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Invariant property for discontinuous mean-type mappings

By PAWEŁ PASTECZKA (Kraków)

Abstract. It is known that if M, N are continuous two-variable means such that $|M(x,y) - N(x,y)| < |x - y|$ for every x, y with $x \neq y$, then there exists a unique invariant mean (which is continuous too).

We are looking for invariant means for pairs satisfying the inequality above, but their continuity is not assumed.

In this setting the invariant mean is no longer uniquely defined, but we prove that there exist the smallest and the biggest one. Furthermore, it is shown that there exists at most one continuous invariant mean related to each pair.

1. Introduction

The idea of invariant means was first introduced by Gauss [11], who considered the so-called arithmetic-geometric mean. It was obtained as a limit in the iteration process

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_n y_n} \qquad (n \in \mathbb{N}_+ \cup \{0\}),$$

where x_0, y_0 are two positive arguments. Then it is known that both (x_n) and (y_n) are convergent to a common limit, which is called the *arithmetic-geometric mean* (of the initial arguments $x_0 := x$ and $y_0 := y$).

In a more general setting a *mean* is an arbitrary function $M\colon I^2 \to I$ (from now on I stands for an arbitrary interval) such that

$$\min(x,y) \le M(x,y) \le \max(x,y) \quad \text{for all} \quad x,\, y \in I.$$

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If the inequalities above remain strict unless $x = y$, then the mean M itself is called *strict*.

For two means M, N on I, we define a selfmapping $(M, N)\colon I^2 \to I^2$ by

$$(M, N)(x, y) := (M(x, y), N(x, y)).$$

A mean K on I is said to be an (M,N)-*invariant mean* if $K = K \circ (M,N)$; more precisely,

$$K(x,y) = K\big(M(x,y), N(x,y)\big) \quad \text{for all} \quad x,\, y \in I.$$

In this setting the arithmetic-geometric mean is an invariant mean for arithmetic and geometric means. In fact, it was proved [5, Theorem 8.2] that if M and N are continuous and strict, then such K always exists and is uniquely determined. Later Matkowski [14] proved that the strictness assumption can be relax to

$$|M(x,y) - N(x,y)| < |x - y| \quad \text{for all} \quad x,\, y \in I,\ x \neq y. \tag{1.1}$$

Finally, similarly like in the case of the arithmetic-geometric mean, we know (see, e.g. [5]) that the (M,N)-invariant mean is obtained as a common limit of iterates of the mean-type mapping (M,N) given by

$$\begin{aligned} x_0 &= x, & y_0 &= y; \\ x_{n+1} &= M(x_n, y_n), & y_{n+1} &= N(x_n, y_n) \qquad \text{for all } n \geq 0; \end{aligned} \tag{1.2}$$

where x and y are its arguments. In fact, these sequences of iterates are used so often that whenever the quadruple (M,N,x,y) is defined, sequences (x_n) and (y_n) are also given.

Invariant means were extensively studied during recent years, see, for example, papers by Baják–Páles [1]–[4], by Daróczy–Páles [8]–[10], by Głazowska [12]–[13], by Matkowski [14]–[18], by Matkowski–Páles [19], by the author [20]–[22], and in the seminal book Borwein–Borwein [5].

We will consider (M,N)-invariant means where M, N satisfies inequality (1.1) but continuity is replaced by symmetry (i.e., $M(x,y) = M(y,x)$ for all $x,\, y \in I$).

Let us just mention that we do not require the means to be discontinuous. On the other hand, if both of them are continuous, then our consideration reduces to the one which was already done many times (see references above).

2. Invariant means with no continuity assumption

In this section we are going to present some examples of constructions which provide (M,N)-invariant means, where M and N are not necessarily continuous.

There are two somehow independent ways of defining such means. The first idea is to extend the meaning of limit in the definition of invariant mean (for example, to $\liminf$ or $\limsup$). We realize this idea in Section 2.1. The second one is related with transfinite iterations (Section 2.2).

Let us begin with two elementary, however useful, results

Lemma 1. *If $M,N\colon I^2 \to I$ are symmetric means, then every (M,N)-invariant mean is symmetric.*

Indeed, if K is an arbitrary (M,N)-invariant mean, then for every $x,\, y \in I$ we get

$$K(x,y) = K(M(x,y),N(x,y)) = K(M(y,x),N(y,x)) = K(y,x).$$

Lemma 2. *If $M,N\colon I^2 \to I$ are symmetric means, then a mean is (M,N)-invariant if and only if it is $(M\wedge N, M\vee N)$-invariant, where*

$$(M\wedge N)(x,y) := \min(M(x,y),N(x,y)), \quad x,\, y \in I,$$
$$(M\vee N)(x,y) := \max(M(x,y),N(x,y)), \quad x,\, y \in I.$$

By the previous lemma, every (M,N)-invariant (or $(M\wedge N, M\vee N)$-invariant) mean is symmetric. Furthermore, for every symmetric function $K\colon I^2 \to I$ we have $K\circ(M,N) = K\circ(M\wedge N, M\vee N)$.

2.1. Boundary invariant means. This idea is motivated by generalized limit functions. Our consideration covers all standard type of limits (i.e., $\lim$, $\liminf$, $\limsup$) but also more general functionals such as Banach limit[1]. A function $\phi\colon \ell^\infty(I) \to I$ is called *2-limit-like* if for every $a=(a_1,a_2,\dots)\in\ell^\infty(I)$,

(i) $\phi(a_1,a_2,a_3,\dots) = \phi(a_3,a_4,a_5,\dots)$, and

(ii) $\liminf_{n\to\infty} a_n \le \phi(a_1,a_2,\dots) \le \limsup_{n\to\infty} a_n$.

Note that whenever the sequence a is convergent, then $\phi(a)=\lim_{n\to\infty} a_n$.

[1]Banach limit is a linear functional $L\colon \ell^\infty \to \mathbb{R}$ such that $\|L\|_\infty = 1$; L is shift invariant, i.e., $L(a_2,a_3,\dots) = L(a_1,a_2,a_3,\dots)$ for every $a\in\ell^\infty$; $L(a)\ge 0$ whenever $a_n \ge 0$ for all n; and $L(a)=\lim_{n\to\infty} a_n$ for every convergent sequence (a_n) (cf. Conway [7]).

Let us emphasize that 2-limit-like functions are much more general objects than common (or even Banach) limits. In fact, we can construct $2^{\mathfrak{c}}$ different 2-limit-like functions. Indeed, each function $w\colon [0,1] \to [0,1]$ leads to a 2-limit-like function on $\ell^\infty[0,1]$ given by

$$\phi_w(a) := \liminf_{n\to\infty} a_n + w\big(\liminf_{n\to\infty} a_{2n}\big) \cdot \Big(\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n\Big).$$

Furthermore, by taking a family of 4-periodic sequences $(0, x, 0, 1, \dots)$ for $x \in [0,1]$, it can be verified that the mapping $w \mapsto \phi_w$ is one-to-one.

At the moment, we can use this definition to introduce the wide class of (M,N)-invariant means.

Proposition 1. *Let $M, N\colon I^2 \to I$ be two means, and $\phi\colon \ell^\infty(I) \to I$ be a 2-limit-like function. Then the function $\mathcal{B}_\phi\colon I^2 \to \mathbb{R}$ given by*

$$\mathcal{B}_\phi(x,y) := \phi(x_0, y_0, x_1, y_1, x_2, y_2, \dots)$$

is a mean on I, which is (M,N)-invariant.

Conversely, every (M,N)-invariant mean equals $\mathcal{B}_\phi$ for some 2-limit-like function ϕ.

Proof. By the definition of mean we have, for all $n \ge 0$,

$$\max(x_{n+1}, y_{n+1}) \le \max(x_n, y_n).$$

Thus the sequence $(\max(x_n, y_n))_{n\in\mathbb{N}}$ is nondecreasing and

$$\limsup\,(x_0, y_0, x_1, y_1, x_2, y_2, \dots) = \limsup_{n\to\infty} \max(x_n, y_n) \le \max(x_0, y_0) = \max(x,y).$$

Similarly, we obtain $\liminf\,(x_0, y_0, x_1, y_1, x_2, y_2, \dots) \ge \min(x,y)$.

Now, as ϕ is between $\liminf$ and $\limsup$, we obtain that $\mathcal{B}_\phi$ is a mean. Moreover,

$$\begin{aligned}
&\mathcal{B}_\phi(M(x,y), N(x,y))\\
&= \phi\Big(M(x_0,y_0), N(x_0,y_0), M\big(M(x_0,y_0), N(x_0,y_0)\big), N\big(M(x_0,y_0), N(x_0,y_0)\big), \dots\Big)\\
&= \phi(x_1, y_1, x_2, y_2, \dots) = \phi(x_0, y_0, x_1, y_1, x_2, y_2, \dots)\\
&= \mathcal{B}_\phi(x_0, y_0) = \mathcal{B}_\phi(x,y),
\end{aligned}$$

which concludes the proof.

To prove the converse, for an arbitrary (M, N)-invariant mean K, we define function ϕ on the orbit of (x, y) by

$$\phi(x_0, y_0, x_1, y_1 \dots) := K(x, y), \qquad x,\, y \in I, \tag{2.1}$$

fulfilled by

$$\phi(a_1, a_2, a_3, a_4, \dots) = \liminf_{n\to\infty} a_n. \tag{2.2}$$

By the definition of sequences (x_n), (y_n) and elementary properties of $\liminf$, we obtain that ϕ satisfies (i). Moreover, in view of (i) and the easy-to-check inequality $\inf(a) \le \phi(a) \le \sup(a)$, the property (ii) is also valid. □

In two particular cases $\phi = \liminf$ and $\phi = \limsup$, as

$$[\min(M(x, y), N(x, y)),\, \max(M(x, y), N(x, y))] \subset [x,\, y]$$

is valid for every $x,\, y \in I$ with $x < y$, we obtain two very important (M, N)-invariant means. Define *lower-* and *upper-invariant means* $\mathcal{L}, \mathcal{U}\colon I^2 \to I$ by

$$\mathcal{L}(x, y) := \mathcal{B}_{\liminf}(x, y) = \lim_{n\to\infty} \min(x_n, y_n),$$
$$\mathcal{U}(x, y) := \mathcal{B}_{\limsup}(x, y) = \lim_{n\to\infty} \max(x_n, y_n).$$

In fact, $\mathcal{L}$ and $\mathcal{U}$ are the smallest and the greatest (M, N)-invariant means, respectively, as every (M, N)-invariant mean is bounded from below by $\min(x_n, y_n)$ and from above by $\max(x_n, y_n)$ (for all $n \in \mathbb{N}$).

2.2. Transfinite invariant means. The transfinite invariant mean is the third (after lower- and upper-) natural invariant mean. In order to define it, we assume the comparability of means M and N — more precisely, $M(x, y) \le N(x, y)$ for all $x,\, y \in I$. Moreover, we assume that inequality (1.1) is valid.

Let us consider two transfinite sequences[2] (x_α) and (y_α) by fulfilling convention (1.2) in the following way:

$$x_\alpha := \lim_{\beta\nearrow\alpha} x_\beta, \qquad y_\alpha := \lim_{\beta\nearrow\alpha} y_\beta \qquad \text{for all limit ordinals } \alpha. \tag{2.3}$$

To provide the correctness of this definition we observe that (x_α) is nondecreasing while (y_α) is nonincreasing. Still, whenever M, N, x, and y are given, these sequences are automatically provided.

[2] that is sequences which are enumerated by ordinal numbers; cf. Cantor [6].

Inequality $M \le N$ implies that $x_\alpha \le y_\alpha$ for every $\alpha > 0$. In particular, by the definition of $\mathcal{L}$ and $\mathcal{U}$, we get

$$\mathcal{L}(x,\, y) = x_\omega \quad \text{and} \quad \mathcal{U}(x,\, y) = y_\omega\,. \tag{2.4}$$

Thus

$$A_\alpha\colon I^2 \ni (x,y) \mapsto x_\alpha \in I \qquad \text{and} \qquad B_\alpha\colon I^2 \ni (x,y) \mapsto y_\alpha \in I$$

are expressed as a function of $\mathcal{L}(x,\, y)$ and $\mathcal{U}(x,\, y)$ for all $\alpha > \omega$. In particular, they are all (M,N)-invariant. Moreover, $A_\omega = \mathcal{L}$ and $B_\omega = \mathcal{U}$.

The next lemma shows that iteration sequences (A_α) and (B_α) are eventually fixed. They reach that state after at most ω_1 iterations (ω_1 stands for the first uncountable ordinal). This implies that there is no point to consider indexes greater than ω_1 as no new means are obtained.

Lemma 3. *Let M, $N\colon I^2 \to I$ be two means having property* (1.1) *such that $M \le N$. Then $A_{\omega_1}(x,y) = B_{\omega_1}(x,y)$ for all $x,\, y \in I$.*

Proof. We need to prove that $x_{\omega_1} = y_{\omega_1}$. Inequality $M \le N$ implies $x_\alpha \le y_\alpha$ for all $\alpha \ge 1$. Moreover, (1.1) yields that for every $\alpha < \omega_1$ either $y_\alpha = x_\alpha$ (equivalently, $y_\alpha - x_\alpha = 0$) or $y_{\alpha+1} - x_{\alpha+1} < y_\alpha - x_\alpha$.

If $x_{\alpha_0} = y_{\alpha_0}$ for some $\alpha_0 < \omega_1$, then, by reflexivity of means, we obtain $x_\alpha = y_\alpha$ for all $\alpha \in [\alpha_0, \omega_1]$. In particular, $x_{\omega_1} = y_{\omega_1}$.

From now on, we may assume that $(y_\alpha - x_\alpha)_{\alpha<\omega_1}$ is strictly decreasing. As $x_\alpha \le y_\alpha$, we know that this sequence consists of nonnegative entries only. This leads to a contradiction as every strictly decreasing sequence of nonnegative numbers is countable. □

Remark 1. As both M and N are means, we obtain, applying the inequality $M \le N$, that the sequence $(x_\alpha)_{\alpha\le\omega_1}$ is nondecreasing, while $(y_\alpha)_{\alpha\le\omega_1}$ is nonincreasing.

Based on the lemma above, we can define, for $M \le N$, a *transfinite invariant mean* $\mathcal{T}\colon I^2 \to I$ by

$$\mathcal{T}(x,y) := A_{\omega_1}(x,y) = B_{\omega_1}(x,y). \tag{2.5}$$

By virtue of Lemma 2, we can skip the comparability assumption whenever both means are symmetric (as it was already done in the case of $\mathcal{L}$ and $\mathcal{U}$).

Let us now present the following important property of a transfinite invariant mean.

Theorem 1. *Let I be an interval, M, $N\colon I^2 \to I$ be means with $M \le N$ satisfying* (1.1). *Either $\mathcal{T}$, defined in* (2.5), *is a unique continuous (M,N)-invariant mean or there are no continuous (M,N)-invariant means.*

Proof. Let K be an arbitrary continuous (M,N)-invariant mean. We show that $K = \mathcal{T}$.

Fix x, $y \in I$. Using the definition of $\mathcal{T}$, it suffices to prove that $K(x,y) = x_{\omega_1}$. We will prove by transfinite induction that

$$K(x,y) = K(x_\alpha, y_\alpha) \quad \text{for all} \quad \alpha \ge 0. \tag{2.6}$$

Indeed, as K is (M,N)-invariant, we obtain

$$K(x_{\alpha+1}, y_{\alpha+1}) = K\big(M(x_\alpha,y_\alpha), N(x_\alpha,y_\alpha)\big) = K(x_\alpha,y_\alpha).$$

Furthermore, as K is continuous, for every limit ordinal number α, we get

$$K(x_\alpha,\, y_\alpha) = K\Big(\lim_{\beta\nearrow\alpha} x_\beta,\, \lim_{\beta\nearrow\alpha} y_\beta\Big) = \lim_{\beta\nearrow\alpha} K(x_\beta,\, y_\beta).$$

Now (2.6) easily follows. Finally, reflexivity of K binded with equality $x_{\omega_1} = y_{\omega_1}$ concludes the proof. □

Remark. By Lemma 2, we can skip the comparability assumption whenever both M and N are symmetric.

3. Applications and conclusions

3.1. Example of invariant property for noncontinuous means. Fix an interval I with $|I| > 1$ and functions M, $N\colon I^2 \to I$ defined by

$$M(x,y) := \begin{cases} \frac{1}{2}(x+y) & \text{for } |x-y| \le 1, \\ \frac{1}{2}\big(x+y-\sqrt{|x-y|}\,\big) & \text{for } |x-y| > 1, \end{cases} \qquad x,\, y \in I;$$

$$N(x,y) := \begin{cases} \frac{1}{2}(x+y) & \text{for } |x-y| \le 1, \\ \frac{1}{2}\big(x+y+\sqrt{|x-y|}\,\big) & \text{for } |x-y| > 1, \end{cases} \qquad x,\, y \in I.$$

It is easy to check that both M and N are symmetric and strict means on I. Furthermore, the arithmetic mean is (M,N)-invariant. Whence, by Theorem 1, it is a transfinite invariant mean for this pair.

Let (x_α) and (y_α) be two transfinite sequences corresponding to the iteration (M,N). Obviously, as $N \ge M$, we have $y_\alpha \ge x_\alpha$ for all $\alpha > 0$. Thus, for all $\alpha \ge 0$,

$$y_{\alpha+1} - x_{\alpha+1} = \begin{cases} 0, & \text{if } |y_\alpha - x_\alpha| \le 1, \\ \sqrt{|y_\alpha - x_\alpha|}, & \text{if } |y_\alpha - x_\alpha| > 1. \end{cases}$$

However, the iteration of square root is well known, so we obtain

$$y_\omega - x_\omega = \begin{cases} 0, & \text{if } |x-y| \le 1, \\ 1, & \text{if } |x-y| > 1. \end{cases} \tag{3.1}$$

On the other hand, we can check by simple induction that

$$x_\alpha + y_\alpha = x + y \quad \text{for all } \alpha \ge 0. \tag{3.2}$$

We now bind (2.4), (3.1), and (3.2) for $\alpha = \omega$ to obtain

$$\mathcal{L}(x,y) = \begin{cases} \frac{x+y}{2}, & \text{if } |x-y| \le 1, \\ \frac{x+y-1}{2}, & \text{if } |x-y| > 1; \end{cases}$$

$$\mathcal{U}(x,y) = \begin{cases} \frac{x+y}{2}, & \text{if } |x-y| \le 1, \\ \frac{x+y+1}{2}, & \text{if } |x-y| > 1. \end{cases}$$

To express it briefly, for every $c \in [-1,1]$, define the mean $K_c \colon I^2 \to I$ by

$$K_c(x,y) := \begin{cases} \frac{x+y}{2}, & |x-y| \le 1, \\ \frac{x+y+c}{2}, & |x-y| > 1. \end{cases}$$

Having this new notation, we can simply write $\mathcal{L} = K_{-1}$, $\mathcal{U} = K_1$, and $\mathcal{T} = K_0$.

If we now continue this in inductive steps, we get $A_{\omega+1} = B_{\omega+1} = K_0 = \mathcal{T}$. Thus (in this example) the sequences $(A_\alpha)_{\alpha \ge \omega}$ and $(B_\alpha)_{\alpha \ge \omega}$ contain the lower-, upper-, and transfinite- invariant means only.

On the other hand, every convex combination of invariant means is again an invariant mean. Thus K_c is (M,N)-invariant for all $c \in [-1,1]$. This shows that not every (M,N)-invariant mean is obtained in sequences (A_α), (B_α).

3.2. Application to functional equations. There appears a natural problem: which results known for continuous means can be adapted to the discontinuous setting?

In this section we are going to prove just a single result inspired by Matkowski [17, Theorem 4].

Proposition 2. *Let $M, N\colon I^2 \to I$ be two means with $M \le N$, having property* (1.1)*, and $\Phi\colon I^2 \to \mathbb{R}$ be a continuous function. Then*

$$\Phi(x, y) = \Phi(M(x, y), N(x, y)) \qquad \text{for all } x,\ y \in I \tag{3.3}$$

if and only if there exists a continuous function $f\colon I \to \mathbb{R}$ such that

$$\Phi = f \circ \mathcal{T}.$$

Moreover, if $x \mapsto \Phi(x, x)$ is an injective function, then $\mathcal{T}$ is continuous.

Recall that, like in many other results, comparability may be replaced by symmetry.

Proof. Take x, $y \in I$ arbitrarily. Using (2.3), equality (3.3) can be rewritten as $\Phi(x_\alpha, y_\alpha) = \Phi(x_{\alpha+1}, y_{\alpha+1})$ for every α.

By continuity of Φ, we may extend the inductive proof to limit ordinals and obtain $\Phi(x_\alpha, y_\alpha) = \Phi(x_0, y_0)$ for every α. If we put $\alpha = \omega_1$, by (2.5), we obtain

$$\Phi(\mathcal{T}(x, y), \mathcal{T}(x, y)) = \Phi(x, y). \tag{3.4}$$

To complete the first implication, we can simply define $f(x) := \Phi(x, x)$.

The converse implication is immediate in view of (M, N)-invariance of $\mathcal{T}$.

Additionally, if $x \mapsto \Phi(x, x)$ is injective, then so is f. In particular, f^{-1} exists and it is a continuous function.

Consequently, $\mathcal{T} = f^{-1} \circ \Phi$ is continuous, too. □

3.3. Conclusions. In this paper we discussed some invariant means which naturally emerged in a case of two noncontinuous mean, either comparable or both symmetric (sometimes additionally satisfying condition (1.1)).

There appear some natural problems concerning this new aspect. For example: (i) find out the 'noncontinuous counterpart' of results which are stated for continuous means; (ii) find out some additional assumption(s) to invariant mean which can be made in order to obtain the uniqueness of the solution (we presented three of those: minimality, maximality, and continuity); (iii) generalize this concept to multivariable means (it is relatively natural in case of $\mathcal{L}$ and $\mathcal{U}$ only).

Some progress toward (i) and (ii) was presented while the third aspect is outside the scope of the present paper.

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PAWEŁ PASTECZKA
INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY OF CRACOW
PODCHORĄŻYCH STR. 2
30-084 KRAKÓW
POLAND

E-mail: pawel.pasteczka@up.krakow.pl

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