

On the zeros of reciprocal polynomials

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Dedicated to Professor Zoltán Daróczy on the occasion of his 80th birthday

Abstract. The purpose of this paper is to study reciprocal polynomials whose zeros are located in certain subsets of the complex plane. Of particular interest are the half planes $\Re z < 0$, $\Re z > 0$, the positive and negative half-lines and the unit circle. Our main tool is the Chebyshev transform (see, e.g., LAKATOS [8]) and a Viéta-like formula for reciprocal polynomials (see LOSONCZI [12]). Using these, we find necessary conditions, in some cases necessary and sufficient conditions for the reciprocal polynomials to have their zeros in the above sets.

1. Introduction

The aim of this paper is to study reciprocal polynomials whose zeros are located in certain subsets of the complex plane. Of particular interest are the half planes $\Re z < 0$, $\Re z > 0$, the positive and negative half-lines and the unit circle. Our main tool is the Chebyshev transform (see e.g., LAKATOS [8]) and a Viéta-like formula for reciprocal polynomials (see, LOSONCZI [12]). Using these we find necessary conditions, in some cases necessary and sufficient conditions for the reciprocal polynomials to have their zeros in the above sets.

There is an extensive literature dealing with polynomials all of whose zeros are on the unit circle. There are *necessary and sufficient conditions* by COHN [2] (see also [13, p. 14, Theorem 2.1.6]), and several *sufficient conditions*

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(see CHEN [1], DiPIPO and HOWE [3], Lakatos [8], SCHINZEL [16], LAKATOS and LOSONCZI [9]–[11], KIM and PARK [5], PETERSEN and SINCLAIR [15], SINCLAIR and VAALER [17], KWON [6]–[7]). VIERA [18] found sufficient conditions for the coefficients of self-inversive polynomials to have a definite number of zeros on the unit circle. The book of OBRESCHKOFF [14] contains a number of results on polynomials with real (positive or negative) zeros.

Definition 1. A polynomial $P_m(z) = \sum_{j=0}^m a_j z^j$, where $m \in \mathbb{N}$, $a_m \neq 0$, $a_0, \dots, a_m \in \mathbb{C}$ and

$$a_j = a_{m-j} \quad (j = 0, \dots, m),$$

is called a *reciprocal polynomial* of degree m .

A reciprocal polynomial P_{2n} of degree $2n$ ($n \in \mathbb{N}$) can be written as

$$P_{2n}(z) = \sum_{k=0}^{n-1} a_k (z^{2n-k} + z^k) + a_n z^n = z^n \left[a_0 \left(z^n + \frac{1}{z^n} \right) + \dots + a_{n-1} \left(z + \frac{1}{z} \right) + a_n \right],$$

which shows that if β is a zero of P_{2n} , then so is $\frac{1}{\beta}$. Hence in the factorization of this polynomial, there are factors of the form

$$(z - \beta) \left(z - \frac{1}{\beta} \right) = z^2 - \alpha z + 1$$

where $\alpha = \beta + \frac{1}{\beta}$. Arranging the zeros of P_{2n} into pairs $(\beta_k, \frac{1}{\beta_k})$, $k = 1, \dots, n$ and multiplying the corresponding factors, we obtain

$$P_{2n}(z) = a_0 \prod_{k=1}^n (z^2 - \alpha_k z + 1),$$

with $\alpha_k = \beta_k + \frac{1}{\beta_k} \in \mathbb{C}$.

We remark that the polynomial $\tilde{P}_{2n}(x) := a_0 \prod_{k=1}^n (x - \alpha_k)$ is exactly the Chebyshev transform $\mathcal{T}P_{2n}$ of P_{2n} (concerning Chebyshev transforms, see Lakatos [8]).

2. Characterizations

We would like to characterize reciprocal polynomials whose zeros are located in some subsets D of the complex plane. We can deal with subsets for which

- (i) $0 \neq \beta \in D$ implies $\frac{1}{\beta} \in D$ and $\alpha = \beta + \frac{1}{\beta} \in D^*$;
- (ii) $0 \neq \beta \in D$ if and only if $\alpha = \beta + \frac{1}{\beta} \in D^*$.

Lemma 1. *Each of the following pairs (D_j, D_j^*) , $(j = 1, \dots, 6)$ satisfies conditions (i), (ii):*

- $D_1 = D_1^* = \{z \in \mathbb{C} : \Re z < 0\}$;
- $D_2 = D_2^* = \{z \in \mathbb{C} : \Re z > 0\}$;
- $D_3 = \mathbb{R}_+ =]0, \infty]$, $D_3^* = [2, \infty[$;
- $D_4 = \mathbb{R}_- =]-\infty, 0[$, $D_4^* =]-\infty, -2[$;
- $D_5 = \mathbb{R}$, $D_5^* =]-\infty, -2] \cup [2, \infty[$;
- $D_6 = \{z \in \mathbb{C} : |z| = 1\}$, $D_6^* = [-2, 2]$;

where $\Re z$ denotes the real part of $z \in \mathbb{C}$.

PROOF. The proofs for (D_j, D_j^*) , $(j = 1, 2)$ are almost obvious, hence we omit them. The proofs for (D_j, D_j^*) , $(j = 3, 4, 5)$ are similar, we give the proof for $j = 3$ only.

If $0 \neq \beta \in D_3 = \mathbb{R}_+$, then $\frac{1}{\beta} \in \mathbb{R}_+$ and $\alpha = \beta + \frac{1}{\beta} \geq 2\sqrt{\beta \frac{1}{\beta}} = 2$, showing that $\alpha \in D_3^*$.

Conversely, suppose that $\alpha = \beta + \frac{1}{\beta} \geq 2$. Writing $\beta \neq 0$ as $\beta = |\beta|e^{ib}$ ($b \in \mathbb{R}$), we get

$$\alpha = \beta + \frac{1}{\beta} = |\beta|e^{ib} + \frac{1}{|\beta|}e^{-ib} = \left(|\beta| + \frac{1}{|\beta|}\right) \cos b + i \left(|\beta| - \frac{1}{|\beta|}\right) \sin b \in D_3^*,$$

hence $\left(|\beta| - \frac{1}{|\beta|}\right) \sin b = 0$. Then either $|\beta| - \frac{1}{|\beta|} = 0$ or $\sin b = 0$. In the first case, we conclude that $|\beta| = 1$, $\beta = e^{ib}$, $\alpha = \beta + \frac{1}{\beta} = 2 \cos b \geq 2$, hence $b = 2k\pi$, ($k \in \mathbb{Z}$) and $\beta = 1 \in D_3$. In the second case, $\sin b = 0$, $b = k\pi$, ($k \in \mathbb{Z}$) and $\beta = (-1)^k|\beta|$. Here k cannot be odd, since then $\beta < 0$ and $\alpha = -\left(-\beta + \frac{1}{-\beta}\right) \leq -2 < 0$, contradicting to our assumption. Hence k is even and $\beta > 0$.

Although the proof for (D_6, D_6^*) can be obtained by a slight modification of the proof of Lakatos [8, Lemma 1], for the sake of completeness we give its proof.

If $\beta \in D_6$, then $\beta = e^{ib}$ with a suitable $b \in \mathbb{R}$, hence $|\frac{1}{\beta}| = |e^{-ib}| = 1$, $\frac{1}{\beta} \in D_6$ and

$$\alpha = \beta + \frac{1}{\beta} = e^{ib} + e^{-ib} = 2 \cos b \in D_6^*.$$

Conversely, if $\alpha = \beta + \frac{1}{\beta} \in D_6^* = [-2, 2]$, then with $\beta = |\beta|e^{ib}$ ($b \in \mathbb{R}$) we get

$$\alpha = \beta + \frac{1}{\beta} = \left(|\beta| + \frac{1}{|\beta|}\right) \cos b + i \left(|\beta| - \frac{1}{|\beta|}\right) \sin b \in [-2, 2].$$

Hence either $|\beta| - \frac{1}{|\beta|} = 0$ or $\sin b = 0$. Arguing similarly as in the previous case ($j=3$), we conclude that either $\beta = e^{ib}$ thus $|\beta| = 1$, or $\beta = (-1)^k|\beta|$ and

$(|\beta| + \frac{1}{|\beta|})(-1)^k \in [-2, 2]$. From this we get that $|\beta| + \frac{1}{|\beta|} \leq 2$, $(|\beta| - 1)^2 \leq 0$, $|\beta| = 1$. In both cases $\beta \in D_6$, completing our proof. \square

An important consequence of this Lemma is

Theorem 1 (First Characterization Theorem). *A complex monic reciprocal polynomial*

$$P_{2n}(z) = \sum_{k=0}^{n-1} A_k(z^{2n-k} + z^k) + A_n z^n \quad (A_0 = 1, A_k \in \mathbb{C} \ (k = 1, \dots, n))$$

of even degree $2n$ has $0 \leq l \leq n$ pairs $(\beta_i, \frac{1}{\beta_i})$ of zeros in D_j if and only if in its factorization

$$P_{2n}(z) = \prod_{k=1}^n (z^2 - \alpha_k z + 1)$$

there are l values $\alpha_i \in D_j^*$ ($j = 1, \dots, 6$).

In the terms of Chebyshev transforms we can reformulate this theorem as follows.

Corollary 1. *The complex monic reciprocal polynomial P_{2n} of even degree $2n$ has $0 \leq l \leq n$ pairs $(\beta_i, \frac{1}{\beta_i})$ of zeros in D_j if and only if its Chebyshev transform*

$$\mathcal{T}P_{2n}(x) = \prod_{k=1}^n (x - \alpha_k)$$

has l zeros $\alpha_i \in D_j^*$ ($j = 1, \dots, 6$).

Corollary 1 can also be easily formulated for monic polynomials of odd degree. For odd degree polynomials, $z = -1$ is always a zero, hence $P_{2n+1}(z) = (z + 1)P_{2n}^*(z)$, where $P_{2n}^*(z)$ is also monic reciprocal, thus for P_{2n}^* Corollary 1 is applicable, and we can easily state the result analogous to Corollary 1.

We remark that there is a classical theorem of HURWITZ [4] which gives necessary and sufficient conditions (in terms of the coefficients) of an arbitrary polynomial (with real coefficients) to have all zeros in D_1 (or in D_2). As far as we know, these are the only cases when necessary and sufficient conditions in terms of the coefficients are known for a polynomial to have all zeros in D_j ($j = 1, \dots, 6$).

To formulate another characterization theorem, we need the Viéta-like formula proved in [12].

Lemma 2. For every $n \in \mathbb{N}$ and $z, \alpha_1, \dots, \alpha_n \in \mathbb{C}$, we have the identity

$$\prod_{k=1}^n (z^2 - \alpha_k z + 1) = \sum_{k=0}^{n-1} A_k (z^{2n-k} + z^k) + A_n z^n, \quad (1)$$

where

$$A_k = (-1)^k \sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \sigma_{k-2l}^{(n)} \quad (k = 0, 1, \dots, n), \quad (2)$$

and $\sigma_k^{(n)} = \sigma_k^{(n)}(\alpha_1, \dots, \alpha_n)$ ($k = 0, 1, \dots, n$) denotes the k -th elementary symmetric function of the variables $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\sigma_0^{(n)} = \sigma_0^{(n)}(\alpha_1, \dots, \alpha_n) := 1$.

By the help of this lemma, we can formulate a stronger characterization theorem.

Theorem 2 (Second Characterization Theorem). *A complex monic reciprocal polynomial*

$$P_{2n}(z) = \sum_{k=0}^{n-1} A_k (z^{2n-k} + z^k) + A_n z^n \quad (A_0 = 1, A_k \in \mathbb{C} \ (k = 1, \dots, n)) \quad (3)$$

of even degree $2n$ ($n \in \mathbb{N}$) has all of its zeros on the positive or negative half-line or the real line if and only if there exist real numbers $\alpha_k \in [2, \infty[$ ($k = 1, \dots, n$) or $\alpha_k \in]-\infty, -2]$ ($k = 1, \dots, n$) or $|\alpha_k| \geq 2$ ($k = 1, \dots, n$), respectively, such that

$$A_k = (-1)^k \sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \sigma_{k-2l}^{(n)}(\alpha_1, \dots, \alpha_n) \quad \text{for } k = 0, 1, \dots, n \quad (4)$$

holds.

A complex monic reciprocal polynomial $P_{2n+1}(z)$ of odd degree $2n+1$ ($n \in \mathbb{N}$) has all of its zeros on the negative half-line or the real line if and only if there exist real numbers $\alpha_k \in]-\infty, -2[$ ($k = 1, \dots, n$) or $|\alpha_k| \geq 2$ ($k = 1, \dots, n$), respectively, such that $P_{2n+1}(z) = (z+1)P_{2n}^*(z)$ and for the coefficients of the monic reciprocal polynomial P_{2n}^* the equations in (4) hold.

PROOF. For even degree polynomials, this follows from Lemmas 1 and 2. For odd degree polynomials, we have $P_{2n+1}(z) = (z+1)P_{2n}^*(z)$, where $P_{2n}^*(z)$ is also reciprocal, thus the first part of the statement applies. \square

3. Bounds for the coefficients, further necessary conditions

In this section, we find bounds for the coefficients of reciprocal polynomials all of whose zeros are either positive or negative, i.e., they are in one of the sets D_j ($j = 3, 4$). Unfortunately, if all roots are in D_j ($j = 1, 2, 5$), then by our method, we cannot find reasonable bounds or conditions for the coefficients. The case of D_6 , however, has already been dealt with in [12].

Theorem 3. *If all zeros of the complex monic reciprocal polynomial (3) of even degree are positive, then*

$$(-1)^k A_k \geq \binom{2n}{k} \quad (k = 0, 1, \dots, n). \quad (5)$$

In (5), there is always equality for $k = 0$. Moreover, if equality holds in (5) for some k , ($1 \leq k \leq n$), then equality holds for all $k = 0, 1, \dots, n$, and in this case $P_{2n}(x) = (x - 1)^{2n}$.

PROOF. We have $\alpha_j \geq 2$ ($j = 1, \dots, n$), hence using the second characterization theorem and the estimate

$$\sigma_k^{(n)}(\alpha_1, \dots, \alpha_n) \geq \binom{n}{k} 2^k, \quad (6)$$

we get from (2) for $k = 0, 1, \dots, n$,

$$\begin{aligned} (-1)^k A_k &= \sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \sigma_{k-2l}^{(n)}(\alpha_1, \dots, \alpha_n) \\ &\geq \sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \binom{n}{k-2l} 2^{k-2l} = \binom{2n}{k}. \end{aligned} \quad (7)$$

In the last step above we used the identity

$$\sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \binom{n}{k-2l} 2^{k-2l} = \binom{2n}{k}. \quad (8)$$

This can easily be obtained by expanding $(1+x)^{2n} = (1+2x+x^2)^n$ by the binomial and by the polynomial theorem, rearranging the sums according to the powers of x and comparing the coefficients of x^k (for details, see [12]).

From (7) it can be seen that if equality holds in (5) for some k , ($1 \leq k \leq n$), then equality holds for this k in (6). This is, however, possible only if $\alpha_j = 2$ for all $j = 1, \dots, n$, hence there is equality in (6) for all $k = 0, 1, \dots, n$ implying that $P_{2n}(x) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} x^k = (x - 1)^{2n}$ as claimed. \square

Remark 1. For $k = 1$, (5) gives $-A_1 \geq 2n$. This inequality can also be easily obtained from the usual Viéta formula applied for P_{2n} . In this case, the zeros of P_{2n} can be arranged into pairs $\beta_j, 1/\beta_j$ ($j = 1, \dots, n$) of positive numbers and by Viéta's formula

$$-A_1 = (\beta_1 + 1/\beta_1) + \dots + (\beta_n + 1/\beta_n) \geq 2\beta_1 1/\beta_1 + \dots + 2\beta_n 1/\beta_n = 2n.$$

For the other values of k , this is, however, not the case.

Remark 2. If (5) holds, then there are $2n$ sign changes in the sequence of coefficients of (3), hence by Descartes sign rule (see, e.g., [14, p. 54, Theorem 13.1]) our polynomial has $2n - 2l$ positive zeros where $0 \leq l \leq n$.

Theorem 4. *If all zeros of the complex monic reciprocal polynomial (3) of even degree are negative, then*

$$A_k \geq \binom{2n}{k} \quad (k = 0, 1, \dots, n). \quad (9)$$

If equality holds in (9) for some k , ($1 \leq k \leq n$), then equality holds for all $k = 0, 1, \dots, n$ and in this case $P_{2n}(x) = (x + 1)^{2n}$.

PROOF. If our polynomial is P_{2n} , then applying Theorem 3 for the polynomial $x \rightarrow P_{2n}(-x)$, we obtain Theorem 4. \square

4. Demonstration of the results by degree four reciprocal polynomials

Here we consider the reciprocal polynomial $p_4(z) := z^4 + A_1 z^3 + A_2 z^2 + A_1 z + 1$ ($A_1, A_2 \in \mathbb{R}$) of degree four with real coefficients. Using the method described at the end of the introduction, we easily obtain for the Chebyshev transform of p_4 (see also pp. 659–660 [8])

$$\mathcal{T}p_4(x) = x^2 + A_1 x + A_2 - 2.$$

For p_4 , we can relatively easily find the regions of the coefficients (A_1, A_2) for which **all zeros** lie in the sets D_i , ($i = 1, \dots, 6$) and compare them to the existing conditions.

Zeros in D_1 and D_2 . All zeros of p_4 have negative (positive) real parts if and only if all zeros of $\mathcal{T}p_4$ have negative (positive) real parts. By Hurwitz theorem all

zeros of $\mathcal{T}p_4$ have negative real parts exactly if $A_1 > 0$, and $\begin{vmatrix} A_1 & 0 \\ 1 & A_2 - 2 \end{vmatrix} > 0$, or if $A_1 > 0$ and $A_2 > 2$. All zeros of $\mathcal{T}p_4$ have positive real parts if and only if all zeros of $\mathcal{T}p_4(-x) = x^2 - A_1x + A_2 - 2$ have negative real parts, and from this we get that D_2^* is given by $A_1 < 0$ and $A_2 > 2$.

The next figure shows the domains D_1^* and D_2^* (colored in light and dark grey; in green and red in the online version) in the plane (A_2, A_1) . By continuity arguments on the half-line $\{(A_2, A_1) : A_1 = 0, A_2 \geq 2\}$ (colored black) all zeros of p_4 are imaginary.

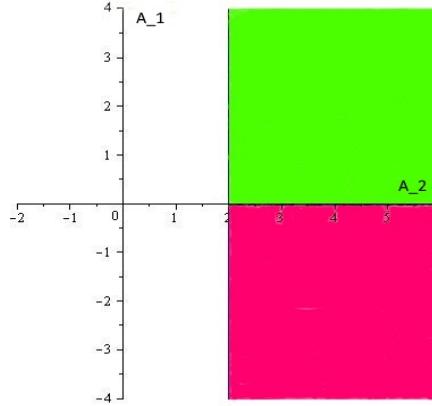


Figure 1. The domains D_1^* and D_2^* .

Zeros in D_3 and D_4 . All zeros of p_4 are positive (negative) if and only if both zeros

$$x_1 = \left(-A_1 + \sqrt{A_1^2 - 4(A_2 - 2)} \right) / 2, \quad x_2 = \left(-A_1 - \sqrt{A_1^2 - 4(A_2 - 2)} \right) / 2$$

of $\mathcal{T}p_4$ are in the interval $[2, \infty[$ ($(-\infty, -2]$). These zeros are real if and only if the discriminant $\Delta := A_1^2 - 4(A_2 - 2)$ is non-negative, which holds if and only if $2\sqrt{\max\{A_2 - 2, 0\}} \leq |A_1|$ (see [8, pp. 659–660]). An elementary calculation shows that $(x_1 \geq) x_2 \geq 2$ holds if and only if

$$-(A_2 + 2)/2 \leq A_1 \leq \min\{-4, 2\sqrt{\max\{A_2 - 2, 0\}}\}. \quad (10)$$

Similarly, $(x_2 \leq) x_1 \leq -2$ holds if and only if

$$\max\{4, 2\sqrt{\max\{A_2 - 2, 0\}}\} \leq A_1 \leq (A_2 + 2)/2. \quad (11)$$

Thus the domains D_3^* and D_4^* are given by (10) and (11). Below these domains are colored in light grey (and green in the online version). *The infinite rectangles with corners $(6, -4)$ and $(6, 4)$ indicate the sets obtained from Theorem 3 and Theorem 4.*

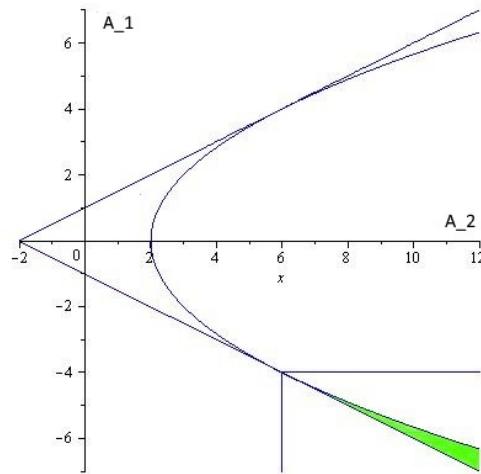


Figure 2. The domain D_3^* .

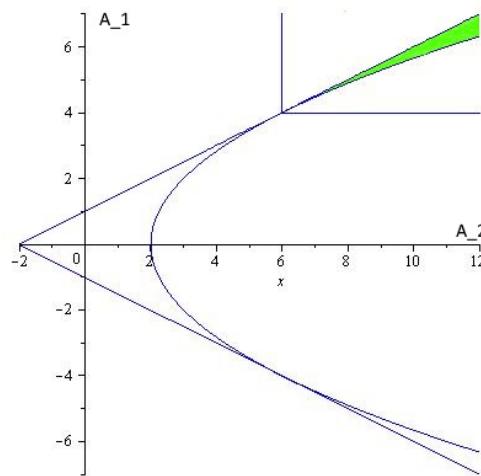


Figure 3. The domain D_4^* .

Zeros in D_5 . All zeros of p_4 are real if and only if all zeros of $\mathcal{T}p_4$ are in $\mathbb{R} \setminus [-2, 2]$. The latter condition holds if and only if $\Delta \geq 0$ and $|x_1| \geq 2$, $|x_2| \geq 2$. For the zeros there are three possibilities (j) $x_2 \geq 2$, (jj) $x_1 \leq -2$, or (jjj) $x_1 \geq 2$ and $x_2 \leq -2$. The cases (j) and (jj) have been dealt with previously. In the case (jjj), p_4 has two positive and two negative zeros as the zeros are pairwise positive or negative. Again, some elementary (but in this case somewhat longer) calculations give that $\Delta \geq 0$, and $x_1 \geq 2$ holds exactly if

$$A_1 \leq \begin{cases} -(A_2 + 2)/2, & \text{if } A_2 \leq 6, \\ -2\sqrt{A_2 - 2}, & \text{if } A_2 > 6. \end{cases} \quad (12)$$

Similarly, $\Delta \geq 0$ and $x_2 \leq -2$ holds exactly if

$$A_1 \geq \begin{cases} (A_2 + 2)/2, & \text{if } A_2 \leq 6, \\ 2\sqrt{A_2 - 2}, & \text{if } A_2 > 6. \end{cases} \quad (13)$$

Finally, we get that the region D^* where p_4 has two positive and two negative zeros is given by (12) and (13), or in equivalent form, by

$$|A_1| \leq -(A_2 + 2)/2 \quad \text{and} \quad A_2 \leq -2.$$

Clearly, $D_5^* = D_3^* \cup D_4^* \cup D^*$.

The set D^* is shown in the next picture. It is remarkable that this set is given by linear inequalities.

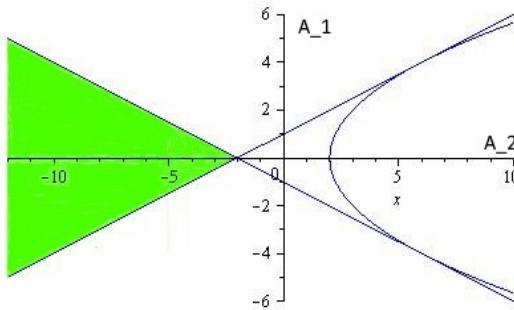


Figure 4. The domain D^* .

Zeros in D_6 . This case has been considered in [8, pp. 659–660], and there it was proved that all zeros of p_4 are on the unit circle if and only if

$$2\sqrt{\max\{A_2 - 2, 0\}} \leq |A_1| \leq \min\{4, (A_2 + 2)/2\}.$$

Our last two pictures show D_6^* and the set S^* of pairs (A_2, A_1) , which can be obtained from the *sufficient conditions* of Lakatos and Losonczi [11, Remark 3, p. 763] for all zeros of p_4 to be on the unit circle.

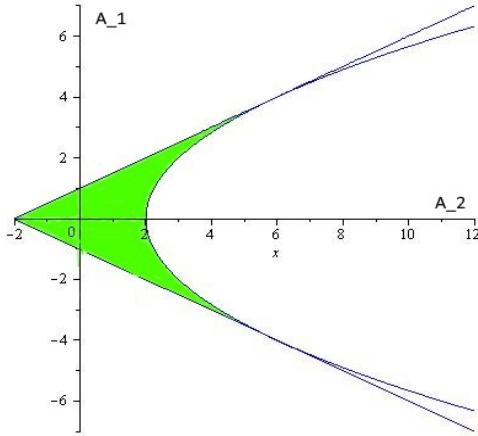


Figure 5. The domain D_6^* .

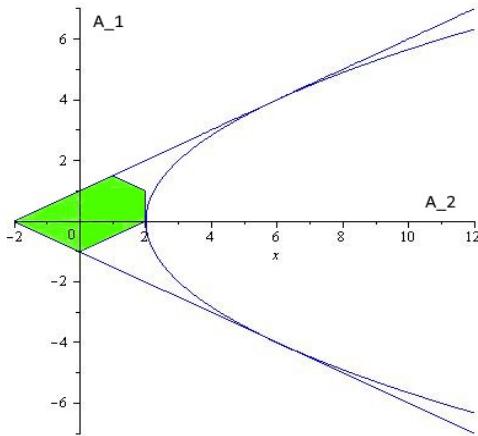


Figure 6. The domain S^* .

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