d}\sum_{k_j=0}^{n_j}\prod_{i=1}^{d}A_{n_i-k_i}^{\alpha_i-1}S_k^{\psi}(f;x).$$

It is known that

$$K_n^{\psi,\alpha}(x)=K_{n_1}^{\psi,\alpha_1}(x^1)\cdots K_{n_d}^{\psi,\alpha_d}(x^d).$$

In 1948, Šneider [22] introduced the Walsh–Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty}\frac{D_n^{\kappa}(x)}{\log n}\geq C>0 \tag{2}$$

holds a.e. This inequality shows a big difference between the two arrangements of the Walsh system.

In 1974, Schipp [17] and Young [31] proved that the Walsh–Kaczmarz system is a convergence system. Skvortsov in 1981 showed that the Fejér means with respect to the Walsh–Kaczmarz system converge uniformly to f for any continuous functions f, see [21]. Gát [5] proved for any integrable functions that the Fejér means with respect to the Walsh–Kaczmarz system converge almost everywhere to the function. Moreover, he showed that the maximal operator $\sigma^{\kappa,*}$ of Walsh–Kaczmarz–Fejér means is of weak type $(1,1)$ and of type (p,p) for all $1<p\leq\infty$. Gát's result was generalized by Simon [19], who showed that the maximal operator $\sigma^{\kappa,*}$ is of type (H_p,L_p) for $p>1/2$. In the endpoint case $p=1/2$, Goginava [10] proved that the maximal operator $\sigma^{\kappa,*}$ is not of type $(H_{1/2},L_{1/2})$, and Weisz [28] showed that the maximal operator is of weak type $(H_{1/2},L_{1/2})$. Recently, the rate of the deviant behaviour in the endpoint case $p=1/2$ was discussed by Goginava and the first author [12]. The case $0<p<1/2$ can be found in the papers of Tephnadze [23], [24].

In 2004, the boundedness of the maximal operator of the Cesàro means $(0<\alpha\leq 1)$ was investigated by Simon [20]. It was showed that the maximal operator is bounded from the Hardy space H_p to the space L_p for $p>p_0:=1/(1+\alpha)$ and of weak type (1,1). Moreover, he showed the inequality

$$\sup_n\|K_n^{\kappa,\alpha}\|_1<\infty\quad\text{ for all } 0<\alpha\leq 1. \tag{3}$$

In the endpoint case, Gát and Goginava showed that the endpoint $p_0:=1/(1+\alpha)$ is essential. That is, they proved that the maximal operator of Cesàro means is not bounded from the Hardy space H_{p_0} to the space L_{p_0} [8]. Other properties of the (C,α)-means in the endpoint case p_0 are discussed in [3], and in the case $p < p_0$ are investigated in [4].

For $x = (x^1, x^2, \dots, x^d) \in G^d$ and $n = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$, the d-dimensional rectangles are defined by $I_n(x) := I_{n_1}(x^1) \times \cdots \times I_{n_d}(x^d)$. For $n \in \mathbb{N}^d$, the σ-algebra generated by the rectangles $\{I_n(x), x \in G^d\}$ is denoted by $\mathcal{F}_n$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n.

Suppose that for all $j = 2, \dots, d$ the functions $\gamma_j \colon [1,+\infty) \to [1,+\infty)$ are strictly monotone increasing continuous functions with properties $\lim_{+\infty} \gamma_j = +\infty$ and $\gamma_j(1) = 1$. Moreover, suppose that there exist $\zeta, c_{j,1}, c_{j,2} > 1$ such that the inequality

$$c_{j,1}\gamma_j(x) \le \gamma_j(\zeta x) \le c_{j,2}\gamma_j(x) \tag{4}$$

holds for each $x \ge 1$. In this case, the functions γ_j are called CRF (cone-like restriction functions). Let $\gamma := (\gamma_2, \dots, \gamma_d)$ and $\beta_j \ge 1$ be fixed $(j = 2, \dots, d)$. In this paper we investigate the maximal operator of the multi-dimensional (C,α) means and the convergence over a cone-like set L (with respect to the first dimension), where

$$L := \{n \in \mathbb{N}^d : \beta_j^{-1}\gamma_j(n_1) \le n_j \le \beta_j\gamma_j(n_1), j = 2, \dots, d\}.$$

If each γ_j is the identical function, then we get a cone. The cone-like sets were introduced by Gát in dimension two [6]. The condition (4) on the function γ is natural, because Gát [6] proved that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if the inequality (4) holds.

Weisz defined a new type of martingale Hardy space depending on the function γ (see [27]). For a given $n_1 \in \mathbb{N}$, set $n_j := |\gamma_j(2^{n_1})|$ $(j = 2, \dots, d)$, that is, n_j is the order of $\gamma_j(2^{n_1})$ (this means that $2^{n_j} \le \gamma_j(2^{n_1}) < 2^{n_j+1}$ for $j = 2, \dots, d$). Let $\overline{n}_1 := (n_1, \dots, n_d)$. Since, the functions γ_j are increasing, the sequence $(\overline{n}_1,\ n_1 \in \mathbb{N})$ is increasing, too. Then there exists a class of one-parameter martingales $f = (f_{\overline{n}_1},\ n_1 \in \mathbb{N})$ with respect to the σ-algebras $(\mathcal{F}_{\overline{n_1}},\ n_1 \in \mathbb{N})$. The maximal function of a martingale f is defined by $f^* = \sup_{n_1 \in \mathbb{N}} |f_{\overline{n}_1}|$. For $0 < p \le \infty$, the dyadic martingale Hardy space $H_p^\gamma(G^d)$ consists of all martingales for which $\|f\|_{H_p^\gamma} := \|f^*\|_p < \infty$. It is known (see [26]) that $H_p^\gamma \sim L_p$ for $1 < p \le \infty$, where $\sim$ denotes the equivalence of norms and spaces.

If $f \in L_1(G^d)$, then it is easily shown that the sequence $(S_{2^{n_1},\dots,2^{n_d}}(f) : \overline{n_1} = (n_1, \dots, n_d), n_1 \in \mathbb{N})$ is a one-parameter martingale with respect to the σ-algebras

$(\mathcal{F}_{\overline{n_1}},\ n_1 \in \mathbb{N})$. In this case, the maximal function can also be given by

$$f^*(x) = \sup_{n_1\in\mathbb{N}} \frac{1}{\operatorname{mes}(I_{\overline{n_1}}(x))} \left| \int_{I_{\overline{n_1}}(x)} f(u)d\mu(u) \right| = \sup_{n_1\in\mathbb{N}} |S_{2^{n_1},\dots,2^{n_d}}(f,x)|$$

for $x \in G^d$.

We define the maximal operator $\sigma_L^{\kappa,\alpha,*}$ by

$$\sigma_L^{\kappa,\alpha,*} f(x) := \sup_{n\in L} |\sigma_n^{\kappa,\alpha} f(x)|.$$

For double Walsh–Paley–Fourier series, Móricz, Schipp and Wade [14] proved that $\sigma_n f$ converge to f a.e. in the Pringsheim sense (that is, no restriction on the indices other than $\min(n_1, n_2) \to \infty$) for all functions $f \in L\log^+ L$. The a.e. convergence of Fejér means $\sigma_n f$ of integrable functions, where the set of indices is inside a positive cone around the identical function, that is $\beta^{-1} \le n_1/n_2 \le \beta$ is provided with some fixed parameter $\beta \ge 1$, was proved by Gát [7] and Weisz [29]. A common generalization of results of Móricz, Schipp, Wade [14] and Gát [7], Weisz [29] for cone-like set was given by the first author and Gát in [9]. Namely, a necessary and sufficient condition for cone-like sets in order to preserve the convergence property, was given. The trigonometric case was treated by Gát [6].

Relating to the original paper [6] on trigonometric systems, Gát asked the following: What could we state for other systems, for example, Walsh–Paley, Walsh–Kaczmarz and Vilenkin systems, and for other means, for example, logarithmic means, Riesz means, (C,α) means? Some parts of Gát's question was answered by Weisz [27], Blahota and the first author (see [2], [15]), and by Gát [9].

In 2011, the properties of the maximal operator of the (C,α) and Riesz means of a multi-dimensional Vilenkin–Fourier series, provided that the supremum in the maximal operator is taken over a cone-like set, were discussed by Weisz [27]. Namely, it was proved that the maximal operator is bounded from dyadic Hardy space H_p to the space L_p for $p_0 < p \le \infty$ ($p_0 := \max\{1/(1+\alpha_k) : k = 1,\dots,d\}$) and is of weak type $(1,1)$. Recently, it was shown that the index p_0 is sharp. Namely, it was proved that the maximal operator is not bounded from the dyadic Hardy space H_{p_0} to the space L_{p_0} [2]. A detailed list of the known results for one- and several dimensional Walsh-like systems can be found in [30].

For the two-dimensional Walsh–Kaczmarz–Fourier series Simon proved [18] that the cone-restricted maximal operator of the Fejér means is bounded from the Hardy space H_p to the space L_p for all $1/2 < p$ (here the set of indices is inside a positive cone around the identical function). That is, the a.e. convergence of cone-restricted Fejér means holds for the Walsh–Kaczmarz system, as well. Moreover, it was proved that $p = 1/2$ is essential. In 2007, Goginava and the first author proved that the cone-restricted maximal operator is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$ [13]. The cone-like restricted two-dimensional maximal operator of Fejér means was discussed in [15].

Motivated by the works of Weisz [27], Simon [18] and the above-mentioned question of Gát, we prove that the maximal operator $\sigma_L^{\kappa,\alpha,*}$ is bounded from the dyadic Hardy space H_p^γ to the Lebesgue space L_p for $p_0 < p$ ($p_0 := \max\{1/(1+\alpha_k) : k = 1, \dots, d\}$) and is of weak type $(1,1)$. As a corollary, we get the theorem of Simon [18] on the a.e. convergence of cone-restricted Fejér means. In the endpoint case $p = p_0$, we show that the maximal operator $\sigma_L^{\kappa,\alpha,*}$ is not bounded from the Hardy space $H_{p_0}^\gamma$ to the space L_{p_0}.

In dimension 2, the case $\alpha_1 = \alpha_2 = 1$ was discussed in [15]. Unfortunately, the counterexample martingale presented in [15] and the method are not suitable for case $0 < \alpha_i < 1$ ($i = 1, \dots, d$).

2. Auxiliary propositions and main results

First, we formulate our main theorems.

Theorem 1. *Let γ be CRF. The maximal operator $\sigma_L^{\kappa,\alpha,*}$ is bounded from the dyadic Hardy space H_p^γ to the space L_p for $p_0 < p \le 1$ ($p_0 := \max\{1/(1+\alpha_i); i = 1, \dots, d\}$).*

By standard argument we have that if $1 < p \le \infty$, then $\sigma_L^{\kappa,\alpha,*}$ is of type (p,p) and of weak type $(1,1)$.

Theorem 2. *Let γ be CRF. Then for any $f \in L^1$,*

$$\lim_{\substack{\wedge n \to \infty \\ n \in L}} \sigma_n^{\kappa,\alpha} f = f$$

holds almost everywhere.

We immediately have the theorem of Simon [18] as a corollary.

Corollary 1 (Simon [18])**.** *Let $f \in L^1$ and $\beta \ge 1$ be a fixed parameter. Then*

$$\lim_{\substack{\wedge n \to \infty \\ \beta^{-1} \le n_1/n_2 \le \beta}} \sigma_n^\kappa f = f$$

holds a.e.

Theorem 3. *Let γ be CRF and $\alpha_1 \le \cdots \le \alpha_d$. The maximal operator $\sigma_L^{\kappa,\alpha,*}$ is not bounded from the Hardy space $H_{p_0}^\gamma$ to the space L_{p_0}.*

To prove our Theorems 1, 2 and 3, we need the following Lemma of Weisz [26], the concept of the atoms (for more details, see [27]) and a Lemma of Goginava [11].

A bounded measurable function a is a p-atom, if there exists a dyadic d-dimensional rectangle $I \in \mathcal{F}_{\overline{n}_1}$, such that

(a) $\operatorname{supp} a \subseteq I$,

(b) $\|a\|_\infty \le \mu(I)^{-1/p}$,

(c) $\int_I a d\mu = 0$.

Lemma 1 (Weisz [26]). *Suppose that the operator T is σ-sublinear and p-quasilocal for any $0 < p < 1$. If T is bounded from L_∞ to L_∞, then*

$$\|Tf\|_p \le c_p \|f\|_{H_p} \quad \text{for all } f \in H_p.$$

Lemma 2 (Goginava [11]). *Let $n \in \mathbb{N}$ and $0 < \alpha \le 1$. Then*

$$\int_G \max_{1 \le N < 2^n} (A_{N-1}^\alpha |K_N^\alpha(x)|)^{1/(1+\alpha)} d\mu(x) \ge c(\alpha) \frac{n}{\log(n+2)}.$$

3. Proofs of the theorems

Now, we prove our main Theorems.

Proof of Theorem 1. Using Lemma 1 of Weisz, we have to prove that the operator $\sigma_L^{\kappa,\alpha,*}$ is bounded from the space L_∞ to the space L_∞. It immediately follows from inequality (3).

Let a be a p-atom, with support I. We can assume that $I = I_{N_1} \times \cdots \times I_{N_d}$ (with $2^{N_j} \le \gamma_j(2^{N_1}) < 2^{N_j+1}$, $j = 2, \dots, d$), $\|a\|_\infty \le 2^{(N_1+\dots+N_d)/p}$ and $\int_I a d\mu = 0$.

In the next steps, we use the next inequality and the monotonicity of CRF functions γ_j ($j = 2, \dots, d$). Then

$$c_{j,1}^l \gamma_j\left(\frac{2^{N_1}}{\zeta^l}\right) \le \gamma_j(2^{N_1}) = \gamma_j\left(\frac{2^{N_1}}{\zeta^l}\zeta^l\right) \le c_{j,2}^l \gamma_j\left(\frac{2^{N_1}}{\zeta^l}\right)$$

holds for all $l \in \mathbb{P}$ ($j = 2, \dots, d$).

Set $\delta := \max\{\zeta^{\log_{c_{j,1}} 2\beta_j+1} : j = 2, \dots, d\}$. If $n_1 \le 2^{N_1}/\delta$, then

$$n_j \le \beta_j \gamma_j(n_1) \le \beta_j \gamma_j(2^{N_1} \zeta^{-\log_{c_{j,1}} 2\beta_j - 1}) \le \beta_j \frac{1}{c_{j,1}^{\log_{c_{j,1}} 2\beta_j+1}} \gamma_j(2^{N_1}) \le \frac{\gamma_j(2^{N_1})}{2} \le 2^{N_j}.$$

$\zeta, c_{j,1}, c_{j,2} > 1$, $\beta_j \geq 1$ imply $n_1 < 2^{N_1}$ and $n_j \leq \gamma_j(2^{N_1})/2 < 2^{N_j}$ ($j = 2, \ldots, d$). In this case, the $(m_1, \ldots, m_d)$-th Fourier coefficients are zeros for $m_1 \leq n_1, \ldots, m_d \leq n_d$. This gives $\sigma_n a = 0$ ($n = (n_1, \ldots, n_d)$).

That is, we could suppose that $n_1 > 2^{N_1}/\delta$. This yields that

$$n_j \geq \frac{\gamma_j(n_1)}{\beta_j} \geq \frac{\gamma_j(2^{N_1}/\delta)}{\beta_j} \geq \frac{1}{\beta_j c_{j,2}^{\max\{\log_{c_{j,1}} 2\beta_j+2 : j=2,\ldots,d\}}} \gamma_j(2^{N_1}) \geq \frac{\gamma_j(2^{N_1})}{\delta'_j} \geq \frac{2^{N_j}}{\delta'}$$

with the notion $\delta'_j := c_{j,2}^{\max\{\log_{c_{j,1}} 2\beta_j+2 : j=2,\ldots,d\}}$ and $\delta' := \max_{j=2,\ldots,d} \delta'_j$ for all $j = 2, \ldots, d$. $\delta' > 1$ can be assumed.

The proof will be complete if we show that the maximal operator $\sigma_L^{\kappa,\alpha,*}$ is p-quasilocal for $p_0 < p \leq 1$. That is, there exists a constant c_p such that the inequality

$$\int_{\overline{I}} |\sigma_L^{\kappa,\alpha,*} a|^p d\mu \leq c_p < \infty$$

holds for all atom a in H_p^γ with support $I = I_{N_1} \times \cdots \times I_{N_d}$ (with $\overline{N_1} = (N_1, \ldots, N_d)$). It is well known that the concept of p-quasi-locality of the maximal operator $\sigma^{\kappa,\alpha,*}$ can be modified as follows [18]: there exists $r = 0, 1, \ldots$, such that

$$\int_{\overline{I^r}} |\sigma_L^{\kappa,\alpha,*} a|^p d\mu \leq c_p < \infty, \tag{5}$$

where $I^r := I^r_{N_1} \times \cdots \times I^r_{N_d} := I_{N_1-r} \times \cdots \times I_{N_d-r}$ ($N_j - r \geq 0$ for all $j = 1, \ldots, d$). We will give the value of r later.

Let us set $x = (x^1, \ldots, x^d) \in \overline{I^r}$.

$$\begin{aligned} |\sigma_n^{\kappa,\alpha} a(x)| &= \left| \int_I a(t^1, \ldots, t^d) K_{n_1}^{\kappa,\alpha_1}(x^1 + t^1) \cdots K_{n_d}^{\kappa,\alpha_d}(x^d + t^d) d\mu(t) \right| \\ &\leq 2^{(N_1+\cdots+N_d)/p} \int_{I_{N_1}} |K_{n_1}^{\kappa,\alpha_1}(x^1 + t^1)| d\mu(t^1) \cdots \int_{I_{N_d}} |K_{n_d}^{\kappa,\alpha_d}(x^d + t^d)| d\mu(t^d) \end{aligned}$$

Now, we decompose the set $\overline{I^r} = \overline{I^r_{N_1} \times \cdots \times I^r_{N_d}}$ as the following disjoint union:

$$\begin{aligned} \overline{I^r} = &(\overline{I^r_{N_1}} \times \cdots \times \overline{I^r_{N_d}}) \\ &\cup (I^r_{N_1} \times \overline{I^r_{N_2}} \times \cdots \times \overline{I^r_{N_d}}) \cup \cdots \cup (\overline{I^r_{N_1}} \times \cdots \times \overline{I^r_{N_{d-1}}} \times I^r_{N_d}) \\ &\vdots \\ &\cup (\overline{I^r_{N_1}} \times I^r_{N_2} \times \cdots \times I^r_{N_d}) \cup \cdots \cup (I^r_{N_1} \times \cdots \times I^r_{N_{d-1}} \times \overline{I^r_{N_d}}). \end{aligned}$$

Let us set $\delta'' := \max\{\delta, \delta'\}$, set $r \in \mathbb{P}$ such that $2^{-r} \leq 1/\delta'' \leq 2^{-r+1}$, and $L^{r,l} := I^r_{N_1} \times \cdots \times I^r_{N_l} \times \overline{I^r_{N_{l+1}}} \times \cdots \times \overline{I^r_{N_d}}$ for $l = 0, \ldots, d$. We define

$$J_i := \int_{I^r_{N_i}} \left(\sup_{n_i \geq 2^{N_i}/\delta''} \int_{I_{N_i}} |K^{\kappa,\alpha_i}_{n_i}(x^i + t^i)| d\mu(t^i) \right)^p \mu(x^i), \quad i = 1, \ldots, l,$$

$$\overline{J_j} := \int_{\overline{I^r_{N_j}}} \left(\sup_{n_j \geq 2^{N_j}/\delta''} \int_{I_{N_j}} |K^{\kappa,\alpha_j}_{n_j}(x^j + t^j)| d\mu(t^j) \right)^p \mu(x^j), \quad j = l+1, \ldots, d.$$

Now, we get

$$\int_{L^{r,l}} |\sigma^{\kappa,\alpha,*}_L a|^p d\mu \leq 2^{N_1 + \cdots + N_d} J_1 \times \cdots \times J_l \times \overline{J_{l+1}} \times \cdots \times \overline{J_d}.$$

First, we discuss the integrals J_i $(i = 1, \ldots, l)$. Inequality (3) and the definitions of δ'', r immediately yield that

$$J_i \leq 2^{-(N_i - r)} \left(\sup_{n_1 \in \mathbb{N}} \|K^{\kappa,\alpha_i}_{n_i}\|_1 \right)^p \leq c_p 2^{-N_i}. \tag{6}$$

Second, we discuss the integrals $\overline{J_j}$ $(j = l+1, \ldots, d)$. In the paper [20, pages 48–59], Simon showed that

$$\int_{\overline{I_N}} \left(\sup_{n \geq 2^N} \int_{I_N} |K^{\kappa,\alpha}_n(x + t)| d\mu(t) \right)^p \mu(x) \leq c_p 2^{-N}, \quad \text{if } p > \frac{1}{1+\alpha} \ (0 < \alpha \leq 1). \tag{7}$$

Using this, we write

$$\overline{J_j} \leq \int_{\overline{I^r_{N_j}}} \left(\sup_{n_j \geq 2^{N_j - r}} \int_{I^r_{N_j}} |K^{\kappa,\alpha_j}_{n_j}(x^j + t^j)| d\mu(t^j) \right)^p \mu(x^j) \leq c_p 2^{-N_j}, \quad \text{if } p > \frac{1}{1+\alpha_j}. \tag{8}$$

Inequalities (6), (8) yield

$$\int_{L^{r,l}} |\sigma^{\kappa,\alpha,*}_L a|^p d\mu \leq c_p \quad \text{for all } l \text{ if } p > p_0.$$

The decomposition of $\overline{I^r}$ gives

$$\int_{\overline{I^r}} |\sigma^{\kappa,\alpha,*}_L a|^p d\mu \leq c_{p,d}, \quad \text{if } p > p_0.$$

This completes the proof of Theorem 1. □

The weak type $(1,1)$ inequality follows by interpolation. The set of Walsh–Kaczmarz polynomials is dense in L_1. The weak-type (1,1) inequality and the usual density argument imply Theorem 2.

Proof of Theorem 3. In the present proof we use a counterexample martingale. Let us set

$$f_{\overline{n_1}}(x) := (D_{2^{n_1+1}}(x^1) - D_{2^{n_1}}(x^1)) \prod_{j=2}^{d} w_{2^{n_j-1}-1}(x^j),$$

where $n_2, \ldots, n_d$ is defined to n_1 earlier.

Now, we calculate $S_j^{\kappa}(f_{\overline{n_1}}; x)$. Since, $\omega_{2^{n-1}-1}(x) = r_{n-2}(x)(-1)^{\sum_{i=0}^{n-3} x_i} = \kappa_{2^{n-1}-1}(x)$, we have

$$\hat{f}^{\kappa}_{\overline{n_1}}(k) = \begin{cases} 1, & \text{if } k_1 = 2^{n_1}, \ldots, 2^{n_1+1}-1, \text{ and } k_j = 2^{n_j-1}-1 \text{ for all } j = 2, \ldots, d; \\ 0, & \text{otherwise.} \end{cases}$$

$$S_j^{\kappa}(f_{\overline{n_1}}, x) = \sum_{\nu=0}^{j_1-1} \hat{f}^{\kappa}_{\overline{n_1}}(\nu, 2^{n_2-1}-1, \ldots, 2^{n_d-1}-1)\kappa_\nu(x^1) \prod_{l=2}^{d} \kappa_{2^{n_l-1}-1}(x^l)$$
$$= \begin{cases} (D^{\kappa}_{j_1}(x^1) - D_{2^{n_1}}(x^1)) \prod_{l=2}^{d} w_{2^{n_l-1}-1}(x^l) & \text{if } j_1 = 2^{n_1}+1, \ldots, 2^{n_1+1}-1, \\ & \text{and } j_l \geq 2^{n_l-1} \text{ for all } l = 2, \ldots, d; \\ f_{\overline{n_1}}(x) & \text{if } j_1 \geq 2^{n_1+1} \text{ and } j_l \geq 2^{n_l-1} \\ & \text{for all } l = 2, \ldots, d; \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

We immediately have that

$$f^*_{\overline{n_1}}(x) = \sup_{m_1 \in \mathbb{N}} |S_{2^{m_1}, \ldots, 2^{m_d}}(f_{\overline{n_1}}, x)| = |f_{\overline{n_1}}(x)|,$$

where $\overline{m_1} = (m_1, \ldots, m_d)$.

$$\|f_{\overline{n_1}}\|_{H^{\gamma}_{p_0}} = \|f^*_{\overline{n_1}}\|_{p_0} = \|D_{2^{n_1}}\|_{p_0} = 2^{(1-1/p_0)n_1} < \infty. \tag{10}$$

That is $f_{\overline{n_1}} \in H^{\gamma}_{p_0}$.

We can write the n-th Dirichlet kernel with respect to the Walsh–Kaczmarz system in the following form:

$$D^{\kappa}_n(x) = D_{2^{|n|}}(x) + \sum_{k=2^{|n|}}^{n-1} r_{|k|}(x) w_{k-2^{|n|}}(\tau_{|k|}(x))$$
$$= D_{2^{|n|}}(x) + r_{|n|}(x) D^{w}_{n-2^{|n|}}(\tau_{|n|}(x)) \tag{11}$$

We set $L_1^N := 2^{n_1} + N$, where $0 < N < 2^{n_1}$ and $L_j^N := [\gamma_j(2^{n_1} + N)]$ for $j = 2, \dots, d$ (where $[x]$ denotes the integer part of x). In this case, $L^N := (L_1^N, \dots, L_d^N) \in L$. Let us calculate $\sigma_{L^N}^{\kappa,\alpha} f_{\overline{n_1}}$.

By inequalities (11) and (9), we get

$$
\begin{aligned}
&|\sigma_{L^N}^{\kappa,\alpha} f_{\overline{n_1}}(x)| \\
&= \frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}} \left| \sum_{j=1}^d \sum_{k_j=0}^{L_j^N} \prod_{i=1}^d A_{L_i^N - k_i}^{\alpha_i - 1} S_k^\kappa(f_{\overline{n_1}}; x) \right| \\
&= \frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}} \left| \sum_{j=2}^d \sum_{k_j=2^{n_j-1}}^{L_j^N} \sum_{k_1=2^{n_1}+1}^{L_1^N} \prod_{i=1}^d A_{L_i^N - k_i}^{\alpha_i - 1} S_k^\kappa(f_{\overline{n_1}}; x) \right| \\
&= \frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}} \left| \sum_{j=2}^d \sum_{k_j=2^{n_j-1}}^{L_j^N} \sum_{k_1=2^{n_1}+1}^{L_1^N} \prod_{i=1}^d A_{L_i^N - k_i}^{\alpha_i - 1} \right. \\
&\qquad \left. \times \prod_{l=2}^d w_{2^{n_l-1}-1}(x^l)(D_{k_1}^\kappa(x^1) - D_{2^{n_1}}(x^1)) \right| \\
&= \frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}} \left| \sum_{j=2}^d \sum_{k_j=0}^{L_j^N - 2^{n_j-1}} \prod_{i=2}^d A_{L_i^N - 2^{n_i-1} - k_i}^{\alpha_i - 1} \sum_{k_1=1}^{L_1^N - 2^{n_1}} A_{L_1^N - 2^{n_1} - k_1}^{\alpha_1 - 1} D_{k_1}^w(\tau_{n_1}(x^1)) \right| \\
&= \frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}} \left| \sum_{j=2}^d \sum_{k_j=0}^{L_j^N - 2^{n_j-1}} \prod_{i=2}^d A_{L_i^N - 2^{n_i-1} - k_i}^{\alpha_i - 1} \right| \left| A_{L_1^N - 2^{n_1}}^{\alpha_1} K_{L_1^N - 2^{n_1}}^{w,\alpha_1}(\tau_{n_1}(x^1)) \right| \\
&\geq \frac{c(\alpha)}{2^{n_1 \alpha_1}} A_N^{\alpha_1} |K_N^{w,\alpha_1}(\tau_{n_1}(x^1))|.
\end{aligned}
$$

Now, we write that

$$
\sigma_L^{\kappa,\alpha,*} f_{\overline{n_1}}(x) \geq \max_{1 \leq N < 2^{n_1}} |\sigma_{L^N}^{\kappa,\alpha} f_{\overline{n_1}}(x)| \geq \frac{c(\alpha)}{2^{n_1 \alpha_1}} \max_{1 \leq N < 2^{n_1}} A_N^{\alpha_1} |K_N^{w,\alpha_1}(\tau_{n_1}(x^1))|.
$$

Inequality (10), properties of the transformation τ_{n_1} and Lemma 2 yield

$$
\begin{aligned}
\frac{\|\sigma_L^{\kappa,\alpha,*} f_{\overline{n_1}}\|_{p_0}}{\|f_{\overline{n_1}}\|_{H_{p_0}^\gamma}} &\geq \frac{1}{2^{(1-1/p_0)n_1}} \left(\int_{G^d} \max_{1 \leq N < 2^{n_1}} |\sigma_{L^N}^{\kappa,\alpha} f_{\overline{n_1}}(x)|^{p_0} d\mu(x) \right)^{1/p_0} \\
&\geq \frac{c(\alpha) 2^{n_1 \alpha_1}}{2^{n_1 \alpha_1}} \left(\int_G \max_{1 \leq N < 2^{n_1}} \left(A_N^{\alpha_1} |K_N^{w,\alpha_1}(\tau_{n_1}(x^1))| \right)^{p_0} d\mu(x^1) \right)^{1/p_0}
\end{aligned}
$$

$$= \frac{c(\alpha)2^{n_1\alpha_1}}{2^{n_1\alpha_1}} \left(\int_G \max_{1 \le N < 2^{n_1}} \left(A_N^{\alpha_1} |K_N^{w,\alpha_1}(x^1)| \right)^{1/(1+\alpha_1)} d\mu(x^1) \right)^{1+\alpha_1}$$

$$\ge c(\alpha) \left(\frac{n_1}{\log(n_1+2)} \right)^{1+\alpha_1} \to \infty, \quad \text{while } n_1 \to \infty.$$

This completes the proof of Theorem 3. □

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