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Local characterization of Jordan *-derivations on $\mathcal{B}(H)$

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Abstract. Let H be an infinite-dimensional real Hilbert space, and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. Assume that $\delta : \mathcal{B}(H) \to \mathcal{B}(H)$ is a real linear map and $P \in \mathcal{B}(H)$ is zero, or the unit element, or a nontrivial idempotent with infinite-dimensional range and infinite-dimensional kernel. It is shown that δ satisfies $\delta(A^2) = \delta(A)A^* + A\delta(A)$ for all $A \in \mathcal{B}(H)$ with $A^2 = P$ if and only if δ is an inner Jordan *-derivation. An example is also given to illustrate that this is not necessarily true when H is finite-dimensional.

1. Introduction

An additive map $x \mapsto x^*$ on a ring $\mathcal{R}$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ is called an involution. A ring $\mathcal{R}$ equipped with an involution is called a *-ring. The pioneering paper [16] settled the definition and first results on Jordan *-derivations. An additive map $\delta : \mathcal{R} \to \mathcal{R}$ is called a Jordan *-derivation if the identity $\delta(a^2) = \delta(a)a^* + a\delta(a)$ holds for all $a \in \mathcal{R}$. It is easily verified that, for every $r \in \mathcal{R}$, the map δ defined by $\delta(a) = ar - ra^* = [a, r]_*$ is a Jordan *-derivation; and such δ is called an inner Jordan *-derivation. Here we point out that the term is also used in the literature with another meaning. For example, if $\mathcal{A}$ is a C*-algebra, a *-derivation (respectively, a Jordan *-derivation) on $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ satisfying $D(ab) = D(a)b + aD(b)$ (respectively,

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$D(a^2) = D(a)a + aD(a))$ and $D(a^*) = D(a)^*$ for all $a, b \in \mathcal{A}$. A similar definition is also valid when $\mathcal{A}$ is a JB*-algebra.

The study of Jordan *-derivations has been motivated by the problem of the representability of quasi-quadratic functionals by sesquilinear ones. It turns out that the question of whether each quasi-quadratic functional is generated by some sesquilinear functional is intimately connected with the structure of Jordan *-derivations (see [9], [14]–[15], and the references therein). Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on a real or complex Hilbert space H with $\dim H > 1$, and let $\mathcal{A}$ be a standard operator algebra on H. Šemrl proved in [16] that every additive Jordan *-derivation $\delta : \mathcal{A} \to \mathcal{B}(H)$ is of the form $\delta(A) = AT - TA^*$ for some $T \in \mathcal{B}(H)$. Recently, Lee, Wong and Zhou [10]–[11] gave a characterization of additive Jordan *-derivations on noncommutative prime *-rings. For other related results, see [2]–[3], [6], and the references therein.

A wide number of papers have been devoted to study conditions under which Jordan derivations of operator algebras can be completely determined by the action on some sets of operators. For example, Jing in [8] proved that if a linear map δ on $\mathcal{B}(H)$ (here H is an infinite-dimensional complex Hilbert space) satisfies $\delta(A^2) = \delta(A)A + A\delta(A)$ for each operator $A \in \mathcal{B}(H)$ with $A^2 = 0$ and $\delta(I) = 0$ (resp. $\delta(A^2) = \delta(A)A + A\delta(A)$ for each operator $A \in \mathcal{B}(H)$ with $A^2 = I$), then there exists a bounded linear operator T on H such that $\delta(A) = TA - AT$ for each $A \in \mathcal{B}(H)$. Dolinar *et al.* in [4] gave a characterization of linear maps satisfying $\delta(A^2) = \delta(A)A + A\delta(A)$ for each operator $A \in \mathcal{B}(H)$ with $A^2 = P$, where P is a fixed idempotent operator.

In some studies, such as in [13], the notion "derivable at" is used with a slightly different meaning. To avoid confusions, we shall employ the term "pairs-derivable". One of the mains results in [12] shows that if X is a Banach A-bimodule over a Banach algebra A, then a linear map $\delta : A \to X$ is a derivation whenever it is continuous and pairs-derivable at an element z which is left (or right) invertible in the sense that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$ with $ab = z$. Zhu, Xiong and Li showed that, for a Hilbert space H, a linear map $\delta : \mathcal{B}(H) \to \mathcal{B}(H)$ is a derivation if and only if it is pairs-derivable at a non-zero point in $\mathcal{B}(H)$ (see [17]). The identity on $\mathcal{B}(H)$ is pairs-derivable at zero but it is not a derivation. In a recent contribution (see [5]), Essaleh and Peralta proved that a continuous linear map T on a unital C*-algebra A is a generalized derivation (i.e. $T(ab) = T(a)b + aT(b) - aT(1)b$ for all $a, b \in A$) whenever it is a triple derivation at the unit element (i.e. $\delta(\{a, b, c\}) = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}$ for every $a, b, c \in A$ with $\{a, b, c\} = 1$, where $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$). If under the same hypothesis we also assume that $T(1) = 0$, then T is a derivation, a symmetric map and a triple derivation. Furthermore, a continuous linear map on a unital C*-algebra which is a triple derivation at the unit element is a triple derivation. Similar conclusions are obtained for continuous linear maps which are derivations or triple derivations at zero. The same authors establish an automatic continuity result by showing that generalized derivations on a von Neumann algebra and linear maps on a von Neumann algebra which are pairs-derivable or triple derivations at zero are all continuous.

Motivated by these, we say that a map δ on a *-ring $\mathcal{R}$ is Jordan *-derivable at a point $Z \in \mathcal{R}$ if $\delta(A^2) = \delta(A)A^* + A\delta(A)$ holds for all $A \in \mathcal{R}$ with $A^2 = Z$. The purpose of the present paper is to give a characterization of real linear maps Jordan *-derivable at an idempotent operator in $\mathcal{B}(H)$. Note that if $\mathcal{R}$ is 2-torsion free, then a Jordan *-derivation on $\mathcal{R}$ can be equivalently defined by the identity $\delta(AB + BA) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A)$. In [13], the authors studied additive maps δ on prime *-ring $\mathcal{R}$ that satisfy $\delta(AB + BA) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A)$ whenever $AB = 0$ for $A, B \in \mathcal{R}$. We remark here that the question of characterizing maps Jordan *-derivable at some point Z is relatively more difficult than the question of characterizing maps satisfying $\delta(AB + BA) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A)$ for any A, B with $AB = Z$, since it is more difficult to find A satisfying $A^2 = Z$.

The paper is organized as follows. In Section 2, we give some useful propositions and lemmata, which are of independent interest. Let H be an infinite-dimensional real Hilbert space, and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. Assume that $\delta : \mathcal{B}(H) \to \mathcal{B}(H)$ is a real linear map. In Section 3, we show that δ is Jordan *-derivable at zero (respectively, at the unit operator, or at an idempotent operator with infinite rank and co-rank) if and only if δ is an inner Jordan *-derivation (Theorems 3.1–3.3). There is a counterexample to illustrate that the results may not be true when H is finite-dimensional (Remark 3.4).

Finally let us fix some notations. For any Hilbert space H, denote by $\mathcal{F}(H)$ the set of all finite rank linear operators in $\mathcal{B}(H)$. An operator $P \in \mathcal{B}(H)$ is an idempotent operator if $P^2 = P$ and is a square zero operator if $P^2 = 0$. For any $A \in \mathcal{B}(H)$, $\operatorname{ran} A$ and $\ker A$ stand for the range and the kernel of A, respectively.

2. Preliminaries

In this section, we will give some propositions and lemmata, which are useful in the proofs of the main theorems and are of independent interest.

Proposition 1. *Let $\mathcal{A} = M_n(\mathbb{R})$ be the algebra of all $n \times n$ matrices over the* *real field $\mathbb{R}$ with $n \geq 2$, and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Assume that $\delta : \mathcal{A} \to \mathcal{M}$ is an $\mathbb{R}$-linear map. If δ satisfies $\delta(P) = \delta(P)P^* + P\delta(P)$ for all idempotent operators $P \in \mathcal{A}$, then $\delta(A^2) = \delta(A)A^* + A\delta(A)$ holds for all $A \in \mathcal{A}$. Furthermore, there exists some $T \in \mathcal{M}$ such that $\delta(A) = AT - TA^*$ for all $A \in \mathcal{A}$.*

Proof. It should be acknowledged that the proof of Proposition 1 owes too much to the ideas in [16]. For the sake of completeness, we will give a brief proof here.

Define a multiplication in $\mathcal{B} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{A} \end{pmatrix}$ by

$$\begin{pmatrix} A_1 & M_1 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} A_2 & M_2 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 & A_1M_2 + M_1B_2 \\ 0 & B_1B_2 \end{pmatrix}.$$

It is easy to check that $\mathcal{B}$ becomes an algebra over $\mathbb{R}$. Define an $\mathbb{R}$-linear map $\phi : \mathcal{A} \to \mathcal{B}$ by

$$\phi(A) = \begin{pmatrix} A & \delta(A) \\ 0 & A^* \end{pmatrix} \quad \text{for all } A \in \mathcal{A}.$$

By the assumption about δ and the definition of ϕ, we see that

$$\phi(P)^2 = \phi(P) \text{ holds for all idempotent operators } P \in \mathcal{A}.$$

By [1, Theorem 2.1], ϕ is a Jordan homomorphism, that is, $\phi(A^2) = \phi(A)^2$ for all $A \in \mathcal{A}$. It follows from the definition of ϕ that the identity $\delta(A^2) = \delta(A)A^* + A\delta(A)$ holds for all $A \in \mathcal{A}$.

Moreover, by [1, Theorem 2.1], ϕ is an orthogonal sum of φ and ψ, where $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism and $\psi : \mathcal{A} \to \mathcal{B}$ is an antihomomorphism. So we can respectively write φ and ψ as

$$\varphi(A) = \begin{pmatrix} \varphi_1(A) & \varphi_2(A) \\ 0 & \varphi_3(A) \end{pmatrix}, \qquad \psi(A) = \begin{pmatrix} \psi_1(A) & \psi_2(A) \\ 0 & \psi_3(A) \end{pmatrix}.$$

By a similar argument to that in [16], one can show that $\varphi_3 \equiv 0$ and $\psi_3(A) = A^*$ for all $A \in \mathcal{A}$. So

$$\varphi(A) = \begin{pmatrix} \varphi_1(A) & \varphi_2(A) \\ 0 & 0 \end{pmatrix}, \qquad \psi(A) = \begin{pmatrix} 0 & \psi_2(A) \\ 0 & A^* \end{pmatrix}.$$

By the properties of φ and ψ, it is easily checked that φ_2 and ψ_2 are $\mathbb{R}$-linear maps satisfying

$$\varphi_2(AB) = A\varphi_2(B) \quad \text{and} \quad \psi_2(AB) = \psi_2(B)A^* \quad \text{for all } A, B \in \mathcal{A}.$$

Particularly, we get

$$\varphi_2(A) = A\varphi_2(I) \quad \text{and} \quad \psi_2(A) = \psi_2(I)A^* \quad \text{for all } A \in \mathcal{A}.$$

In addition, note that $\delta(A) = \varphi_2(A) + \psi_2(A)$ for all $A \in \mathcal{A}$; and $\delta(I) = 0$ as $\delta(I) = \delta(I)I^* + I\delta(I)$. It follows that $\varphi_2(I) + \psi_2(I) = 0$. Now, let $T = \varphi_2(I) \in \mathcal{M}$; then $\delta(A) = AT - TA^*$, as desired. □

Lemma 2.1 ([7, Lemma 7]). *Let H be an infinite-dimensional Hilbert space. Then every operator $A \in \mathcal{B}(H)$ is a sum of a finite number of idempotent operators in $\mathcal{B}(H)$, and is also a finite sum of square zero operators in $\mathcal{B}(H)$.*

By using Lemma 2.1, we can prove the following proposition.

Proposition 2. *Let H be an infinite-dimensional Hilbert space over the field $\mathbb{F}$ of all real or complex numbers. Assume that $\delta : \mathcal{B}(H) \to \mathcal{B}(H)$ is an additive map. If δ satisfies $\delta(N)N^* + N\delta(N) = 0$ for $N \in \mathcal{B}(H)$ with $N^2 = 0$, then $\delta(P) = \delta(P)P^* + P\delta(P)$ holds for all $P \in \mathcal{B}(H)$ with $P^2 = P$.*

Proof. The proof is similar to that of [8, Theorem 2.2]. Here, we will give a sketch of the proof.

Pick any idempotent operator $P \in \mathcal{B}(H)$ with infinite-dimensional range and infinite-dimensional kernel. Write $P_1 = P$ and $P_2 = I - P$. We first give a claim.

Claim. For any operators $A, B \in \mathcal{B}(H)$ satisfying $P_1AP_1 = A$ and $P_2BP_2 = B$, we have $\delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A) = 0$.

Take any operators $A, B \in \mathcal{B}(H)$ with $P_1AP_1 = A$ and $P_2BP_2 = B$. Under the space decomposition $H = P_1H \oplus P_2H$, A and B have the forms $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix}$, where $A_1 \in \mathcal{B}(P_1H)$ and $B_2 \in \mathcal{B}(P_2H)$. By using Lemma 2.1 to A_1 and B_2, one can find square zero operators $N_k, M_l \in \mathcal{B}(H)$ with $N_k + M_l$ still square-zero operators ($k = 1, \dots, n$; $l = 1, \dots, m$) such that $A = \sum_{k=1}^{n} N_k$ and $B = \sum_{l=1}^{m} M_l$. By the assumption, one can get

$$\delta(N_k)M_l^* + \delta(M_l)N_k^* + N_k\delta(M_l) + M_l\delta(N_k) = 0, \quad k = 1, \dots, n;\ l = 1, \dots, m.$$

Hence, by the additivity of δ, the claim is true.

Now, take any idempotent operator $P \in \mathcal{B}(H)$. If $\dim \operatorname{ran} P = \dim \ker P = \infty$, applying Claim to P and $I - P$, we have

$$2\delta(P) = 2\delta(P)P^* + 2P\delta(P) - P\delta(I) - \delta(I)P^*. \tag{2.1}$$

Multiplying by P and P^* from the left and the right in equation (2.1), respectively, one can obtain $P\delta(I) = \delta(I)P^*$. So equation (2.1) reduces to $\delta(P) = \delta(P)P^* + P\delta(P) - \delta(I)P^*$.

If $\dim \operatorname{ran} P < \infty$, choose two idempotent operators $R_1, R_2 \in \mathcal{B}(H)$ such that $\dim \operatorname{ran} R_1 = \dim \operatorname{ran} R_2 = \infty$, $PR_i = R_iP = 0$ $(i = 1, 2)$, $R_1R_2 = R_2R_1 = 0$ and $I = P + R_1 + R_2$. Obviously, $\dim \ker R_1 = \dim \ker R_2 = \infty$. From Claim and what has been proved above, one can easily show that $\delta(P) = \delta(P)P^* + P\delta(P) - P\delta(I) = \delta(P)P^* + P\delta(P) - \delta(I)P^*$.

If $\dim \ker P < \infty$, a similar argument still can achieve $\delta(P) = \delta(P)P^* + P\delta(P) - \delta(I)P^*$ and $P\delta(I) = \delta(I)P^*$.

To sum up, we have proved that $\delta(P) = \delta(P)P^* + P\delta(P) - \delta(I)P^*$ and $\delta(I)P^* = P\delta(I)$ hold for all $P \in \mathcal{B}(H)$ with $P^2 = P$.

To complete the proof, one still needs to check $\delta(I) = 0$. In fact, by Lemma 2.1, $\delta(I)A^* = A\delta(I)$ holds for all $A \in \mathcal{B}(H)$. Particularly, $\delta(I)S = S\delta(I)$ holds for all self-adjoint operators $S \in \mathcal{B}(H)$. This implies $\delta(I)A = A\delta(I)$ for each $A \in \mathcal{B}(H)$, and so $\delta(I) = \lambda I$ for some scalar λ. It follows from $\delta(I)A^* = A\delta(I)$ that λ must be zero, as desired. □

3. Real linear maps Jordan *-derivable at an idempotent operator

In this section, we will give a characterization of real linear maps which are Jordan *-derivable at an idempotent operator P.

If P is a trivial idempotent operator, that is, $P = 0, I$, we have

Theorem 3.1. *Let H be an infinite-dimensional real Hilbert space. Assume that $\delta : \mathcal{B}(H) \to \mathcal{B}(H)$ is a real linear map. Then δ is Jordan *-derivable at zero, that is, δ satisfies $\delta(N)N^* + N\delta(N) = 0$ for all $N \in \mathcal{B}(H)$ with $N^2 = 0$, if and only if there exists some $T \in \mathcal{B}(H)$ such that $\delta(A) = AT - TA^*$ for all $A \in \mathcal{B}(H)$, that is, δ is an inner Jordan *-derivation.*

Theorem 3.2. *Let H be an infinite-dimensional real Hilbert space. Assume that $\delta : \mathcal{B}(H) \to \mathcal{B}(H)$ is a real linear map. Then δ is Jordan *-derivable at I if and only if there exists some $T \in \mathcal{B}(H)$ such that $\delta(A) = AT - TA^*$ for all $A \in \mathcal{B}(H)$.*

For a nontrivial idempotent operator, we have

Theorem 3.3. *Let H be an infinite-dimensional real Hilbert space. Assume that $\delta : \mathcal{B}(H) \to \mathcal{B}(H)$ is a real linear map and $P \in \mathcal{B}(H)$ is any fixed nontrivial idempotent with infinite-dimensional range and infinite-dimensional kernel. Then*

*δ is Jordan *-derivable at P if and only if there exists some $T \in \mathcal{B}(H)$ and a real linear map $f : \mathcal{B}(H) \to \mathcal{B}(H)$ such that $\delta(X) = XT - TX^* + f(PX(I-P)) + f((I-P)XP)$ for all $X \in \mathcal{B}(H)$.*

Remark 3.1. We remark that Theorems 3.1–3.3 may not be true on finite-dimensional Hilbert spaces.

Let H be the real Hilbert space $\mathbb{R}^2$ with a standard basis $\{e_1, e_2\}$. Then $\mathcal{B}(H)$ can be identified with the space $M_2(\mathbb{R})$ of 2×2 real matrices. For any $A \in M_2(\mathbb{R})$, if $A^2 = 0$, then A is of rank at most one, and so it can be written as

$$A = \begin{pmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{pmatrix} \in M_2(\mathbb{R}) \quad \text{with } a_1b_1 + a_2b_2 = 0.$$

Define a map δ on $M_2(\mathbb{R})$ by

$$\delta \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ -2a_{11} & -a_{21} \end{pmatrix} \quad \text{for all} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{R}).$$

Clearly, δ is real linear. In addition, it is not difficult to check that δ is Jordan *-derivable at the idempotent $P = 0$, that is, $\delta(N^2) = \delta(N)N^* + N\delta(N)$ for any $N \in M_2(\mathbb{R})$ with $N^2 = 0$. However, if there exists some $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in M_2(\mathbb{R})$ such that $\delta(A) = AT - TA^*$ for all $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{R})$, then T must satisfy the conditions $a_{12} = a_{21}(t_{21} - t_{12})$ and $a_{21} = a_{12}(t_{21} - t_{12})$, which is impossible. So δ cannot be of the form $A \mapsto [A, T]_*$.

Now we are at a position to give our proofs of Theorems 3.1–3.3.

Proof of Theorem 3.1. The "if" part is obvious. For the "only if" part, by Proposition 2, we have

$$\delta(P) = \delta(P)P^* + P\delta(P) \quad \text{for all idempotent operators } P \in \mathcal{B}(H). \tag{3.1}$$

Pick any $A, B \in \mathcal{F}(H)$. Then there is a finite rank projection $Q \in \mathcal{F}(H)$ such that $QAQ = A$ and $QBQ = B$. Let $\{x_1, x_2, \dots, x_n\}$ be a basis of $\operatorname{ran} Q$, and let $\mathcal{C} \subseteq \mathcal{F}(H)$ be the algebra of all operators C of the form $C = \sum_{i,j=1}^n t_{ij} x_i \otimes x_j$, $t_{ij} \in \mathbb{R}$, and note that $\mathcal{C}$ is isomorphic to $M_n(\mathbb{R})$ via the isomorphism $C \mapsto (t_{ij})_{n \times n}$. Thus, for the restriction of δ to $\mathcal{C}$, Proposition 1 entails that there exists some $S \in \mathcal{B}(H)$ depending on n such that $\delta(A) = AS - SA^*$ for all $A \in \mathcal{F}(H)$ with

$QAQ = A$, which implies that $\delta|_{\mathcal{F}(H)}$ (i.e. the restriction of δ to $\mathcal{F}(H)$) is a Jordan *-derivation. Now, by [16], there exists some $T \in \mathcal{B}(H)$ such that

$$\delta(A) = AT - TA^* \quad \text{holds for all } A \in \mathcal{F}(H). \tag{3.2}$$

Next, take any idempotent operator $P \in \mathcal{B}(H)$ and any rank-one idempotent operator $Q \in \mathcal{B}(H)$. If $PQ = QP = 0$, then $(P+Q)^2 = P+Q$. By equation (3.1), we have $\delta((P+Q)^2) = \delta(P+Q)(P+Q)^* + (P+Q)\delta(P+Q)$, that is,

$$\delta(P)Q^* + P\delta(Q) + \delta(Q)P^* + Q\delta(P) = 0.$$

Since Q is rank-one, by equation (3.2), the equation implies

$$\delta(P)Q^* - PTQ^* + QTP^* + Q\delta(P) = 0,$$

and so $Q\delta(P)Q^* = 0$. Hence

$$[\delta(P) - PT + TP^*]Q^* = PTQ^* - QTP^*Q^* - Q\delta(P)Q^* - PTQ^* + TP^*Q^* = 0. \tag{3.3}$$

If $PQ = QP = Q$, then $(P-Q)^2 = P - Q$ and $(P-Q)Q = Q(P-Q) = 0$. By what has been proved, one gets $[\delta(P-Q) - (P-Q)T + T(P-Q)^*]Q^* = 0$, which and equation (3.2) yield

$$[\delta(P) - PT + TP^*]Q^* = 0. \tag{3.4}$$

Combining equations (3.3)–(3.4) entails that $[\delta(P) - PT + TP^*]Q^* = 0$ holds for each rank-one idempotent $Q \in \mathcal{B}(H)$, which entails $\delta(P) = PT - TP^*$ for all idempotent operators $P \in \mathcal{B}(H)$.

Finally, by Lemma 2.1, every operator in $\mathcal{B}(H)$ can be written as a finite sum of idempotents. Hence $\delta(A) = AT - TA^*$ holds for all $A \in \mathcal{B}(H)$, completing the proof. □

Proof of Theorem 3.2. For any idempotent operator $P \in \mathcal{B}(H)$, it is obvious that $(I - 2P)^2 = I$. By the assumption, one has

$$\delta(I) = \delta(I - 2P)(I - 2P)^* + (I - 2P)\delta(I - 2P).$$

Note that $\delta(I) = 0$, as $\delta(I) = \delta(I)I^* + I\delta(I)$. The above equation reduces to $\delta(P) = \delta(P)P^* + P\delta(P)$ for all idempotent operators $P \in \mathcal{B}(H)$.

Now, by a similar proof to that of Theorem 3.1, one can check that the theorem is true. □

Proof of Theorem 3.3. For the "if" part, assume that $A^2 = P$. Then $AP = AA^2 = A^2A = PA$, and so $(I-P)AP = PA(I-P) = 0$. Note that $f(0) = 0$ by the additivity of f. Hence

$$\begin{aligned}\delta(A)A^* + A\delta(A) &= (AT - TA^* + f(PA(I-P)) + f((I-P)AP))A^* \\ &\quad + A(AT - TA^* + f(PA(I-P)) + f((I-P)AP)) \\ &= A^2T - T(A^2)^* = PT - TP^* = \delta(P).\end{aligned}$$

That is, δ is Jordan *-derivable at P.

For the "only if" part, assume that δ satisfies $\delta(A)A^* + A\delta(A) = \delta(P)$ for any $A \in \mathcal{B}(H)$ with $A^2 = P$.

Denote by $H_1 = \operatorname{ran} P$ and $H_2 = \ker P$ the range and the kernel of P, respectively. For P and any $X \in \mathcal{B}(H)$, according to the space decomposition $H = H_1 \oplus H_2$, we may write $P = \begin{pmatrix} I_{H_1} & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ and $\delta(X) = \begin{pmatrix} \delta_{11}(X) & \delta_{12}(X) \\ \delta_{21}(X) & \delta_{22}(X) \end{pmatrix}$, where $X_{ij} \in \mathcal{B}(H_j, H_i)$ and $\delta_{ij} : \mathcal{B}(H) \to \mathcal{B}(H_j, H_i)$ are real linear maps.

Observe that $\delta(P) = \delta(P)P^* + P\delta(P)$. By a simple matrix computation, it is easy to see that

$$P\delta(P)P = (I-P)\delta(P)(I-P) = 0, \quad \text{that is, } \delta_{11}(P) = \delta_{22}(P) = 0. \tag{3.5}$$

Define two linear operators $\tau_{11} : \mathcal{B}(H_1) \to \mathcal{B}(H_1)$ and $\tau_{22} : \mathcal{B}(H_2) \to \mathcal{B}(H_2)$ respectively by

$$\tau_{11}(X_{11}) = \delta_{11}\begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau_{22}(X_{22}) = \delta_{22}\begin{pmatrix} I_{H_1} & 0 \\ 0 & X_{22} \end{pmatrix}$$

for each $X_{11} \in \mathcal{B}(H_1)$ and $X_{22} \in \mathcal{B}(H_2)$.

Take any operator $X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}$ with $X_{11}^2 = I_{H_1}$ and $X_{22}^2 = 0$. Since δ is real linear, by the definitions of τ_{11}, τ_{22} and noting that equation (3.5), τ_{11} and τ_{22} are also real linear. In addition, as δ is Jordan *-derivable at P, a simple calculation entails that τ_{11} is Jordan *-derivable at I_{H_1} and τ_{22} is Jordan *-derivable at 0. Note that both H_1 and H_2 are infinite-dimensional Hilbert spaces. So, by Theorems 3.1 and 3.2, there exist some $A_{11} \in \mathcal{B}(H_1)$ and $A_{22} \in \mathcal{B}(H_2)$ such that

$$\tau_{11}(X_{11}) = [X_{11}, A_{11}]_* \quad \text{and} \quad \tau_{22}(X_{22}) = [X_{22}, A_{22}]_*$$

for all $X_{11} \in \mathcal{B}(H_1)$ and all $X_{22} \in \mathcal{B}(H_2)$. Thus, for any

$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} = \begin{pmatrix} X_{11} - I_{H_1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I_{H_1} & 0 \\ 0 & X_{22} \end{pmatrix} \doteq X_1 + X_2 \in \mathcal{B}(H),$$

we have

$$\begin{aligned}
\delta(X) &= \delta(X_1) + \delta(X_2) \\
&= \begin{pmatrix} \delta_{11}(X_1) & 0 \\ 0 & \delta_{22}(X_2) \end{pmatrix} + \begin{pmatrix} \delta_{11}(X_2) & 0 \\ 0 & \delta_{22}(X_1) \end{pmatrix} + \begin{pmatrix} 0 & \delta_{12}(X) \\ \delta_{21}(X) & 0 \end{pmatrix} \\
&= \begin{pmatrix} \tau_{11}(X_{11} - I_{H_1}) & 0 \\ 0 & \tau_{22}(X_{22}) \end{pmatrix} + \begin{pmatrix} \delta_{11}(X_2) & 0 \\ 0 & \delta_{22}(X_1) \end{pmatrix} + \begin{pmatrix} 0 & \delta_{12}(X) \\ \delta_{21}(X) & 0 \end{pmatrix} \\
&= \begin{pmatrix} [X_{11}, A_{11}]_* & 0 \\ 0 & [X_{22}, A_{22}]_* \end{pmatrix} + \begin{pmatrix} \delta_{11}(X_2) & 0 \\ 0 & \delta_{22}(X_1) \end{pmatrix} + \begin{pmatrix} 0 & \delta_{12}(X) \\ \delta_{21}(X) & 0 \end{pmatrix}. \qquad (3.6)
\end{aligned}$$

Take any $A = \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix}$ with $A_1^2 = I_H$ and $N_2^2 = 0$. Then $A^2 = P$, and so

$$\delta \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix}^* + \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} \delta \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} = \delta(P). \qquad (3.7)$$

By using equations (3.5)–(3.6) to equation (3.7), we have

$$\delta_{11} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} A_1^* + A_1 \delta_{22} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} = 0, \qquad (3.8)$$

$$\delta_{12} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} N_2^* + A_1 \delta_{12} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} = \delta_{12}(P), \qquad (3.9)$$

$$\delta_{21} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} A_1^* + N_2 \delta_{21} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} = \delta_{21}(P) \qquad (3.10)$$

and

$$\delta_{22} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} N_2^* + N_2 \delta_{22} \begin{pmatrix} A_1 & 0 \\ 0 & N_2 \end{pmatrix} = 0 \qquad (3.11)$$

for all $A_1 \in \mathcal{B}(H_1)$ and all $N_2 \in \mathcal{B}(H_2)$ with $A_1^2 = I_{H_1}$ and $N_2^2 = 0$.

For equation (3.8), by taking $A_1 = I_{H_1}$, one easily gets $\delta_{11}\begin{pmatrix} I_{H_1} & 0 \\ 0 & N_2 \end{pmatrix} = 0$ for all square-zero operators $N_2 \in \mathcal{B}(H_2)$. Since $\ker P$ is infinite-dimensional, by Lemma 2.1, we have

$$\delta_{11}(X_2) = \delta_{11}\begin{pmatrix} I_{H_1} & 0 \\ 0 & X_{22} \end{pmatrix} = 0 \quad \text{for every } X_{22} \in \mathcal{B}(H_2). \tag{3.12}$$

For equation (3.9), letting $N_2 = 0$, one has $A_1\delta_{12}\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = \delta_{12}(P)$, and so

$$\delta_{12}\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = A_1\delta_{12}(P) \quad \text{for all } A_1 \in \mathcal{B}(H_1) \text{ with } A_1^2 = I_{H_1}.$$

Take any idempotent $Q \in \mathcal{B}(H_1)$. As $(I_{H_1} - 2Q)^2 = I_{H_1}$, the above equation implies

$$\delta_{12}\begin{pmatrix} I_{H_1} - 2Q & 0 \\ 0 & 0 \end{pmatrix} = (I_{H_1} - 2Q)\delta_{12}(P),$$

that is, $\delta_{12}\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} = Q\delta_{12}(P)$. Note that each operator is a sum of finite idempotents (Lemma 2.1). Hence

$$\delta_{12}\begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} = X_{11}\delta_{12}(P) \quad \text{for all } X_{11} \in \mathcal{B}(H_1). \tag{3.13}$$

Taking $A_1 = I_{H_1}$ in equation (3.9), one gets

$$\delta_{12}(P)N_2^* + \delta_{12}\begin{pmatrix} 0 & 0 \\ 0 & N_2 \end{pmatrix} N_2^* + \delta_{12}(P) + \delta_{12}\begin{pmatrix} 0 & 0 \\ 0 & N_2 \end{pmatrix} = \delta_{12}(P).$$

Multiplying by N_2^* from the right in the above equation, we have $\delta_{12}\begin{pmatrix} 0 & 0 \\ 0 & N_2 \end{pmatrix} N_2^* = 0$, and so $\delta_{12}\begin{pmatrix} 0 & 0 \\ 0 & N_2 \end{pmatrix} = -\delta_{12}(P)N_2^*$ for all square-zero operators N_2.

By Lemma 2.1 again, one obtains

$$\delta_{12}\begin{pmatrix} 0 & 0 \\ 0 & X_{22} \end{pmatrix} = -\delta_{12}(P)X_{22}^* \quad \text{for all operators } X_{22} \in \mathcal{B}(H_2). \tag{3.14}$$

Combining equation (3.13) and (3.14), we achieve that

$$\delta_{12}\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} = X_{11}\delta_{12}(P) - \delta_{12}(P)X_{22}^* \quad \text{for all } X_{11} \in \mathcal{B}(H_1),\ X_{22} \in \mathcal{B}(H_2). \tag{3.15}$$

Likewise, by an argument for equation (3.10), one can show that

$$\delta_{21}\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} = -X_{22}\delta_{21}(P) + \delta_{21}(P)X_{11}^* \quad \text{for all } X_{11} \in \mathcal{B}(H_1),\ X_{22} \in \mathcal{B}(H_2). \tag{3.16}$$

Note that equation (3.11) can be written as

$$\delta_{22}\begin{pmatrix} A_1 - I_{H_1} & 0 \\ 0 & 0 \end{pmatrix} N_2^* + \delta_{22}\begin{pmatrix} I_{H_1} & 0 \\ 0 & N_2 \end{pmatrix} N_2^*$$
$$+ N_2\delta_{22}\begin{pmatrix} A_1 - I_{H_1} & 0 \\ 0 & 0 \end{pmatrix} + N_2\delta_{22}\begin{pmatrix} I_{H_1} & 0 \\ 0 & N_2 \end{pmatrix} = 0,$$

and equation (3.11) also implies $\delta_{22}\begin{pmatrix} I_{H_1} & 0 \\ 0 & N_2 \end{pmatrix} N_2^* + N_2\delta_{22}\begin{pmatrix} I_{H_1} & 0 \\ 0 & N_2 \end{pmatrix} = 0.$ Thus

$$\delta_{22}\begin{pmatrix} A_1 - I_{H_1} & 0 \\ 0 & 0 \end{pmatrix} N_2^* + N_2\delta_{22}\begin{pmatrix} A_1 - I_{H_1} & 0 \\ 0 & 0 \end{pmatrix} = 0$$

holds for all $A_1 \in \mathcal{B}(H_1)$ and all $N_2 \in \mathcal{B}(H_2)$ with $A_1^2 = I_{H_1}$ and $N_2^2 = 0$. By Lemma 2.1, one can easily see that

$$\delta_{22}\begin{pmatrix} A_1 - I_{H_1} & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{holds for all } A_1 \in \mathcal{B}(H_1) \text{ with } A_1^2 = I_{H_1}.$$

Thus, by a similar argument to that of equation (3.13), we obtain

$$\delta_{22}\begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{for all } X_{11} \in \mathcal{B}(H_1). \tag{3.17}$$

Now, let $T = \begin{pmatrix} A_{11} & \delta_{12}(P) \\ -\delta_{21}(P) & A_{22} \end{pmatrix}$. Then, combining equation (3.12) and equations (3.15)–(3.17), one has proved that

$$\delta(X) = XT - TX^* \quad \text{for all } X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \in \mathcal{B}(H).$$

Hence, for any $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathcal{B}(H)$, we have

$$\begin{aligned} \delta(X) &= \delta \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} + \delta \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \\ &= XT - TX^* - \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} T + T \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}^* + \delta \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}. \end{aligned}$$

Define a map $f : \mathcal{B}(H) \to \mathcal{B}(H)$ by $f(X) = \delta(X) - (XT - TX^*)$ for all $X \in \mathcal{B}(H)$. It is obvious that f is real linear and $\delta(X) = XT - TX^* + f(PX(I-P)) + f((I-P)XP)$ for all $X \in \mathcal{B}(H)$, completing the proof. □

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