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Constructing solvable groups with derived length four and four character degrees

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Abstract. In this paper, we present a new method to construct solvable groups with derived length four and four character degrees. We then use this method to present a number of new families of groups with derived length four and four character degrees.

1. Introduction

Throughout this paper, all groups will be finite and solvable. The Taketa problem or alternatively, the Isaacs–Seitz conjecture, conjectures when G is solvable that the derived length of G is less than or equal to the number of character degrees. In general, this conjecture is still open, but it was settled when G has four character degrees by Garrison in his dissertation [2]. In this paper, we are interested in solvable groups with exactly four character degrees and derived length four. We will use $\mathrm{dl}(G)$ to denote the derived length of G and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ for the set of character degrees of G.

In Section 2 of [1], we listed all of the solvable groups with four character degrees and derived length four that we knew at that time. So far as we know, no other examples have been published since that time. All of the examples in Section 2 of [1] have Fitting height 3. Riedl proved in Theorem A of [14] that if G is a solvable group with four character degrees, then G has Fitting height at most 3. It makes sense to ask whether or not there exist solvable groups with four character degrees and derived length four whose Fitting height is 1 or 2. We now resolve the question regarding examples with Fitting height 2.

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Theorem 1.1. *There exists a solvable group G with* $\mathrm{dl}(G) = 4$, $|\mathrm{cd}(G)| = 4$, *and Fitting height* 2.

The question related to this problem that attracts the most attention is whether or not there exists an example of Fitting height 1. It is easy to see that this is equivalent to the question: does there exist a p-group with four character degrees and derived length four? A closely related question is whether or not there exists a nonnilpotent group with four character degrees and derived length where all of the degrees are powers of a given prime p. It is not difficult to see that such a group would have to be a semi-direct product of a p-group with three character degrees and derived length three acting on an abelian p'-group where the orbit sizes are highly restricted. Unfortunately, we are not able to touch on either of these problems in this current work. At this time, we know of no published research that touches on the existence of nonnilpotent groups with derived length four and four character degrees that are powers of a prime p, and the published research regarding the existence p-groups with derived length four and four character degrees is minimal. On the other hand, researchers are familiar with the problem for p-groups and there seems to be universal agreement that these appear to be difficult problems. We would love to see progress on these problems.

We also mention that none of the examples in the list in [1] have a nonabelian normal Sylow p-subgroup for some prime p. And every example in that list has at least one character degree that is a prime. The method of constructing these groups in this paper has a nonabelian normal Sylow p-subgroup for some prime p. We will produce examples where this normal Sylow subgroup is an extra-special group and examples where this normal Sylow subgroup is a Heisenberg group (i.e., a Sylow p-subgroup of $\mathrm{GL}_3(p^a)$ for some positive integer a). We will also provide examples where no character degree is a prime (see Theorem 3.6).

One might ask: given that a number of examples of such groups already exist in the literature, why publish more such groups? As we will note throughout the paper, the examples we present here have somewhat different properties than the ones in the literature. While this has value, we do not believe that this is the main reason to produce these groups. The main reason is as follows. In [1], we suggested that it might be possible to classify the solvable groups with four character degrees and derived length four. In particular, we hoped that we might be able to list the possible degree sets that could arise in this situation, but the construction in this paper shows that such an approach is probably not feasible. In fact, we believe that this provides an argument for the counter position that classifying these groups is either not possible or not worth the effort needed to

produce it. In any case, if one were to attempt to obtain a classification of these groups, then one "family" in the classification would be groups that satisfy the hypotheses of Lemma 2.1.

We will use our construction to produce several different "families" of solvable groups with derived length four and four character degrees with the idea of showing the range of properties and character degree sets that could arise from this construction. We have put the example with Fitting height 2 in Section 4 so that readers that are not interested in any other examples can skip directly to them. Since Section 2 of [1] had seven families, one could view that we have doubled the number of families of examples. On the other hand, we have in no way exhausted the groups that can be produced from our construction. In particular, all of the families we produce have that the normal Sylow p-subgroup is either an extraspecial group or is a Heisenberg group. The normal Sylow p-subgroup found in the construction is more general, and we would expect that there will be many examples that do not have P as one of these two groups. We decided that seven families was more than sufficient to make our point.

2. The Key Lemma

In this section, we present our construction for new solvable groups having derived length 4 and four character degrees.

Lemma 2.1. *Let p be a prime, and let P be a p-group so that $\mathrm{cd}(P) = \{1, p^\alpha\}$ for some positive integer α. Suppose the group H acts via automorphisms on P and that H satisfies the following hypotheses: p does not divide $|H|$ and $\mathrm{cd}(H) = \{1, a\}$ for some positive integer a. Let $C = C_H(P')$ and $D = C_P(C)$. Assume one of the following:*

(1) *$C = 1$ and H acts Frobeniusly on P.*

(2) *$C > 1$ is abelian, $D < P$, H acts Frobeniusly on P/D, every nonlinear character in $\mathrm{Irr}(P)$ is fully ramified with respect to P/D, and H/C acts Frobeniusly on D. When $P' < D$, assume that $|H : C| = a$.*

If $G = P \rtimes H$, then $\mathrm{dl}(G) = 4$ and $\mathrm{cd}(G) = \{1, a, |H|, |H : C|p^\alpha\}$.

Proof. Suppose first that H acts Frobeniusly on P. Since H is nonabelian, we have $H' > 1$, so H' acts Frobeniusly on P. It follows that $P = [P, H'] \leq G'$, so $P = [P, H'] \leq [G', G'] = G''$. On the other hand, since $|\mathrm{cd}(G/P)| = 2$, we have $G'' \leq P$; so $G'' = P$. Because $P'' = 1$, we conclude that $\mathrm{dl}(G) = 4$ in this case.

Suppose (2). Since C centralizes P', it follows that $P' \leq D$. This implies that D is normal in P. Also, because C is normal in H, we see that D is normalized by H, so D is normal in G. Observe that C is abelian and H is nonabelian, which imply that $C < H$. Since H/C acts Frobeniusly on D, so $D = [D, H] \leq G'$. The fact that H acts Frobeniusly on P/D implies that $P/D = [P/D, H] = [P, H]D/D \leq (G/D)' = G'D/D = G'/D$. This implies that $P \leq G'$. We now have that $G' = PH'$. We have $H' > 1$ and H acts Frobeniusly on P/D, so H' acts Frobeniusly on P/D. We have two cases to deal with, when $H' \leq C$ and when $H' \not\leq C$.

Suppose that $H' \leq C$. It follows that H' centralizes D. Since every nonlinear irreducible character of P is fully-ramified with respect to P/D, this implies that H' fixes every nonlinear irreducible character of P. Applying [6, Theorem 3.3], we have $[P, H']' = P'$. Observe that $[P, H'] \leq G'' \leq P$. This implies that $[P, H']' \leq G''' \leq P'$. We deduce that $G''' = P'$, and since $P' > 1$ is abelian, we conclude that $\mathrm{dl}(G) = 4$.

Now, we consider the case where $H' \not\leq C$. In particular, $H' \cap C < H'$. It follows that $H'/H' \cap C$ acts Frobeniusly on D. Hence, we have $D = [D, H'] \leq G'$. Since H acts Frobeniusly on P/D and $H' > 1$, it follows that H' acts Frobeniusly on P/D. This implies that $P/D = [P/D, H']$, and so, $P = [P, H']D = [P, H'][D, H'] = [P, H']$. Observe that $[P, H'] \leq G'$; thus, $P \leq G'$. We then obtain $P = [P, H'] \leq [G', G'] = G'' \leq P$, and hence, $G'' = P$. Because $P' > 1$ and $P'' = 1$, we conclude that $\mathrm{dl}(G) = 4$.

We now compute $\mathrm{cd}(G)$. We have $\mathrm{cd}(G/P) = \mathrm{cd}(H) = \{1, a\}$. If H acts Frobeniusly on P, then $\theta^G \in \mathrm{Irr}(G)$ for all $\theta \in \mathrm{Irr}(P)$ (see [5, Theorem 6.34 (b)]), and so $\mathrm{cd}(G) = \{1, a, |H|, |H|p^\alpha\}$, which is the desired result since $C = 1$. Thus, we assume we have hypothesis (2).

Since H acts Frobeniusly on P/D, we have if $1 \neq \lambda \in \mathrm{Irr}(P/D)$, then $\lambda^G \in \mathrm{Irr}(G)$, and so, $|H| = \lambda^G(1) \in \mathrm{cd}(G)$. We deduce that $\mathrm{cd}(G/D) = \{1, a, |H|\}$. When $D = P'$, we have $\mathrm{cd}(G/P') = \mathrm{cd}(G/D)$.

Suppose that $P' < D$. Consider $\delta \in \mathrm{Irr}(D/P')$. Observe that δ is C-invariant, and since H/C acts Frobeniusly on D and thus on D/P', we see that C is the stabilizer of δ in H. We know that δ extends to $\mathrm{Irr}(P/P')$. Note that C acts on the extensions of δ to P, and using Gallagher's theorem ([5, Corollary 6.17]), $\mathrm{Irr}(P/D)$ acts transitively by right multiplication on the extensions of δ to P. Applying Glauberman's lemma ([5, Lemma 13.8]), we see that δ has a C-invariant extension $\mu \in \mathrm{Irr}(P/P')$. Since C acts Frobeniusly on $\mathrm{Irr}(P/D)$, we may apply [5, Corollary 13.9] to see that μ is the unique G-invariant extension of δ to μ. Note that C will be the stabilizer of μ in H, and so, PC is the stabilizer of μ

in G. Applying [5, Corollary 6.27], μ extends to PC and by Gallagher's theorem, μ only has extensions to PC since C is abelian. This implies that $\mathrm{cd}(G \mid \mu) = \{|G : PC|\} = \{|H : C|\} = \{a\}$, since $a = |H : C|$ in this case.

Note that any extension of δ to P will have the form $\mu\lambda$ for some character $\lambda \in \mathrm{Irr}(P/D)$. If $1 \neq \lambda$, we see that if $h \in C_H(\mu\lambda)$, then h stabilizes $(\mu\lambda)_D = \delta$, and so, $C_H(\mu\lambda) \leq C_H(\delta) = C = C_H(\mu)$, and so, $\mu\lambda = (\mu\lambda)^h = \mu\lambda^h$. Applying Gallagher's theorem, we have that h stabilizes λ. Since C acts Frobeniusly on $\mathrm{Irr}(P/D)$ and $\lambda \neq 1$, we conclude that $h = 1$. It follows that $C_H(\mu\lambda) = 1$, and so, $(\mu\lambda)^G \in \mathrm{Irr}(G)$. We deduce that $\mathrm{cd}(G \mid \delta) = \{a, |H|\}$. This yields $\mathrm{cd}(G/P') = \{1, a, |H|\} = \mathrm{cd}(G/D)$ in this case.

Finally, suppose $1 \neq \gamma \in \mathrm{Irr}(P')$, and consider $\hat{\gamma} \in \mathrm{Irr}(D \mid \gamma)$. Notice that the irreducible constituents of $\hat{\gamma}^P$ will be nonlinear, so $\hat{\gamma}$ is fully-ramified with respect to P/D, and let $\epsilon \in \mathrm{Irr}(P)$ be the unique irreducible constituent of $(\hat{\gamma})^P$. We know that $\epsilon(1) = p^\alpha$. It follows that ϵ and $\hat{\gamma}$ have the same stabilizer in G. Because C centralizes D and H/C acts Frobeniusly on D, it follows that PC will be the stabilizer of $\hat{\gamma}$ in G. Applying [5, Corollary 6.28], we see that ϵ extends to PC, and since C is abelian, we may use Gallagher's theorem to see that ϵ only has extensions to PC. We obtain $\mathrm{cd}(G \mid \hat{\gamma}) = \{|G : PC|\epsilon(1)\} = \{|H : C|p^\alpha\}$. We conclude that $\mathrm{cd}(G) = \{1, a, |H|, |H : C|p^\alpha\}$, as desired. □

3. Various constructions

We now find specific families of groups that meet the parameters of Lemma 2.1. We begin with a family of groups based on the Heisenberg group. In this first example, G will be a Frobenius group where the Frobenius kernel is a Heisenberg group and a Frobenius complement is a nonnilpotent metacyclic group. This example is an example where hypothesis (1) of Lemma 2.1 is applied.

Theorem 3.1. *Let p be a prime, and let q be an odd prime that divides $p-1$. Then there exists a group G with $\mathrm{dl}(G) = 4$, Fitting height 3, and $\mathrm{cd}(G) = \{1, q, (p^q-1)/(p-1))_{q'}(p-1)_q q, p^q((p^q-1)/(p-1))_{q'}(p-1)_q q\}$.*

Proof. We are going to take P to be the Heisenberg group of order p^{3q}. We represent P as follows. Let F be the field of order p^q. Then we can view P as

$$\left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in F \right\}.$$

We write F^* for the multiplicative group of F and $\mathcal{G}$ for the Galois Group of F with respect to Z_p. It is easy to see that $\mathcal{G}$ acts on F^* and that the resulting semi-direct product $F^*\mathcal{G}$ is isomorphic to the semi-linear group $\Gamma(F)$. (See [12, page 37] for the definition of the semilinear group.) We can define an action by automorphisms for $\Gamma(F)$ on P as follows: if $\lambda \in F^*$, then

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \lambda = \begin{bmatrix} 1 & \lambda a & \lambda^2 c \\ 0 & 1 & \lambda b \\ 0 & 0 & 1 \end{bmatrix}$$

where the multiplication is in F, and if $\sigma \in \mathcal{G}$, then

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \sigma = \begin{bmatrix} 1 & a^\sigma & c^\sigma \\ 0 & 1 & b^\sigma \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that if λ has odd order, then the action of λ on P is Frobenius.

Now, let γ be an element of F^* of order $((p^q-1)/(p-1))_{q'}$. Notice that the order of γ is odd, so $\gamma^2 \neq 1$. Let λ be a generator for the Sylow q-subgroup of F^*, and let $\sigma \in \mathcal{G}$ correspond to the Frobenius automorphism for F, so σ has order q. Recall that the fixed field for σ in F has order $p-1$. It follows that γ is not in the fixed field for σ, so γ and σ do not commute. We define $H = \langle \gamma, \lambda\sigma \rangle$. It is not difficult to see since σ does not commute with γ and λ does commute with γ that γ and $\sigma\lambda$ do not commute; so H is not abelian. On the other hand, H has a normal abelian subgroup of index q, so $\mathrm{cd}(H) = \{1, q\}$. We observe that

$$(\lambda\sigma)^q = \lambda^{\sigma^q} \cdots \lambda^{\sigma^2}\lambda^\sigma = \lambda^{p^{q-1}} \cdots \lambda^p \lambda = \lambda^{1+p+\cdots+p^{q-1}} = \lambda^{(p^q-1)/(p-1)}.$$

Since q divides $p-1$, it follows that $\lambda^{(p^q-1)/(p-1)} \neq 1$. Note that $Q = \langle \lambda\sigma \rangle$ is a Sylow q-subgroup of H, is cyclic, has order $(p^q-1)_q$, and acts Frobeniusly on P. Since q is odd, we know that $(p^q-1)_q = (p-1)_q q$. Also, observe that $\langle \gamma \rangle$ is a Hall q-complement of H, has order $((p^q-1)/(p-1))_{q'}$, and also acts Frobeniusly on P. Observe that $C_H(P') = 1$ and $|H| = ((p^q-1)/(p-1))_{q'}(p-1)_q q$. We conclude that H and P satisfy the hypotheses of Lemma 2.1, and we obtain the conclusion from there. Note that P is the Fitting subgroup of G and H has Fitting height 2, so G has Fitting height 3. □

In this next example, the normal Sylow p-subgroup is again a Heisenberg group. In this case, G is not a Frobenius group. This example illustrates where hypothesis (2) of Lemma 2.1 is applied with $D = P'$.

Theorem 3.2. *Let p be a prime, let q be an odd prime that divides $p-1$, and let r be a divisor of $p-1$ that is coprime to q. Then there exists a group G with* $\mathrm{dl}(G) = 4$, *Fitting height* 3, *and*

$$\mathrm{cd}(G) = \{1, q, ((p^q-1)/(p-1))_{q'}(p-1)_q qr, p^q((p^q-1)/(p-1))_{q'}(p-1)_q q\}.$$

Proof. We again take P to be the Heisenberg group of order p^{3q}, and we take K to the subgroup H from Theorem 3.1, so $|K| = ((p^q-1)/(p-1))_{q'}(p-1)_q q$. Let η be an element of F^* with order r. It is known that the gcd of $(p^q-1)/(p-1)$ and $p-1$ is q (see [13]). Thus, since r divides $p-1$ and is coprime to q, it follows that r is coprime to $((p^q-1)/(p-1))_{q'}$. We can define an action of η on P by

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \eta = \begin{bmatrix} 1 & \eta a & c \\ 0 & 1 & \eta^{-1} b \\ 0 & 0 & 1 \end{bmatrix}$$

where the multiplication is in F. Note that the action of η will commute with the actions of λ and γ. Also, notice that η will be in the prime subfield of F, so η and σ commute. We deduce that η centralizes K. Thus, we can take $H = K \times \langle \eta \rangle$. Notice that η acts Frobeniusly on P/P' and centralizes P'. It follows that $C = C_H(P') = \langle \eta \rangle$ and $D = C_P(C) = P'$. In particular, $G = P \rtimes H$ satisfies the hypotheses of Lemma 2.1, and we obtain the derived length and character degrees from that lemma. It is easy to see that G has Fitting height 3. □

This next two groups are based on extraspecial groups of order p^{2q+1} where q is a prime. In both cases, we use hypothesis (2) of Lemma 2.1 with $D = P'$. In the next theorem, we have that $|H : C| = q$ using the notation of Lemma 2.1.

Theorem 3.3. *Let p be a prime, and let q be an odd prime that divides $p-1$. Then there exists a group G with* $\mathrm{dl}(G) = 4$, *Fitting height* 3, *and* $\mathrm{cd}(G) = \{1, q, (p^q-1)_{q'}(p-1)_q q, p^q q\}$.

Proof. Let V be a vector space of dimension q over Z_p, the field of order p. Let $\hat{V}$ be the dual space for V; that is, $\hat{V}$ is the set of all linear transformations from V to Z_p. We define $P = \{(a, \alpha, z) \mid a \in V, \alpha \in \hat{V}, z \in Z_p\}$ where multiplication in P is defined by $(a_1, \alpha_1, z_1)(a_2, \alpha_2, z_2) = (a_1 + a_2, \alpha_1 + \alpha_2, z_1 + z_2 + \alpha_2(a_1))$. It is not difficult to see that P is an extraspecial p-group of order p^{2q+1}.

If δ is an automorphism of V, then we obtain an automorphism for $\hat{V}$ by defining α^δ by $\alpha^\delta(v) = \alpha(v^{\delta^{-1}})$ for all $v \in V$. Note that $\alpha^\delta(v^\delta) = \alpha(v)$. It is not difficult to see that we can define an automorphism on P by $(v, \alpha, z)^\delta = (v^\delta, \alpha^\delta, z)$.

We can identify V with the additive group of the field F of order p^q. If λ is a nonzero element of F, then multiplication by λ yields an automorphism of V, and we use v^λ to denote this map on V and α^λ to denote the associated map on $\hat{V}$. Also, the Galois automorphisms of F will yield automorphisms of V. If σ is a Galois automorphism of F, then we use v^σ to be the automorphism on V and α^σ for the associated map on $\hat{V}$. We take γ to be a generator of the Hall q-complement of F^*. We take λ to be a generator for the Sylow q-subgroup of F^*. We take σ to be the Frobenius automorphism. We use these same letters to denote the automorphisms of P given by each of these elements as above.

Let x be a non-zero element of Z_p, then we can view x as element of F^*, and so we can define the action of x on V and $\hat{V}$ as before. Notice that since the elements of $\hat{V}$ are linear transformations, we have that $\alpha(v^x) = \alpha(xv) = x\alpha(v)$ and $\alpha^{x^{-1}}(v) = \alpha(v^x)$, so $\alpha^{x^{-1}}(v^x) = x^2\alpha(v)$. Thus, we define an automorphism of P by $(v, \alpha, z) \mapsto (v^x, \alpha^{x^{-1}}, x^2 z)$. Let ξ be this automorphism defined for an element x of order q in Z_p^*. It is easy to see that ξ will commute with γ and λ as automorphisms of P. Since as a Galois automorphism, σ fixes the elements in Z_p, it is not difficult to see that σ and ξ will commute. Let $H = \langle \gamma, \lambda\xi\sigma \rangle$. Since γ commutes with λ and ξ, but not σ, we see that H is not abelian. Working as in the proof of the last theorem, we see that $(\lambda\xi\sigma)^q = \lambda^{(p^q-1)/(p-1)}$, which will have order $(p-1)_q$. Notice that $C = \langle \gamma, \lambda^{(p^q-1)/(p-1)} \rangle$ has index q in H, is abelian, and centralizes P'. It follows that $\mathrm{cd}(H) = \{1, q\}$ and $C = C_H(P')$. Observe that $P' = C_P(C)$, which is D in the notation of Lemma 2.1. It is not difficult to see that H acts Frobeniusly on P/D and that H/C acts Frobeniusly on D, so the hypotheses of Lemma 2.1 are met. We obtain the derived length and the character degree set conclusions from that result. Notice that P is the Fitting subgroup of G and $G/P \cong H$ has Fitting height 2, so G has Fitting height 3. □

In this next example, we again have an extraspecial group of order p^{2q+1}, but in this case, we have $|H : C|$ is relatively prime to q, again using the notation of Lemma 2.1. Note in the three previous examples that q divides all of the nontrivial degrees in $\mathrm{cd}(G)$. This provides an example of our construction where there is a degree that is coprime to q.

Theorem 3.4. *Let p be a prime, let q a prime that divides $p-1$, and let r be an odd divisor of $p-1$ that is relatively prime to q. Then there exists a group G with $\mathrm{dl}(G) = 4$, Fitting height 3, and $\mathrm{cd}(G) = \{1, q, (p^q-1)_{\{q,r\}'}(p-1)_q qr, p^q r\}$.*

Proof. As in the proof of Theorem 3.3, we take V to a vector space of dimension q over Z_p, we write $\hat{V}$ for the dual space for V, and P for the associated extraspecial group. Again, we take F to be the field of order p^q, and we have the

same action for elements of F^* and the Galois group of F on P. We take γ to be a generator for the Hall $\{q,r\}$-complement of F^*, λ to be a generator for the Sylow q-subgroup of F^*, and σ to be the Frobenius automorphism of F. We now take x to be an element of order r in Z_p^*, and we let ξ be the automorphism of P defined for x. Take $H = \langle \gamma, \lambda\sigma, \xi\rangle$. Since γ commutes with λ, but not σ, we see that H is not abelian. We see that $(\lambda\sigma)^q = \lambda^{(p^q-1)/(p-1)}$, which will have order $(p-1)_q$. Observe that $\langle \gamma, \lambda^{(p^q-1)/(p-1)}, \xi\rangle$ is a normal, abelian subgroup of index q, so $\mathrm{cd}(H) = \{1,q\}$. Also, $C = C_H(P') = \langle \gamma, \lambda\sigma\rangle$ and $D = C_P(C) = P'$. It is not difficult to see that H acts Frobeniusly on P/D and that H/C acts Frobeniusly on D, so the hypotheses of Lemma 2.1 are met. We obtain the derived length and the character degree set conclusions from that result. Notice that P is the Fitting subgroup of G and $G/P \cong H$ has Fitting height 2, so G has Fitting height 3. □

We now present an example where hypothesis (2) of Lemma 2.1 is used with $P' < D = C_P(C_H(P'))$.

Theorem 3.5. *Let p be a prime, let q an odd prime that divides $p-1$, and let $n > q$ be an integer. Then there exists a group G with* $\mathrm{dl}(G) = 4$, *Fitting height* 3, *and* $\mathrm{cd}(G) = \{1, q, (p^q-1)_{q'}(p-1)_q q, p^n q\}$.

Proof. Let P_1 be the group P from the Theorem 3.3. We take P_2 to be an extraspecial group of order $p^{2(n-q)}$ and exponent p. We will take P to be a central product of P_1 and P_2, and we let H be as in Theorem 3.3. We have H act on P_1 as it acted on P in Theorem 3.3. We will have γ, λ and σ act trivially on P_2, and it is not difficult to see that there is a Frobenius action of x on P_2 so that the action on $Z(P_2)$ matches the action of x on $Z(P_1)$. This then defines an action of H on P. Notice that $C = C_H(P') = \langle \gamma, \lambda^{(p^q-1)/(p-1)}\rangle$ and $D = C_P(C) = P_2$ so $P' < D$. Observe that $|H : C| = q$. Also, all of the nonlinear irreducible characters of P are fully-ramified with respect to D, so that the hypotheses of Lemma 2.1 are met. We obtain the conclusions regarding the derived length and character degrees from there. The Fitting height follows as in Theorem 3.3. □

Next we present an example where n is not a prime, and P is the normal Sylow p-subgroup and $\mathrm{cd}(P) = \{1, p^n\}$. Note that in the previous examples, we have had n as a prime. Recall that q is a Zsigmondy prime divisor of $p^n - 1$ for positive integers p and n if q divides $p^n - 1$ and q does not divide $p^a - 1$ for integers a such that $1 \le a < n$. Observe that none of the character degrees in this example is a prime.

Theorem 3.6. *Let p be a prime, and let n be an odd integer so that every prime divisor of n divides $p-1$. Let π be the set of prime divisors of n, let ρ be the set of Zsigmondy prime divisors of p^n-1 (the definition of Zsigmondy primes can be found as* [12, Definition 6.1]*), and let m be an integer so that m divides $n(p-1)_\rho$, n divides m, and every prime divisor of $(p-1)_\rho$ divides m/n. Then there exists a group G with* $\mathrm{dl}(G) = 4$, *Fitting height* 3, *and* $\mathrm{cd}(G) = \{1, n, (p^n-1)_\rho m, p^n(p^n-1)_\rho m\}$.

Sketch of proof. We take P to be the Heisenberg group of order p^{3n}, and let F be the field of order p^n. Working as in the proof of Theorem 3.1, we can define an action of F^* and $\mathrm{Gal}(F)$ on P. Applying [9, Theorem 11], we can find subgroups K and N of $F^*\mathrm{Gal}(F)$ so that $|K| = (p^n-1)_\rho$, $|N| = m$, $\mathrm{cd}(NK) = \{1, n\}$, K is cyclic, N is nilpotent, and NK acts Frobeniusly on P. Take $G = PNK$. We now apply Lemma 2.1 to obtain the conclusion. □

4. Fitting height 2

We begin with an observation suggested by one of the referees. We obtain some general information regarding examples satisfying Lemma 2.1 with Fitting height 2.

Lemma 4.1. *Let G be a group that satisfies the hypotheses of Lemma 2.1. If G has nilpotence class* 2, *then* $\mathrm{cd}(H) = \{1, 2\}$, *$C > 1$ and $|H : C|_2 = 2$.*

Proof. Notice that under both hypotheses H is a Frobenius complement. This implies that all of the Sylow subgroups of H are either cyclic or generalized quaternion. Also, it is not difficult to see that P is the Fitting subgroup of G. Since G has Fitting height 2, we see that H must be nilpotent. Since all of the Sylow subgroups of H for odd primes will be cyclic, we see that H must have a nonabelian Sylow 2-subgroup. In particular, the Sylow 2-subgroup must be generalized quaternion. Since the degree set for a generalized quaternion group is $\{1, 2\}$ and H is a direct product of its Sylow subgroups, we obtain $\mathrm{cd}(H) = \{1, 2\}$.

Let T be the Sylow 2-subgroup of G; so T is a generalized quaternion group. Suppose $C = 1$ so that we are in hypothesis (1) of Lemma 2.1. Then G is a Frobenius group with Frobenius kernel P and Frobenius complement H. In particular, T would be a Frobenius complement for P. It is well known that any Frobenius kernel whose Frobenius complement has even order must be abelian and P is clearly not abelian, so we have a contradiction. Thus, $C > 1$, and we have hypothesis (2) of Lemma 2.1. Since C is abelian, we know that C does not contain

a full Sylow 2-subgroup of H. It follows that $T \cap C < T$. If $T \cap C = 1$, then T acts Frobeniusly on both P/D and D, and this would imply that T acts Frobeniusly on P, and we have seen that this leads to a contradiction. Thus, we must have $T \cap C > 1$. We know that $T/(T \cap C) \cong TC/C$ is either abelian or dihedral. On the other hand, we know that H/C acts Frobeniusly on D, so it is a Frobenius complement, and thus, TC/C will be a Frobenius complement. Since a 2-group that is a Frobenius complement must be cyclic or generalized quaternion, we conclude that TC/C must be cyclic. However, the only cyclic quotient of a dihedral group has order 2, we conclude that $|H : C|_2 = |TC : C| = 2$. □

It follows that if G has Fitting height 2 and satisfies the hypotheses of Lemma 2.1, then H must have even order, and in fact, both H and G have 2 as a character degree. Obviously, this raises the question of whether there can exist groups of odd order with derived length four and four character degrees. In light of Lemma 4.1, we can come close to determining the character degree set for a group G that satisfies the hypotheses of Lemma 2.1 and has Fitting height 2. When H is a 2-group, we have $\mathrm{cd}(G) = \{1, 2, |H|, 2p^\alpha\}$. The question that arises is what p-groups can arise as the subgroup P in Lemma 2.1. We now produce examples with Fitting height 2 when $p \equiv 3(\mathrm{mod} 8)$ and P is an extraspecial group of order p^5. Notice that this yields Theorem 1.1.

Theorem 4.2. *Let p be a prime that is congruent to 3 modulo 8. Then there exists a group G with $\mathrm{dl}(G) = 4$, Fitting height 2, and $\mathrm{cd}(G) = \{1, 2, 8, 2p^2\}$.*

Proof. We somewhat follow the construction found in the proof of Theorem 3.3. We take V to be a vector space of dimension 2 over Z_p, and we define P as in the proof of Theorem 3.3 to be the extraspecial group of order p^5 arising from pairing V with $\hat{V}$. Viewing V as a field of order p^2, it is not difficult to see that the multiplicative group will have an element of order 8. Let λ be the automorphism of V that is obtained by multiplication from that element, and as in the proof of Theorem 3.3, λ also determines an automorphism of $\hat{V}$ of order 8, and we also use λ to denote the automorphism of P given by $(a, \alpha, z) \mapsto (a^\lambda, \alpha^\lambda, z)$. We let σ be the Frobenius automorphism for V viewed as field, and again, σ defines an automorphism of $\hat{V}$, and we write σ for the automorphism of P given by $(a, \alpha, z) \mapsto (a^\sigma, \alpha^\sigma, z)$. It is not difficult to see that $(\lambda\sigma)^2 = \lambda^{p+1}$, and since $p \equiv 3(\mathrm{\ modulo\ } 8)$, we see that $\lambda^{p+1} = \lambda^4 = -1$.

Let ζ be an element of order 2 in the multiplicative group of Z_p, and observe that the map $(a, \alpha, z) \mapsto (\zeta a, \alpha, \zeta z)$ is an automorphism of order 2 on P and will centralize λ and σ as automorphisms of P. We now take H to be the subgroup of the automorphism group of P given by $\langle \zeta\lambda^2, \lambda\sigma \rangle$. It is not difficult to see that

H will be isomorphic to the quaternion group of order 8. Since $\lambda^4 = -1$, we see that H acts Frobeniusly on P/P'. Observe that $\lambda\sigma$ centralizes P', and $\zeta\lambda^2$ does not centralize P'. Since $|H : \langle\lambda\sigma\rangle| = 2$, we conclude that $C = \langle\lambda\sigma\rangle$. Notice that H/C acts Frobeniusly on P'. Thus, the hypotheses of Lemma 2.1 are met, and we obtain that $\mathrm{dl}(G) = 4$ and the character degrees are as stated. Since H is nilpotent, we see that G has Fitting height 2. □

When H is not a 2-group, things can be more complicated. Take $p = 11$, and let P be an extraspecial group of order 11^5 and exponent 11. We take Q to be the quaternions acting as in the proof of Theorem 4.2, so we can view Q as a subgroup of the automorphisms of Q. It is not difficult to see that P has an automorphism of order 5 that centralizes $Z(P)$, acts Frobeniusly on $P/Z(P)$, and centralizes Q. Let R be the group of order 5 generated by that automorphism. Take $H_1 = Q \times R$ and $G_1 = P \rtimes H_1$. One can compute that $|H_1 : C_{H_1}(P')| = 2$ and $\mathrm{cd}(G) = \{1, 2, 40, 242\}$. On the other hand, P also has an automorphism of order 5 that acts Frobeniusly on P and centralizes Q, and we use S to denote the group of order 5 generated by this automorphism. Define $H_2 = Q \times S$ and $G_2 = P \rtimes H_2$. In this case, we obtain $|H_2 : C_{H_2}(P')| = 10$ and $\mathrm{cd}(G) = \{1, 2, 40, 1210\}$.

To see an even more complicated example, take $p = 19$ and P to be the extraspecial group of order 19^5 and exponent 19. In this case, not only does P have automorphisms as in the previous case, it also has an automorphism of order 9 that acts fixed-point-freely but whose cube centralizes $Z(P)$ and acts Frobeniusly on $P/Z(P)$, and we obtain $\mathrm{cd}(G) = \{1, 2, 72, 6 \cdot 19^2\}$. Thus, while it may be possible to classify the groups with Fitting height 2 that satisfy the hypotheses of Lemma 2.1, such a classification is going to be more complicated than we wish to pursue here.

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