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On two open problems of the theory of permutable subgroups of finite groups

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Abstract. Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, G a finite group and $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$; G is said to be *σ-full* if G possesses a complete Hall σ-set.

A subgroup A of G is said to be *σ-permutable* in G if G possesses a complete Hall σ-set and A permutes with each Hall σ_i-subgroup H of G, that is, $AH = HA$ for all $i \in I$.

We prove that if G is σ-full, then the set $\mathcal{L}_{\sigma\,\mathrm{per}}(G)$, of all σ-permutable subgroups of G, forms a sublattice of the lattice of all subgroups of G. Also, answering to [9, Question 6.13], we describe the conditions under which the lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(G)$ is distributive.

1. Introduction

Throughout this paper, G always denotes a finite group. Moreover, we use $\mathcal{L}(G)$ to denote the lattice of all subgroups of G, and $\mathcal{L}_n(G)$ is the lattice of all normal subgroups of G. The symbol $\mathbb{P}$ denotes the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. As usual, $\pi(G)$ is the set of all primes dividing the order $|G|$ of G.

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The subgroups A and B of G are said to be *permutable* if $AB = BA$. In this case they also say that A *permutes* with B. If A permutes with all Sylow subgroups of G, then A is called S*-permutable* in G [1]. Recall also that an element a of the lattice $\mathcal{L}$ is called *meet-distributive* [8, p. 136] if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $b, c \in \mathcal{L}$.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

A set $\mathcal{H}$ of subgroups of G is a *complete Hall* σ*-set* of G [9] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$; G is said to be σ*-full* [9] if G possesses a complete Hall σ-set.

Recall that a subgroup A of G is said to be σ*-permutable* in G [10] if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$.

Our first observations are the following useful facts.

Proposition 1.1. *Suppose that G is σ-full and A is a σ-permutable subgroup of G. Then A permutes with all Hall σ_i-subgroups of G for all i.*

Theorem A. *Suppose that G is σ-full. Then the set $\mathcal{L}_{\sigma\,\mathrm{per}}(G)$, of all σ-permutable subgroups of G, forms a sublattice of the lattice $\mathcal{L}(G)$.*

Note that Theorem A improves Theorem C in [10] and, in fact, gives an alternative proof for the following well-known result.

Corollary 1.2 (Kegel in [5])**.** *The set $\mathcal{L}_S(G)$ of all S-permutable subgroups of G forms a sublattice of the lattice $\mathcal{L}(G)$.*

Example 1.3. (i) G is called σ*-nilpotent* [9] if $G = H_1 \times \cdots \times H_t$, where $\{H_1, \ldots, H_t\}$ is a complete Hall σ-set of G. It is not difficult to show that G is σ-nilpotent if and only if every subgroup of G is σ-permutable in G.

(ii) In the classical case when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$ (we use here the terminology in [11]), a subgroup A of G is σ^1-permutable in G if and only if it is S-permutable in G.

(iii) In the other classical case when $\sigma = \sigma^\pi = \{\pi, \pi'\}$, a subgroup A of a π-separable group G is σ^π-permutable in G if and only if A permutes with all Hall π-subgroups and with all Hall π'-subgroups of G.

(iv) In fact, in the theory of π-soluble groups ($\pi = \{p_1, \ldots, p_n\}$) we deal with the partition $\sigma = \sigma^{1\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$ of $\mathbb{P}$. In view of Proposition 1.1, a subgroup A of G is $\sigma^{1\pi}$-permutable in G if and only if G possesses a Hall

π'-subgroup V, and A permutes with all conjugates of V and with all Sylow p-subgroups of G for all $p \in \pi$.

The conditions under which the lattice $\mathcal{L}_{sn}(G)$ of all subnormal subgroups of G is modular or distributive are known (see [8, Theorems 9.2.3, 9.2.4]). It is well-known also that the lattice $\mathcal{L}_n(G)$ of all normal subgroups of G is modular and this lattice is distributive if and only if in every factor group G/R, any two G/R-isomorphic normal subgroups coincide (see [7] and [8, Theorem 9.1.6]). Kegel proved [5] that the set $\mathcal{L}_S(G)$ of all S-permutable subgroups of G forms a sublattice of the lattice $\mathcal{L}_{sn}(G)$. Since $\mathcal{L}_n(G) \subseteq \mathcal{L}_S(G) \subseteq \mathcal{L}_{sn}(G)$, where both inclusions in general are strict, it seems natural to ask: *Under what conditions the lattice $\mathcal{L}_S(G)$ is modular or distributive?* Moreover, in view of Theorem A, it makes sense to consider the following general

Question 1.4 (see Questions 6.11 and 6.13 in [9]). Under what conditions the lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(G)$ is modular or distributive?

Note that if $K \trianglelefteq H$ and $K, H \in \mathcal{L}_{\sigma_i\,\mathrm{per}}(G)$, where $\mathcal{L}_{\sigma_i\,\mathrm{per}}(G)$ is the set of all σ-permutable σ_i-subgroups of G, then $O^{\sigma_i}(G)$ normalizes both subgroups K and H [10, Lemma 3.1], and hence we can consider $O^{\sigma_i}(G)$ as a group of operators for H/K (assuming, as usual, that $(hK)^a = h^aK$ for all $hK \in H/K$ and $a \in O^{\sigma_i}(G)$).

We do not know under which conditions on G the lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(G)$ is modular. Nevertheless, we give the full answer to the second part of Question 1.4.

Recall that $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

Let G be a σ-full group and $\mathcal{L} = \mathcal{L}_{\sigma\,\mathrm{per}}(G)$. Then we say that the lattice $\mathcal{L}$ *satisfies the weak distributivity condition with respect to σ* (*the $W\sigma D$-condition*, in short) if the following hold: (i) every two members of $\mathcal{L}$ are permutable; (ii) the lattice $\mathcal{L}_n(G)$ is distributive; (iii) $G/G^{\mathfrak{N}_\sigma}$ is cyclic and $G^{\mathfrak{N}_\sigma}$ is a meet-distributive element of $\mathcal{L}$.

Theorem B. *Suppose that G is σ-full. Let $\mathcal{L} = \mathcal{L}_{\sigma\,\mathrm{per}}(G)$. Then the following conditions are equivalent:*

(i) *The lattice $\mathcal{L}$ is distributive.*

(ii) *$\mathcal{L}$ satisfies the $W\sigma D$-condition and in every factor group $\bar{G} = G/R$, any two $O^{\sigma_i}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma_i\,\mathrm{per}}(\bar{G})$ for some i, coincide.*

(iii) *$\mathcal{L}$ satisfies the $W\sigma D$-condition and in every factor group $\bar{G} = G/R$, any two*

$O^{\sigma_i}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma_i \,\mathrm{per}}(\bar{G})$ (for some i) and the subgroups $\bar{H}$ and $\bar{L}$ cover $\bar{K}$ in $\mathcal{L}_{\sigma \,\mathrm{per}}(\bar{G})$, coincide.

In the case when $\sigma = \sigma^{\pi}$, we get from Theorem B the following result.

Corollary 1.5. *Suppose that G is π-separable, and let $\mathcal{L} = \mathcal{L}_{\sigma^{\pi} \,\mathrm{per}}(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L}$ satisfies the $W\sigma^{\pi}D$-condition and the following hold:*

(1) *In every factor group $\bar{G} = G/R$, any two $O^{\pi}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}$ and $\bar{L}$ are σ^{π}-permutable π-subgroups of $\bar{G}$, coincide.*

(2) *In every factor group $\bar{G} = G/R$, any two $O^{\pi'}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}$ and $\bar{L}$ are σ^{π}-permutable π'-subgroups of $\bar{G}$, coincide.*

In the case when $\sigma = \sigma^{1\pi}$, we get from Theorem B the following fact.

Corollary 1.6. *Suppose that G possesses a Hall π'-subgroup and let $\mathcal{L} = \mathcal{L}_{\sigma^{1\pi} \,\mathrm{per}}(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L}$ satisfies the $W\sigma^{1\pi}D$-condition and the following hold:*

(1) *In every factor group $\bar{G} = G/R$, any two $O^{p}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{pS}(\bar{G})$ and $p \in \pi$, coincide;*

(2) *In every factor group $\bar{G} = G/R$, any two $O^{\pi'}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}$ and $\bar{L}$ are σ^{π}-permutable π'-subgroups of $\bar{G}$, coincide.*

In this corollary, $\mathcal{L}_{pS}(\bar{G})$ denotes the set of all S-permutable p-subgroups of $\bar{G}$.

In the case when $\pi = \mathbb{P}$, we get from Corollary 1.6 the following

Corollary 1.7 (see [12, Theorem A]). *Let $\mathcal{L} = \mathcal{L}_{S}(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L}$ satisfies the $W\sigma^{1}D$-condition and in every factor group $\bar{G} = G/R$, any two $O^{p}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{pS}(\bar{G})$ and p is a prime, coincide.*

The proof of Theorem B consists of many steps, and the following result together with Proposition 1.1 and Theorem A are three of them.

Proposition 1.8. *A σ-nilpotent subgroup A of G is σ-permutable in G if and only if each characteristic subgroup of A is σ-permutable in G.*

Corollary 1.9 (see [1, Theorem 1.2.17]). *Let A be a nilpotent subgroup of G. Then the following statements are equivalent:*

(i) *A is S-permutable in G.*

(ii) *Each Sylow subgroup of A is S-permutable in G.*

(iii) *Each characteristic subgroup of A is S-permutable in G.*

2. Proofs of Theorems A and Propositions 1.1 and 1.8

Lemma 2.1 (see [3, A, Lemma 1.6])**.** *Let A, B and H be subgroups of G. If $AH = HA$ and $BH = HB$, then $\langle A, B\rangle H = H\langle A, B\rangle$.*

A subgroup A of G is called *σ-subnormal* in G [10] if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$.

The importance of this concept is related to the following result.

Lemma 2.2 (see [10, Theorem B])**.** *Every σ-permutable subgroup A of G is σ-subnormal in G and A/A_G is σ-nilpotent.*

Proof of Theorem A. In fact, in view of Lemmas 2.1 and 2.2, it is enough to show that if A and B are σ-subnormal subgroups of G such that for a Hall σ_i-subgroup H of G we have $AH = HA$ and $BH = HB$, then $(A\cap B)H = H(A\cap B)$. Assume that this is false and let G be a counterexample of minimal order. Then G is not a σ_i-group, since otherwise we have $H = G$ and so $G = (A \cap B)H = H(A \cap B)$.

Let $E = AH \cap BH$. Then $A\cap E$ and $B\cap E$ are σ-subnormal subgroups of E by [10, Lemma 2.6(1)]. Moreover, $AH \cap E = H(A \cap E) = (A \cap E)H$. Similarly, $(B \cap E)H = H(B \cap E)$. Hence the hypothesis holds for $(A \cap E, B \cap E, H, E)$. Assume that $E < G$. Then the choice of G implies that $A\cap B = (A\cap E)\cap(B\cap E)$ is permutable with H. Hence $E = G$, so $G = AH = BH$. Thus $|G : A|$ and $|G : B|$ are σ_i-numbers. Hence we have $O^{\sigma_i}(A) = O^{\sigma_i}(G) = O^{\sigma_i}(B)$ by [10, Lemma 2.6(8)]. Therefore, since G is not a σ_i-group, it follows that $V = A_G \cap B_G \neq 1$. Moreover, A/V and B/V are σ-subnormal subgroups of G/V by [10, Lemma 2.6(4)]. Also, we have $(A/V)(HV/V) = AH/V = HA/V = (HV/V)(A/V)$ and $(B/V)(HV/V) = (HV/V)(B/V)$, where HV/V is a Hall σ_i-subgroup of G/V. Hence the choice of G implies that

$$\begin{aligned}(A\cap B/V)(HV/V) &= ((A/V)\cap(B/V))(HV/V)\\ &= (HV/V)((A/V)\cap(B/V)) = (HV/V)(A\cap B/V).\end{aligned}$$

But then $(A\cap B)H = (A\cap B)HV = HV(A\cap B) = H(A\cap B)$. This contradiction completes the proof of the result. □

Proposition 2.3. *Let A be a σ-nilpotent σ-subnormal subgroup of G, and B a characteristic subgroup of A. Let H be a Hall σ_i-subgroup of G. If $AH = HA$, then $BH = HB$.*

Proof. Assume that this proposition is false, and let G be a counterexample with $|G|+|B|+|A|$ minimal.

By hypothesis, $A = A_1 \times \cdots \times A_t$, where $\{A_1, \dots, A_t\}$ is a complete Hall σ-set of A. Hence $B = (A_1 \cap B) \times \cdots \times (A_t \cap B)$, where $\{A_1 \cap B, \dots, A_t \cap B\}$ is a complete Hall σ-set of B. We can assume without loss of generality that A_k is a σ_k-subgroup of A for all $k = 1, \dots, t$.

It is clear that $A_i \cap B$ is characteristic in A for all $i = 1, \dots, t$. Therefore, if $A_i \cap B < B$, then $(A_i \cap B)H = H(A_i \cap B)$ by the choice of G and so for some j, $j = 1$ say, we have $A_1 \cap B = B$, since otherwise we have

$$BH = ((A_1 \cap B) \times \cdots \times (A_t \cap B))H = H((A_1 \cap B) \times \cdots \times (A_t \cap B)) = HB.$$

Thus $B \leq A_1$. It is clear that A_1 is a σ-subnormal subgroup of G, so in the case when $i = 1$, we have $B \leq A_1 \leq H$ by [10, Lemma 2.6(7)]. But then $BH = H = HB$, a contradiction. Thus $i > 1$.

Now we show that $A_1 H = HA_1$. First note that A_i is σ-subnormal in G, so $A_i \leq H$ by [10, Lemma 2.6(7)]. Therefore $A = A_1 \times V \times A_i$, where $V = A_2 \cdots A_{i-1} A_{i+1} \cdots A_t$, and so

$$AH = HA = (A_1 \times V \times A_i)H = (A_1 \times V)H = H(A_1 \times V),$$

where $A_1 \times V$ is a σ-subnormal σ_i'-subgroup of G. Then $A_1 \times V$ is σ-subnormal in $(A_1 \times V)H$ by [10, Lemma 2.6(1)]. Hence $H \leq N_G(A_1 \times V)$ by [10, Lemma 2.6(8)]. Since A_1 is a characteristic subgroup of $A_1 \times V$, we have $H \leq N_G(A_1)$, and so $A_1 H = HA_1$. But B is a characteristic subgroup of A_1, since B is characteristic in A by hypothesis and $A = A_1 \times \cdots \times A_t$. Therefore $H \leq N_G(B)$, and so $BH = HB$, a contradiction. The proposition is proved. □

Corollary 2.4. *Let A be a σ-nilpotent subgroup of a σ-full group G. Then the following statements are equivalent:*

(i) *A is σ-permutable in G.*

(ii) *Each Hall σ_i-subgroup of A is σ-permutable in G for all i.*

(iii) *Each characteristic subgroup of A is σ-permutable in G.*

Proof. By hypothesis, $A = A_1 \times \cdots \times A_t$, where $\{A_1, \dots, A_t\}$ is a complete Hall σ-set of A. Then A_i is characteristic in A for all $i = 1, \dots, t$. Therefore (ii), (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii), (iii) This follows from Proposition 2.3.

The corollary is proved. □

Proof of Proposition 1.8. This directly follows from Corollary 2.4. □

Now we are ready to prove Proposition 1.1.

Proof of Proposition 1.1. Assume that this proposition is false, and let G be a counterexample with $|G|+|A|$ minimal. Then for some i and some Hall σ_i-subgroup H of G, we have $AH \neq HA$ but $A_1H = HA_1$ for every σ-permutable subgroup A_1 of G with $A_1 < A$. By hypothesis, G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that $AL^x = L^xA$ for all $L \in \mathcal{H}$ and all $x \in G$. We can assume without loss of generality that H_k is a σ_k-group for all $k = 1, \dots, t$. Let $V = H_i$.

First we show that $A_G = 1$. Indeed, assume that $R = A_G \neq 1$. Then $\mathcal{H}_0 = \{H_1R/R, \dots, H_tR/R\}$ is a complete Hall σ-set of G/R such that

$$AL^x/R = (A/R)(LR/R)^{xR} = (LR/R)^{xR}(A/R) = L^xA/R$$

for all $LR/R \in \mathcal{H}_0$ and all $xR \in G/R$. On the other hand, HR/R is a Hall σ_i-subgroup of G/R. Hence the choice of G implies that

$$AH/R = (A/R)(HR/R) = (HR/R)(A/R) = HA/R,$$

and so $AH = HA$, a contradiction. Therefore $A_G = 1$, hence $A = A_1 \times \cdots \times A_t$, where $\{A_1, \dots, A_t\}$ is a complete Hall σ-set of A by Lemma 2.2. Moreover, Lemma 2.2 implies also that A is σ-subnormal in G.

First assume that $A = A_1$ is a σ_j-group. If $j = i$, then $A \cap H = A$ by [10, Lemma 2.6(7)], and so $AH = H = HA$. Hence $j \neq i$. By hypothesis, $AV^x = V^xA$ for each $x \in G$. Then $V^x \leq N_G(A)$ for all $x \in G$ by [10, Lemma 3.1]. Hence $V^G \leq N_G(A)$. But then $H \leq V^G \leq N_G(A)$, which implies that $AH = HA$. This contradiction shows that $A \neq A_1$.

The subgroups $A_1, \dots, A_t$ are characteristic in A, so $A_iL^x = L^xA_i$ for all $L \in \mathcal{H}$ and all $x \in G$ by Proposition 1.8. Therefore, the minimality of $|G| + |A|$ implies that $A_iH = HA_i$ for all $i = 1, \dots, t$, so $AH = HA$. This contradiction completes the proof of the result. □

3. Proof of Theorem B

Now we use Proposition 1.1 to prove the following fact.

Lemma 3.1. *Let $R \leq V$ be subgroups of a σ-full group G, where R is normal in G. If V/R is σ-permutable in G/R, then V is σ-permutable in G.*

Proof. Let $i \in I$ and H be a Hall σ_i-subgroup of G. Then HR/R is a Hall σ_i-subgroup of G/R, and so

$$VH/R = (V/R)(HR/R) = (HR/R)(V/R) = HV/R$$

by hypothesis and Proposition 1.1, hence $VH = HV$. The lemma is proved. □

Lemma 3.2 (see Lemma 5.2 in [6]). *Let $\mathcal{L}$ be a modular sublattice of the lattice $\mathcal{L}(G)$, and $U, V, N \in \mathcal{L}$ with $N \trianglelefteq \langle U, V\rangle$. If U permutes both with $V \cap UN$ and VN, then U permutes with V.*

Proposition 3.3. *Let G be σ-full and $\mathcal{L} = \mathcal{L}_{\sigma_i \,\mathrm{per}}(G)$. Then: (i) $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\sigma \,\mathrm{per}}(G)$ and (ii) if $\mathcal{L}$ is distributive, then $AB = BA$ for all $A, B \in \mathcal{L}$.*

Proof. (i) Let $A, B \in \mathcal{L}$. By hypothesis, for some Hall σ_i-subgroup H of G and for each $x \in G$, we have $H^x = AH^x = H^xA$, so $A \le H_G \le O_{\sigma_i}(G)$. Similarly, $B \le O_{\sigma_i}(G)$. Thus $\langle A, B\rangle$ is a σ_i-subgroup of G and this subgroup is σ-permutable in G by Lemma 2.1. Finally, $A \cap B$ is also a σ_i-subgroup of G and this subgroup is σ-permutable in G by Theorem A. Thus we have (i).

(ii) Suppose that this assertion is false, and let G be a counterexample with $|G| + |A| + |B|$ minimal. Thus $AB \ne BA$ but $A_1B_1 = B_1A_1$ for all $A_1, B_1 \in \mathcal{L}$ such that $A_1 \le A$, $B_1 \le B$ and either $A_1 \ne A$ or $B_1 \ne B$. Let $V = \langle A, B\rangle O^{\sigma_i}(G)$ and $R = \langle A, B\rangle \cap O^{\sigma_i}(G)$. Then V is σ-subnormal in G.

(1) *The group V is σ-full and $\mathcal{L}_{\sigma_i \,\mathrm{per}}(V)$ is a sublattice of $\mathcal{L}$.*

First note that each Hall σ_j-subgroup of G is contained in V for all $j \ne i$, since $O^{\sigma_i}(G)$ is the subgroup of G generated by all its σ_i'-elements. On the other hand, $H \cap V$ is a Hall σ_i-subgroup of V for each Hall σ_i-subgroup H of G by [10, Lemma 2.6(7)], so V is σ-full.

Now, let $H \in \mathcal{L}_{\sigma_i \,\mathrm{per}}(V)$. Then $H \le O_{\sigma_i}(V) \le O_{\sigma_i}(G)$ by [10, Lemma 2.6(11)]. Therefore H permutes with each Hall σ_i-subgroup of G. On the other hand, for every $j \ne i$ and for every Hall σ_j-subgroup W of G, we have $HW = WH$ since $W \le V$. Hence $H \in \mathcal{L}_{\sigma \,\mathrm{per}}(G)$, which implies (1).

(2) *$V = G$, so $\langle A, B\rangle \trianglelefteq G$.*

Claim (1) implies that the hypothesis holds for $\mathcal{L}_{\sigma_i \,\mathrm{per}}(V)$, and so in the case when $V \ne G$, the choice of G implies that $AB = BA$. Thus $G = \langle A, B\rangle O^{\sigma_i}(G)$. Therefore, since $O^{\sigma_i}(G) \le N_G(\langle A, B\rangle)$ by [10, Lemma 3.1], $\langle A, B\rangle$ is normal in G.

(3) $R = 1$.

Assume that $R = \langle A, B\rangle \cap O^{\sigma_i}(G) \neq 1$. First we show that $BRA = \langle A, B\rangle R$. Indeed, let H/R be a σ_i-subgroup of G/R. Then H is a σ_i-group since $\langle A, B\rangle \leq O_{\sigma_i}(G)$. Moreover, Lemma 3.1 and [10, Lemma 2.8(2)] imply that H/R is σ-permutable in G/R if and only if H is σ-permutable in G. Therefore the lattice $\mathcal{L}_{\sigma_i\,\mathrm{per}}(G/R)$ is isomorphic to the interval $[G/R]$ in the distributive lattice $\mathcal{L}$. Therefore, by the minimality of G, $(AR/R)(BR/R) = (BR/R)(AR/R)$, and so $BRA = \langle A, B\rangle R$.

Now we show that $BRA = BR$. Assume that this is false. Then $A\cap BR < A$. But Theorem A implies that $A \cap BR$ is σ-permutable in G, so the minimality of $|G|+|A|+|B|$ implies that B permutes with $A\cap BR$. Also, B permutes with RA since $B(RA) = \langle A, B\rangle R$, so $AB = BA$ by Lemma 3.2, Part (i) and Theorem A. This contradiction shows that $A \leq BR$, so $BRA = BR$. But $R \leq O^{\sigma_i}(G) \leq N_G(B)$ by [10, Lemma 3.1], hence B is normal in BR, and since $A \leq BR$, it follows that $AB = BA$. This contradiction shows that we have (3).

Final contradiction. Claims (2) and (3) imply that $G = \langle A, B\rangle O^{\sigma_i}(G) = \langle A, B\rangle \times O^{\sigma_i}(G)$, so every subgroup H of $\langle A, B\rangle$ is $O^{\sigma_i}(G)$-invariant since $\langle A, B\rangle \leq O_{\sigma_i}(G)$. It follows that every subgroup of $\langle A, B\rangle$ is σ-permutable in G. Hence $\mathcal{L}(\langle A, B\rangle)$ is a sublattice of the distributive lattice $\mathcal{L}$. Thus $\langle A, B\rangle$ is cyclic by the Ore theorem [8, Theorem 1.2.3], so $AB = BA$, a contradiction. The proposition is proved. □

Corollary 3.4. *If G is σ-full and the lattice $\mathcal{L} = \mathcal{L}_{\sigma\,\mathrm{per}}(G)$ is distributive, then every two members A and B of $\mathcal{L}$ are permutable.*

Proof. Suppose that this corollary is false, and let G be a counterexample with $|G| + |A| + |B|$ minimal.

Let R be a minimal normal subgroup of G. Then the lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(G/R)$ is isomorphic to the interval $[G/R]$ in the distributive lattice $\mathcal{L}$ by Lemma 3.1 and [10, Lemma 2.8(2)]. Therefore [10, Lemma 2.8(2)] and the minimality of G imply that $(AR/R)(BR/R) = (BR/R)(AR/R)$. It follows that $RAB = \langle A, B\rangle R$ is a subgroup of G, so $A_G = 1 = B_G$. Hence, because of Lemma 2.2, A and B are σ-nilpotent. The minimality of $|G| + |A| + |B|$ implies that for some i we have $A, B \leq O_{\sigma_i}(G)$, and so $A, B \in \mathcal{L}_{\sigma_i\,\mathrm{per}}(G)$. But $\mathcal{L}_{\sigma_i\,\mathrm{per}}(G)$ is a sublattice of the distributive lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(G)$ by Proposition 3.3(i). Therefore, $AB = BA$ by Proposition 3.3(ii), a contradiction. The corollary is proved. □

Lemma 3.5 (see [4, p. 59])**.** *A modular lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L}$ has no distinct elements a, b and c such that $a \vee b = a \vee c = b \vee c$ and $a \wedge b = a \wedge c = b \wedge c$.*

Lemma 3.6 (see [8, Theorem 1.6.2]). *Let $G = A \times B$, $f : A \to B$ be an isomorphism and $C = \{aa^f \mid a \in A\}$. Then $G = AC = BC$ and $A \cap C = 1 = B \cap C$.*

Proof of Theorem B. Let $D = G^{\mathfrak{N}_\sigma}$. (i) $\Rightarrow$ (ii) First note that every two members of $\mathcal{L}$ are permutable by Corollary 3.4. Moreover, since the lattice $\mathcal{L}_n(G)$ is a sublattice of the lattice $\mathcal{L}$, it is distributive. Now note that since $G/D = G/G^{\mathfrak{N}_\sigma}$ is σ-nilpotent, every subgroup E of G satisfying $D \le E \le G$ is σ-permutable in G by Lemma 3.1. Hence $\mathcal{L}(G/D) = \mathcal{L}_{\sigma\,\mathrm{per}}(G/D)$. In view of Lemma 3.1 and [10, Lemma 2.8(2)], the lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(G/D)$ is isomorphic to the interval $[G/D]$ in lattice $\mathcal{L}$, so $\mathcal{L}_{\sigma\,\mathrm{per}}(G/D)$ is distributive. Hence G/D is cyclic by the Ore theorem [8, Theorem 1.2.3]. It is clear also that D is a meet-distributive element of $\mathcal{L}$. Thus the lattice $\mathcal{L}$ satisfies the $W\sigma D$-condition.

We show that in every factor group $\bar{G} = G/R$, any two $O^{\sigma_i}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma_i\,\mathrm{per}}(\bar{G})$, coincide. In view of Lemma 3.1 and [10, Lemma 2.8(2)], it is enough to consider the case when $\bar{G} = G$ and $\bar{K} = K, \bar{H} = H, \bar{L} = L \in \mathcal{L}_{\sigma_i\,\mathrm{per}}(G)$.

Suppose that $H \ne L$. Then $H \ne K$. Let $K < H_0 \le H$, where H_0 covers K in $\mathcal{L}$, and let $L_0/K = (H_0/K)^f$, where $f : H/K \to L/K$ is an $O^{\sigma_i}(G)$-isomorphism. For $g \in O^{\sigma_i}(G)$ and $l_0K = (hK)^f \in L_0/K$, where $h \in H_0$, we have

$$(l_0K)^g = ((hK)^f)^g = ((hK)^g)^f = (h^gK)^f = (h_0K)^f,$$

where $h_0 \in H_0$, since H_0 is $O^{\sigma_i}(G)$-invariant by [10, Lemma 3.1]. Hence $(l_0K)^g \in L_0/K$. It follows that L_0 is $O^{\sigma_i}(G)$-invariant, and so L_0 covers K in $\mathcal{L}$, since the inverse map $f^{-1} : L/K \to H/K$ is an $O^{\sigma_i}(G)$-isomorphism too.

First assume that $H_0 \ne L_0$, and let $E_0/K = \{hK(hK)^f | hK \in H_0/K\}$. Then $(H_0/K)(L_0/K) = (H_0/K)\times(L_0/K)$. Indeed, if $H_0^x \ne H_0$ for some $x \in L_0$, then (i) and the fact that H_0 and L_0 cover K in $\mathcal{L}$ would imply that $\{K; H_0; H_0^x; L_0; H_0L_0\}$ would be a diamond in the distributive lattice $\mathcal{L}$, contradicting Lemma 3.5. Hence, by Lemma 3.6, E_0/K is a subgroup of $(H_0/K) \times (L_0/K)$, and we have

$$(H_0/K) \times (L_0/K) = (H_0/K) \times (E_0/K) = (L_0/K) \times (E_0/K).$$

Note that if $g \in O^{\sigma_i}(G)$ and $hK(hK)^f \in E_0/K$, then

$$(hK(hK)^f)^g = (hK)^g((hK)^f)^g = (h^gK)(h^gK)^f \in E_0/K,$$

since $f_{H_0/K}$ is an $O^{\sigma_i}(G)$-isomorphism from H_0/K onto $L_0/K = (H_0/K)^f$. Hence E_0/K is $O^{\sigma_i}(G)$-invariant, so $O^{\sigma_i}(G) \le N_G(E_0)$. Therefore, H_0, L_0 and E_0

are distinct elements of $\mathcal{L}$ such that $H_0 \cap L_0 = H_0 \cap E_0 = L_0 \cap E_0 = K$ and $H_0L_0 = H_0E_0 = L_0E_0$, which is impossible by Lemma 3.5, since H_0L_0 is a σ-permutable subgroup of G. Therefore $H_0 = L_0$. Now f induces an $O^{\sigma_i}(G)$-isomorphism $f' : H/H_0 \to L/H_0$, and an obvious induction yields that $H = L$. Hence the implication (i) $\Rightarrow$ (ii) holds.

(ii) $\Rightarrow$ (iii) This implication is evident.

(iii) $\Rightarrow$ (i) Suppose that this is false, and let G be a counterexample of minimal order.

First note that if $A, B, C \in \mathcal{L}_{\sigma\,\mathrm{per}}(G)$ and $A \leq C$, then

$$C \cap \langle A, B\rangle = C \cap AB = A(C \cap B) = \langle A, C \cap B\rangle$$

by hypothesis, so the lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(G)$ is modular. Hence, by Lemma 3.5, there are distinct σ-permutable subgroups A, B and C of G such that for some σ-permutable subgroups E and T of G, we have $E = A \cap B = A \cap C = B \cap C$ and $T = AB = AC = BC$.

(1) *The lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(G/R)$ is distributive for each non-identity normal subgroup R of G.*

In view of the choice of G, it is enough to show that the hypothesis holds for $\bar{G} = G/R$.

Let $\bar{K}, \bar{H} \in \mathcal{L}_{\sigma\,\mathrm{per}}(\bar{G})$. Then $K, H \in \mathcal{L}$ by Lemma 3.1, and so $KH = HK$ by hypothesis, which implies that $(K/R)(H/R) = (H/R)(K/R)$. It is clear also that the lattice $\mathcal{L}_n(\bar{G})$ is isomorphic to some sublattice of the lattice $\mathcal{L}_n(G)$, so $\mathcal{L}_n(\bar{G})$ is distributive.

In view of [2, Proposition 2.2.8] and [10, Corollary 2.4 and Lemma 2.5], we have $\bar{G}^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}R/R = DR/R$. Thus $\bar{G}/\bar{G}^{\mathfrak{N}_\sigma} = (G/R)/(DR/R) \simeq G/DR \simeq (G/D)/(DR/D)$ is cyclic, since G/D is cyclic by hypothesis.

By hypothesis we have also that $D \cap \langle K, H\rangle = D \cap KH = \langle D \cap K, D \cap H\rangle = (D \cap K)(D \cap H)$, since $D \cap K$ and $D \cap H$ are σ-permutable in G by Theorem A, so

$$\begin{aligned}\bar{G}^{\mathfrak{N}_\sigma} \cap \langle \bar{K}, \bar{H}\rangle &= (DR \cap KH)/R = R(D \cap KH)/R = ((D \cap K)R/R)((D \cap H)R/R)\\ &= ((DR/R) \cap (K/R))((DR/R) \cap (H/R)) = \langle \bar{G}^{\mathfrak{N}_\sigma} \cap \bar{K}, \bar{G}^{\mathfrak{N}_\sigma} \cap \bar{H}\rangle.\end{aligned}$$

Hence $\bar{G}^{\mathfrak{N}_\sigma}$ is a meet-distributive element of $\mathcal{L}_{\sigma\,\mathrm{per}}(\bar{G})$. Thus the lattice $\mathcal{L}_{\sigma\,\mathrm{per}}(\bar{G})$ satisfies the $W\sigma D$-condition.

Finally, let $\bar{N}$ be any normal subgroup of $\bar{G}$. Let $\widehat{G} = \bar{G}/\bar{N}$, and let $\widehat{H}/\widehat{K} = (\bar{H}/\bar{N})/(\bar{K}/\bar{N})$ and $\widehat{L}/\widehat{K} = (\bar{L}/\bar{N})/(\bar{K}/\bar{N})$ be $O^{\sigma_i}(\widehat{G})$-isomorphic sections, where $\widehat{K}, \widehat{H}, \widehat{L} \in \mathcal{L}_{\sigma_i\,\mathrm{per}}(\widehat{G})$ and the subgroups $\widehat{H}$ and $\widehat{L}$ cover $\widehat{K}$ in $\mathcal{L}_{\sigma\,\mathrm{per}}(\widehat{G})$. Then we have $(H/N)/(K/N)$ and $(L/N)/(K/N)$ are $O^{\sigma_i}(G/N)$-isomorphic sections, where $K/N, H/N, L/N \in \mathcal{L}_{\sigma_i \,\mathrm{per}}(G/N)$ and the subgroups H/N and L/N cover K/N in $\mathcal{L}_{\sigma \,\mathrm{per}}(G/N)$. But then $H/N = L/N$ by hypothesis, which implies that $\widehat{H}/\widehat{K} = \widehat{L}/\widehat{K}$. Therefore the hypothesis holds for G/R, so we have (1).

(2) $E_G = 1$.

In view of Lemma 3.1, this follows from Claim (1), Lemma 3.5 and the choice of G.

(3) $A_GB_G \cap A_GC_G \cap B_GC_G = 1$.

Since $A \cap B = E$, we have $B_G \cap A_G \leq E_G = 1$ by Claim (2). Similarly, $B_G \cap C_G = 1$ and $A_G \cap C_G = 1$. Therefore,

$$\begin{aligned}
&(A_GB_G \cap A_GC_G) \cap B_GC_G \\
&\quad = A_G(B_G \cap A_GC_G) \cap B_GC_G = A_G(B_G \cap A_G)(B_G \cap C_G) \cap B_GC_G \\
&\quad = A_G \cap B_GC_G = (A_G \cap B_G)(A_G \cap C_G) = 1
\end{aligned}$$

by hypothesis.

(4) *The subgroup T is σ-nilpotent.*

Note that

$$T/A_GB_G = AB/A_GB_G = (AA_GB_G/A_GB_G)(BA_GB_G/A_GB_G),$$

where

$$AA_GB_G/A_GB_G \simeq A/A \cap A_GB_G = A/A_G(A \cap B_G) \simeq (A/A_G)/(A_G(A \cap B_G)/A_G)$$

and

$$BA_GB_G/A_GB_G \simeq (B/B_G)/(B_G(B \cap A_G)/B_G)$$

are σ-nilpotent by Lemma 2.2. We know that the subgroups AA_GB_G/A_GB_G and BA_GB_G/A_GB_G are σ-subnormal in G/A_GB_G by Lemma 2.2. Hence T/A_GB_G is σ-nilpotent by [10, Lemma 2.6(11)]. Similarly, T/A_GC_G and T/C_GB_G are σ-nilpotent. Hence from Claim (3) it follows that $T \simeq T/(A_GB_G \cap A_GC_G \cap B_GC_G)$ is σ-nilpotent by Corollary 2.4 and [10, Lemma 2.5].

(5) *For some i, there are distinct σ_i-subgroups $A_i, B_i, C_i \in \mathcal{L}$ such that $H_i = A_iB_i = A_iC_i = B_iC_i$ and $K_i = A_i \cap B_i = A_i \cap C_i = B_i \cap C_i$ are σ-permutable subgroups of G.*

Let $\sigma_i \in \sigma(T)$, that is, $\sigma_i \cap \pi(T) \neq \emptyset$. Then, by Claim (4), $H_i = O_{\sigma_i}(T)$ is the Hall σ_i-subgroup of T and $A_i = O_{\sigma_i}(A)$, $B_i = O_{\sigma_i}(B)$ and $C_i = O_{\sigma_i}(C)$ are the Hall σ_i-subgroups of A, B and C, respectively. Hence $H_i = A_iB_i = A_iC_i = B_iC_i$. Moreover, A_i, B_i and C_i are σ-permutable in G by Proposition 1.8. It is clear also that $K_i = A_i \cap B_i = A_i \cap C_i = B_i \cap C_i = O_{\sigma_i}(E)$.

Suppose that $A_i = B_i$. Then $H_i = A_iB_i = A_i = B_i = K_i \leq C_i \leq H_i$. Hence $A_i = B_i = C_i$. Therefore, since $A \neq B \neq C$ and $A \neq C$, there is $\sigma_i \in \sigma(T)$ such that $A_i \neq B_i \neq C_i$ and $A_i \neq C_i$. Finally, H_i and K_i are evidently σ-permutable subgroups of G, so we have (5).

(6) *There are distinct σ_i-subgroups $A_0, B_0, C_0 \in \mathcal{L}$ such that $H_0 = A_0B_0 = A_0C_0 = B_0C_0$ and $K_0 = A_0 \cap B_0 = A_0 \cap C_0 = B_0 \cap C_0$ are σ-permutable subgroups of G and A_0, B_0, C_0 are normal subgroups of $O^{\sigma_i}(G)$.*

Let $A_0 = A_i \cap D$, $B_0 = B_i \cap D$ and $C_0 = C_i \cap D$. Then A_0, B_0 and C_0 are σ-permutable σ_i-subgroups of G by Claim (5) and Theorem A. Moreover, Claim (5) implies that

$$K_0 = A_0 \cap B_0 = A_i \cap B_i \cap D = A_i \cap C_i \cap D = A_0 \cap C_0 = B_i \cap C_i \cap D = B_0 \cap C_0.$$

Since D is a meet-distributive element of $\mathcal{L}$ by hypothesis,

$$H_0 = D \cap A_iB_i = (D \cap A_i)(D \cap B_i) = A_0B_0 = A_0C_0 = D \cap A_iC_i = D \cap B_iC_i = B_0C_0.$$

Now we show that A_0, B_0, C_0 are distinct elements of $\mathcal{L}$. First note that

$$|H_i : K_i| = |A_i : K_i||B_i : K_i| = |A_i : K_i||C_i : K_i| = |B_i : K_i||C_i : K_i|,$$

so $|A_i : K_i| = |B_i : K_i| = |C_i : K_i|$. Hence $|A_i| = |B_i| = |C_i|$. Suppose that $A_0 = B_0$. Then

$$D \cap H_i = D \cap A_iB_i = (D \cap A_i)(D \cap B_i) = A_0B_0 = A_0 = B_0 = D \cap K_i.$$

Hence $K_iD \cap H_i = K_i(D \cap H_i) = K_i$ is normal in H_i and $DH_i/DK_i \simeq H_i/(H_i \cap K_iD) = H_i/K_i(H_i \cap D) = H_i/K_i$ is cyclic, since G/D is cyclic by hypothesis. On the other hand, $H_i/K_i = (A_i/K_i)(B_i/K_i)$, where $|A_i/K_i| = |B_i/K_i|$, so $A_i/K_i = B_i/K_i = 1$, which implies that $A_i = B_i$. This contradiction shows that $A_0 \neq B_0$. Similarly, $A_0 \neq C_0$ and $B_0 \neq C_0$. Finally, A_0, B_0, C_0 are normal subgroups of $O^{\sigma_i}(G)$ by [10, Lemma 3.1]. Finally, Claim (5) and Theorem A imply that K_0 and H_0 are σ-permutable in G.

(7) *A_0/K_0 and B_0/K_0 are $O^{\sigma_i}(G)$-isomorphic.*

From Claim (6) we get that

$$H_0/K_0 = (A_0/K_0) \times (B_0/K_0) = (A_0/K_0) \times (C_0/K_0) = (B_0/K_0) \times (C_0/K_0).$$

Therefore,

$$A_0/K_0 \simeq ((A_0/K_0) \times (C_0/K_0))/(C_0/K_0) = (H_0/K_0)/(C_0/K_0)$$

and

$$B_0/K_0 \simeq ((B_0/K_0) \times (C_0/K_0))/(C_0/K_0) = (H_0/K_0)/(C_0/K_0)$$

are $O^{\sigma_i}(G)$-isomorphisms by [10, Lemma 3.1]. Hence we have (7).

Final contradiction. Let $f : A_0/K_0 \to B_0/K_0$ be an $O^{\sigma_i}(G)$-isomorphism. Let $K_0 < X \leq A_0$, where X covers K_0 in $\mathcal{L}$. Then X/K_0 is a chief factor of $O^{\sigma_i}(G)$ by [10, Lemma 3.1], so $L/K_0 = f(X/K_0)$ is also a chief factor of $O^{\sigma_i}(G)$. Hence L covers K_0 in $\mathcal{L}$. Now f induces an $O^{\sigma_i}(G)$-isomorphism from X/K_0 onto L/K_0, and so $L = X$ by hypothesis. Hence $K_0 < A_0 \cap B_0$, contrary to (6). The implication is proved.

The theorem is proved. □

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