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# Funk functions and constructions of dually flat Finsler metrics

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**Abstract.** Dually flat Finsler metrics arise from $\alpha$-flat information structures on Riemann–Finsler manifolds. Inspired by the theory of Funk functions and Hamel functions due to Li–Shen, we give a new approach to produce dually flat Finsler metrics in this paper. Moreover, we manufacture new dually flat spherically symmetric Finsler metrics by using the standard Euclidean norm on $\mathbb{R}^n$.

## 1. Introduction

A Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is *dually flat* if it satisfies the 'dually flat equations'

$$(F^2)_{x^i y^j} y^i = 2(F^2)_{x^j}. \tag{1.1}$$

As an example, the Funk metric

$$\Theta = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2} + \frac{\langle x, y\rangle}{1-|x|^2} \tag{1.2}$$

is dually flat on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$. This Finsler metric is produced from the Euclidean metric $\Phi(x, y) = |y|$ and the radial field $V_x := x$ by navigation problem, and therefore it has the form $\Theta = \alpha + \beta$, where $\alpha$ is the Klein metric and $\beta$ is the exact form $\beta = -\frac{1}{2} d(ln(1-|x|^2))$. Dually flat Finsler metrics arise from information geometry, and the notion of dually flat metrics in Riemann–Finsler geometry was introduced by Amari–Nagaoka and Z. Shen [1], [12].

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Recently, the study of dually flat Finsler metrics has attracted a lot of attention [2], [6]–[8], [10], [14]. Cheng–Shen–Zhou and Yu characterized dually flat Finsler metrics of Randers type [2], [14]. Liu–Mo constructed explicitly all dually flat Randers metrics by using the bijection between Randers metrics and their navigation representation [10]. Huang–Mo manufactured explicitly new examples of dually flat Finsler metrics by using the fundamental property of dually flat equation. They showed the following:

**Theorem 1.1.** *Let $f$ be a function defined by*

$$f(t,s) = g(t) + h(t)s + g'(t)s^2 + \frac{1}{6}h'(t)s^3 + \sum_{j=2}^{m}(-1)^{j-1}\frac{(2j-3)!!}{(2j+1)!}h^{(j)}(t)s^{2j+1} + b\sqrt{\frac{-s^2}{(c+s)^3} + \frac{s^4}{2(c+s)^4}},$$

*where $b$, $c$ are constants, $g$ is a differentiable function, $h$ is a polynomial function of degree $N$, where $N \le m$, and $h^{(j)}$ denotes the $j$-order derivative of $h$. Then the spherically symmetric Finsler metric*

$$F(x,y) = |y|\sqrt{f\left(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|}\right)} \tag{1.3}$$

*is dually flat on $\mathbb{B}^n(\mu)$, where $\mu$ is the radius of the ball.*

The class (1.3) of Finsler metrics contains the Funk metric on $\mathbb{B}^n$, and any member of this class satisfies

$$F(Ax, Ay) = F(x,y) \tag{1.4}$$

for all $A \in O(n)$. A Finsler metric $F$ is said to be *spherically symmetric* (*orthogonally invariant* in an alternative terminology in [8]), if it satisfies (1.4) for all $A \in O(n)$, or equivalently, if the orthogonal group $O(n)$ acts as isometries of $F$.

Spherically symmetric Finsler metrics form a rich class of Finsler metrics. Many classical Finsler metrics with nice curvature properties are spherically symmetric, such as the Bryant metrics with one parameter, the metric introduced by Berwald in 1929, the generalized fourth root metric given by Li–Shen and the Chern–Shen's metric [3].

Very recently, a significant progress has been made in studying spherically symmetric Finsler metrics. The classification theorem of projectively flat spherically symmetric metrics of constant flag curvature has been completed [11], [15].

Recall that a Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is said to be *projectively flat* if all of its geodesics are straight lines in $U$, or equivalently, if $F$ is a Hamel function. Huang–Liu–Mo produced a lot of new dually flat spherically symmetric Finsler metrics in terms of the bijection between Hamel functions and solutions of dually flat equations [6]. Huang–Mo constructed explicitly two new families of dually flat Finsler metrics with orthogonal invariance in the spirit of Pogorelov's idea [8].

A natural problem is to determine all dually flat spherically symmetric Finsler metrics on $\mathbb{B}^n(\mu) := \{v \in \mathbb{R}^n \mid |v| < \mu\}$. This problem turns out to be very difficult. The first step might be to construct as many examples as possible.

Inspired by the theory of Funk functions and Hamel functions due to Li–Shen, we give a new approach to produce new dually flat Finsler metrics on $\mathbb{B}^n$ in this paper. First, we show that any Minkowski norm $\phi$ on $\mathbb{R}^n$ produces infinitely many solutions of the dually flat equations (see Proposition 3.2 below). Furthermore, when $\phi(y) := |y|$ is the standard Euclidean norm, we prove that the solutions of the corresponding dually flat equations are spherically symmetric. More precisely, we show the following:

**Theorem 1.2.** *For $k \in \{0, 1, \dots\}$, the Finsler metrics*

$$F(x, y) := \sqrt{(\Theta^2)_{x^{i_1}\dots x^{i_k}} x^{i_1} \dots x^{i_k} + k(\Theta^2)_{x^{i_1}\dots x^{i_{k-1}}} x^{i_1} \dots x^{i_{k-1}}}$$

*are dually flat on $\mathbb{B}^n$ (here $\Theta$ is the Funk metric on the unit ball $\mathbb{B}^n$). Moreover, $F$ is spherically symmetric.*

We will prove Theorem 1.2 in Section 4. As an application, we give explicit expressions of the dually flat Finsler metrics with orthogonal invariance for $k \in \{0, 1, 2, 3\}$ by using some interesting properties (see Proposition 5.1, 5.2, 5.3 and Lemma 2.1 below). For recent results on spherically symmetric Finsler metrics, we refer the reader to [5]–[8], [11], [15].

## 2. Preliminaries

A *Minkowski norm* on a vector space $V$ is a nonnegative function $F : V \to [0, +\infty)$ with the following properties:

(i) $F$ is positively homogeneous of degree one, i.e., for any $y \in V$ and any $\lambda > 0$, $F(\lambda y) = \lambda F(y)$;

(ii) $F$ is $C^\infty$ on $V\backslash\{0\}$ and for any tangent vector $y \in V\backslash\{0\}$, the symmetric bilinear form $g_y : V \times V \to \mathbb{R}$ given by

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y+su+tv)]\,|_{s=t=0}$$

is positive definite.

Let $\langle\, ,\,\rangle$ denote the standard inner product on $\mathbb{R}^n$, defined by $\langle u,v\rangle := \sum_{i=1}^n u^i v^i$. Then $|y| := \sqrt{\langle y,y\rangle}$ is the *standard Euclidean norm* on $\mathbb{R}^n$.

Let $M$ be a differentiable manifold, and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of $M$, where $T_xM$ is the tangent space at $x \in M$. A function $F$ on $TM$ is called a *Finsler metric* on $M$ if it has the following properties:

(a) $F$ is $C^\infty$ on $TM_0 := TM\backslash\{0\}$;

(b) $F_x(y) := F(x,\, y)$ is a Minkowski norm on $T_xM$ for any $x \in M$.

The pair $(M,F)$ is called a *Finsler manifold* or a *Finsler space.*

Let $U$ be an open subset in $\mathbb{R}^n$. A scalar function $\Theta$ on $U \times \mathbb{R}^n$ is called a *Funk function* [9] if

$$\Theta(x,\, \lambda y) = \lambda\Theta(x,\, y), \tag{2.1}$$

for all $\lambda > 0$, and it satisfies

$$\Theta_{x^k} = \Theta\Theta_{y^k}. \tag{2.2}$$

As an example, consider a Minkowski norm $\phi : E \to \mathbb{R}$ on a vector space $E$. We obtain a *Funk metric (or Funk function)* $\Theta$ *on the strongly convex domain* $\Omega := \{v \in E|\phi(v) < 1\}$ with the navigation data $\Phi(x,y) = \phi(y)$ and $V_x := x$ [3], [9].

A scalar function $\Theta$ on $U\times\mathbb{R}^n$ is called a *Hamel function* if it satisfies (2.1) and

$$\Theta_{x^k} - \Theta_{y^k x^j} y^j = 0. \tag{2.3}$$

Clearly, any Funk function is a Hamel function and the Hamel functions form a linear space [9].

**Lemma 2.1.** *Let* $|\, ,\,|$ *and* $\langle\, ,\,\rangle$ *be the standard Euclidean norm and inner product on* $\mathbb{R}^n$*, respectively. Then* $(|x|^2|y|^2 - \langle x,y\rangle^2)_{y^j}x^j = 0$.

Proof. By a straightforward computation, one obtains

$$(|x|^2|y|^2 - \langle x,y\rangle^2)_{y^j} = 2(|x|^2y^j - \langle x,y\rangle x^j). \tag{2.4}$$

Contracting (2.4) with $x^j$ gives

$$(|x|^2|y|^2 - \langle x,y\rangle^2)_{y^j}x^j = 2(|x|^2\langle x,y\rangle - \langle x,y\rangle|x|^2) = 0. \qquad \square$$

## 3. Solutions of dually flat equations

In this section, we are going to produce infinitely many solutions of dually flat equations by using a Minkowski norm on $\mathbb{R}^n$.

**Lemma 3.1.** *Let $U \subset \mathbb{R}^n$ be a strongly convex domain determined by a Minkowski norm $\phi$ on $\mathbb{R}^n$. Let $\Theta$ denote the Funk metric on $U$ defined by*

$$\Theta(x,y) = \phi(y + \Theta(x,y)x). \tag{3.1}$$

*Then*

$$(\Theta^n)_{y^{i_1}y^{i_2}\dots y^{i_{n-2}}} = \frac{n}{2}(\Theta^2)_{x^{i_1}x^{i_2}\dots x^{i_{n-2}}} \tag{3.2}$$

*for $n \in \{3,4,\dots\}$.*

Proof. By (1.38) in [3], we get (2.2). It follows that $(\Theta^3)_{y^i} = 3\Theta^2\Theta_{y^i} = 3\Theta\Theta_{x^i} = \frac{3}{2}(\Theta^2)_{x^i}$. Hence (3.2) holds when $n = 3$. Suppose that the Funk metric $\Theta$ satisfies (3.2). Then we have

$$\begin{aligned}(\Theta^{n+1})_{y^{i_1}\dots y^{i_{n-2}}y^{i_{n-1}}} &= (n+1)(\Theta^n\Theta_{y^{i_1}})_{y^{i_2}\dots y^{i_{n-1}}} = (n+1)(\Theta^{n-1}\Theta_{x^{i_1}})_{y^{i_2}\dots y^{i_{n-1}}}\\ &= \frac{n+1}{n}(\Theta^n)_{x^{i_1}y^{i_2}\dots y^{i_{n-1}}} = \frac{n+1}{n}(\Theta^n)_{y^{i_2}\dots y^{i_{n-1}}x^{i_1}}\\ &= \frac{n+1}{n}\frac{n}{2}(\Theta^2)_{x^{i_2}\dots x^{i_{n-1}}x^{i_1}} = \frac{n+1}{2}(\Theta^2)_{x^{i_1}\dots x^{i_{n-1}}},\end{aligned}$$

where we have made use of (2.2) and (3.2). Thus Lemma 3.1 follows by mathematical induction. □

**Proposition 3.2.** *Let $\Theta$ be the Funk metric on a strongly convex domain. Then*

$$F := \sqrt{(\Theta^2)_{x^{i_1}\dots x^{i_k}}x^{i_1}\dots x^{i_k} + k(\Theta^2)_{x^{i_1}\dots x^{i_{k-1}}}x^{i_1}\dots x^{i_{k-1}}} \tag{3.3}$$

*satisfies the dually flat equation* (1.1).

Proof. By Li–Shen's Example 3.1 in [9],

$$P := \frac{1}{k+1}(\Theta^{k+1})_{y^{i_1}\dots y^{i_k}}x^{i_1}\dots x^{i_k} \tag{3.4}$$

is a Hamel function. It follows that $P$ satisfies

$$P_{x^k} - P_{y^k x^j}y^j = 0. \tag{3.5}$$

Theorem 2.3 in [6] tells us that

$$\sqrt{P_{x^i}y^i} \tag{3.6}$$

is a solution of (1.1). From (3.4), one obtains

$$(k+1)P_{x^j} = (\mathrm{I})_j + (\mathrm{II})_j, \tag{3.7}$$

where

$$\begin{aligned}(\mathrm{I})_j :&= (\Theta^{k+1})_{y^{i_1}\ldots y^{i_k}x^j} x^{i_1}\ldots x^{i_k} = (\Theta^{k+1})_{x^j y^{i_1}\ldots y^{i_k}} x^{i_1}\ldots x^{i_k}\\ &= (k+1)(\Theta^k\Theta_{x^j})_{y^{i_1}\ldots y^{i_k}} x^{i_1}\ldots x^{i_k} = (k+1)(\Theta^{k+1}\Theta_{y^j})_{y^{i_1}\ldots y^{i_k}} x^{i_1}\ldots x^{i_k}\\ &= \frac{k+1}{k+2}(\Theta^{k+2})_{y^j y^{i_1}\ldots y^{i_k}} x^{i_1}\ldots x^{i_k} = \frac{k+1}{k+2}(\Theta^{k+2})_{y^{i_1}\ldots y^{i_k}y^j} x^{i_1}\ldots x^{i_k},\end{aligned} \tag{3.8}$$

having made use of (2.2). In (3.7),

$$\begin{aligned}(\mathrm{II})_j :&= (\Theta^{k+1})_{y^{i_1}\ldots y^{i_k}}(x^{i_1}\ldots x^{i_k})_{x^j}\\ &= (\Theta^{k+1})_{y^{i_1}\ldots y^{i_k}}(\delta^{i_1}_j x^{i_2}\ldots x^{i_k} + x^{i_1}\ldots x^{i_{l-1}}\delta^{i_l}_j x^{i_{l+1}}\ldots x^{i_k} + x^{i_1}\ldots x^{i_{k-1}}\delta^{i_k}_j)\\ &= (\Theta^{k+1})_{y^j y^{i_2}\ldots y^{i_k}} x^{i_2}\ldots x^{i_k} + \ldots + (\Theta^{k+1})_{y^{i_1}\ldots y^{i_{l-1}}y^j y^{i_{l+1}}\ldots y^{i_k}} x^{i_1}\ldots \widehat{x^{i_l}}\ldots x^{i_k}\\ &\quad + \ldots + (\Theta^{k+1})_{y^{i_1}\ldots y^{i_{k-1}}y^j} x^{i_1}\ldots x^{i_{k-1}}\\ &= k(\Theta^{k+1})_{y^{i_1}\ldots y^{i_{k-1}}y^j} x^{i_1}\ldots x^{i_{k-1}}.\end{aligned} \tag{3.9}$$

Plugging (3.8) and (3.9) into (3.7) yields

$$P_{x^j} = \frac{1}{k+2}(\Theta^{k+2})_{y^{i_1}\ldots y^{i_k}y^j} x^{i_1}\ldots x^{i_k} + \frac{k}{k+1}(\Theta^{k+1})_{y^{i_1}\ldots y^{i_{k-1}}y^j} x^{i_1}\ldots x^{i_{k-1}}.$$

Note that both $[\Theta^{k+2}]_{y^{i_1}\ldots y^{i_k}}$ and $[\Theta^{k+1}]_{y^{i_1}\ldots y^{i_{k-1}}}$ are positively homogeneous of degree 2. Hence

$$\begin{aligned}P_{x^j}y^j &= \frac{1}{k+2}(\Theta^{k+2})_{y^{i_1}\ldots y^{i_k}y^j} y^j x^{i_1}\ldots x^{i_k} + \frac{k}{k+1}(\Theta^{k+1})_{y^{i_1}\ldots y^{i_{k-1}}y^j} y^j x^{i_1}\ldots x^{i_{k-1}}\\ &= \frac{2}{k+2}(\Theta^{k+2})_{y^{i_1}\ldots y^{i_k}} x^{i_1}\ldots x^{i_k} + \frac{2k}{k+1}(\Theta^{k+1})_{y^{i_1}\ldots y^{i_{k-1}}} x^{i_1}\ldots x^{i_{k-1}}\\ &= \frac{2}{k+2}\frac{k+2}{2}(\Theta^2)_{x^{i_1}\ldots x^{i_k}} x^{i_1}\ldots x^{i_k} + \frac{2k}{k+1}\frac{k+1}{2}(\Theta^2)_{x^{i_1}\ldots x^{i_{k-1}}} x^{i_1}\ldots x^{i_{k-1}}\\ &= (\Theta^2)_{x^{i_1}\ldots x^{i_k}} x^{i_1}\ldots x^{i_k} + k(\Theta^2)_{x^{i_1}\ldots x^{i_{k-1}}} x^{i_1}\ldots x^{i_{k-1}}.\end{aligned}$$

Plugging this into (3.6) yields (3.3), and hence (3.3) is a solution of (1.1). □

**Corollary 3.3.** *Let* $\Theta = \alpha + \beta$ *be the Funk metric expressed in* (1.2). *Then the function* $F$ *given in* (3.3) *satisfies the dually flat equation* (1.1).

Proof. By choosing $\phi(y) = |y|$ in Lemma 3.1, we obtain that $\Theta$ satisfies (1.2). Then, in fact, $\Theta$ is the Funk metric on the unit ball $\mathbb{B}^n$. Hence

$$F := \sqrt{(\Theta^2)_{x^{i_1}\dots x^{i_k}} x^{i_1}\dots x^{i_k} + k(\Theta^2)_{x^{i_1}\dots x^{i_{k-1}}} x^{i_1}\dots x^{i_{k-1}}}$$

satisfies (1.1). □

## 4. Dually flat spherically symmetric Finsler metrics

In this section, we are going to manufacture dually flat spherically symmetric Finsler metrics via the standard Euclidean norm in $\mathbb{R}^n$.

**Lemma 4.1.** *For $k \in \{0, 1, 2, \dots\}$,*

$$(\Theta^2)_{x^{i_1}\dots x^{i_k}} x^{i_1}\dots x^{i_k} \tag{4.1}$$

*is a function of $|x|$, $|y|^2$ and $\langle x, y\rangle$, where $|\,,\,|$ and $\langle\,,\,\rangle$ are the standard Euclidean norm and inner product on $\mathbb{R}^n$, respectively, and $\Theta$ is the Funk metric on the unit ball $\mathbb{B}^n$.*

Proof. By using (3.2), we get $(\Theta^2)_{x^{i_1}\dots x^{i_k}} = \frac{2}{k+2}(\Theta^{k+2})_{y^{i_1}\dots y^{i_k}}$. Hence, it suffices to show that $(\Theta^{k+2})_{y^{i_1}\dots y^{i_k}} x^{i_1}\dots x^{i_k}$ is a function of $|x|$, $|y|^2$ and $\langle x, y\rangle$. By an explicit expression for an arbitrary partial derivative of a product of functions [4], [13], we see that $(\Theta^{k+2})_{y^{i_1}\dots y^{i_k}} x^{i_1}\dots x^{i_k}$ is a homogeneous polynomial (of degree $k+2$) of $\Theta$, $\Theta_{y^i} x^i$, $\Theta_{y^i y^j} x^i x^j, \dots, \Theta_{y^{i_1}\dots y^{i_k}} x^{i_1}\dots x^{i_k}$. It follows that we only need to prove that for $k \in \{0, 1, \dots\}$

$$\Theta_{y^{i_1}\dots y^{i_k}} x^{i_1}\dots x^{i_k} \tag{4.2}$$

is a function of $|x|$, $|y|^2$, $\langle x, y\rangle$. When $k = 0$, $\Theta$ is clearly a function of $|x|$, $|y|^2$ and $\langle x, y\rangle$ from (1.2). Suppose that $\Theta_{y^{i_1}\dots y^{i_{k-1}}} x^{i_1}\dots x^{i_{k-1}} = \psi(|x|, |y|^2, \langle x, y\rangle)$. Then we have

$$\begin{aligned}
(\Theta)_{y^{i_1}\dots y^{i_k}} x^{i_1}\dots x^{i_k} &= (\Theta_{y^{i_1}\dots y^{i_{k-1}}})_{y^{i_k}} x^{i_1}\dots x^{i_k} = (\Theta_{y^{i_1}\dots y^{i_{k-1}}} x^{i_1}\dots x^{i_{k-1}})_{y^{i_k}} x^{i_k}\\
&= \psi(|x|, |y|^2, \langle x, y\rangle)_{y^{i_k}} x^{i_k}\\
&= \left(\frac{\partial\psi}{\partial|x|}\frac{\partial|x|}{\partial y^{i_k}} + \frac{\partial\psi}{\partial|y|^2}\frac{\partial|y|^2}{\partial y^{i_k}} + \frac{\partial\psi}{\partial\langle x, y\rangle}\frac{\partial\langle x, y\rangle}{\partial y^{i_k}}\right) x^{i_k}\\
&= \sum_{i_k}\left(2y^{i_k}\frac{\partial\psi}{\partial|y|^2} + x^{i_k}\frac{\partial\psi}{\partial\langle x, y\rangle}\right) x^{i_k}\\
&= 2\langle x, y\rangle\frac{\partial\psi}{\partial|y|^2}(|x|, |y|^2, \langle x, y\rangle) + |x|^2\frac{\partial\psi}{\partial\langle x, y\rangle}(|x|, |y|^2, \langle x, y\rangle),
\end{aligned}$$

and the assertion follows by mathematical induction for $k \in \{0, 1, \cdots\}$. □

Proof of Theorem 1.2. The statement is an immediate consequence of Corollary 3.3 and Lemma 4.1. □

## 5. Some explicit constructions

In this section, we are going to give some explicit expressions of dually flat spherically symmetric Finsler metrics in Theorem 1.2.

Consider the Funk metric $\Theta$ defined in (1.2). We express it in the form

$$\Theta = \alpha + \frac{\langle x, y\rangle}{1-|x|^2}, \tag{5.1}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2}. \tag{5.2}$$

Differentiating (5.2) with respect to $y^k$, we obtain

$$\alpha_{y^k} = \frac{1}{\alpha}\left[\frac{y^k}{1-|x|^2} + \frac{\langle x, y\rangle x^k}{(1-|x|^2)^2}\right]. \tag{5.3}$$

Contracting (5.3) with $x^k$ gives

$$\alpha_{y^k} x^k = \frac{1}{\alpha}\frac{\langle x, y\rangle}{(1-|x|^2)^2}. \tag{5.4}$$

Thus we obtain

$$\Theta_{y^k} x^k = \alpha_{y^k} x^k + \frac{\langle x, y\rangle_{y^k} x^k}{1-|x|^2} = \frac{1}{\alpha}\frac{\langle x, y\rangle}{(1-|x|^2)^2} + \frac{|x|^2}{1-|x|^2},$$

and hence

$$1 + \Theta_{y^k} x^k = \frac{\Theta}{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}. \tag{5.5}$$

This, together with (2.2) and (1.2), yields

$$\begin{aligned}\Theta^2 + (\Theta^2)_{x^j} x^j &= \Theta^2 + 2\Theta\Theta_{x^j} x^j = \Theta^2 + 2\Theta^2\Theta_{y^j} x^j = 2\Theta^2(1 + \Theta_{y^j} x^j) - \Theta^2 \\ &= \Theta^2\left[\frac{2\Theta}{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}} - 1\right] \\ &= \left[1 + |x|^2 + \frac{2\langle x, y\rangle}{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}\right]\frac{\Theta^2}{1-|x|^2}. \end{aligned} \tag{5.6}$$

Taking $k = 1$ in Theorem 1.2 and using (5.6), we have the following:

**Proposition 5.1.** *Let $f$ be a function defined by*

$$f(t,s)=\left[1+2t+\frac{2s}{\sqrt{1-2t+s^2}}\right]\frac{(\sqrt{1-2t+s^2}+s)^2}{(1-2t)^3}.$$

*Then the spherically symmetric Finsler metric given by*

$$F(x,y)=|y|\sqrt{f\left(\frac{|x|^2}{2},\frac{\langle x,y\rangle}{|y|}\right)}$$

*is dually flat on $\mathbb{B}^n$.*

Now we consider the case $k=2$ in Theorem 1.2. Plugging (5.2) into (5.5) yields

$$\Theta_{y^k}x^k=\frac{\Theta}{\alpha(1-|x|^2)}-1. \tag{5.7}$$

Differentiating (5.7) with respect to $y^l$, we get

$$\Theta_{y^ky^l}x^k=(\Theta_{y^k}x^k)_{y^l}=\frac{1}{1-|x|^2}\left(\frac{\Theta}{\alpha}\right)_{y^l}=\frac{1}{\alpha^2(1-|x|^2)}(\alpha\Theta_{y^l}-\Theta\alpha_{y^l}). \tag{5.8}$$

By (5.4), (5.7) and (5.8), we obtain

$$\begin{aligned}\Theta_{y^ky^l}x^kx^l&=\frac{1}{\alpha^2(1-|x|^2)}(\alpha\Theta_{y^l}x^l-\Theta\alpha_{y^l}x^l)\\&=\frac{1}{\alpha^2(1-|x|^2)}\left\{\alpha\left[\frac{\Theta}{\alpha(1-|x|^2)}-1\right]-\Theta\frac{\langle x,y\rangle}{\alpha(1-|x|^2)^2}\right\}=\frac{(\mathrm{I})}{\alpha^2(1-|x|^2)}\end{aligned} \tag{5.9}$$

where $(\mathrm{I}):=-\alpha+\frac{\Theta}{1-|x|^2}-\frac{\Theta\langle x,y\rangle}{\alpha(1-|x|^2)^2}$. It follows that

$$\begin{aligned}\alpha(1-|x|^2)^2(\mathrm{I})&=\alpha(1-|x|^2)\Theta-\alpha^2(1-|x|^2)^2-\Theta\langle x,y\rangle\\&=\alpha(1-|x|^2)\left(\alpha+\frac{\langle x,y\rangle}{1-|x|^2}\right)-\alpha^2(1-|x|^2)^2-\left(\alpha+\frac{\langle x,y\rangle}{1-|x|^2}\right)\langle x,y\rangle\\&=\alpha^2(1-|x|^2)-\alpha^2(1-|x|^2)^2-\frac{\langle x,y\rangle^2}{1-|x|^2}=\alpha^2|x|^2(1-|x|^2)-\frac{\langle x,y\rangle^2}{1-|x|^2}\\&=|x|^2\left(|y|^2+\frac{\langle x,y\rangle^2}{1-|x|^2}\right)-\frac{\langle x,y\rangle^2}{1-|x|^2}=|x|^2|y|^2-\langle x,y\rangle^2.\end{aligned}$$

Plugging this into (5.9) yields

$$\Theta_{y^ky^l}x^kx^l=\frac{|x|^2|y|^2-\langle x,y\rangle^2}{\alpha^3(1-|x|^2)^3}. \tag{5.10}$$

By (5.6) and (5.7), we have

$$(\Theta^2)_{x^j}x^j = 2\Theta^2\Theta_{y^j}x^j = 2\Theta^2\left[\frac{\Theta}{\alpha(1-|x|^2)} - 1\right]. \tag{5.11}$$

Taking $n = 4$ in (3.2), we get $(\Theta^2)_{x^ix^j} = \frac{1}{2}(\Theta^4)_{y^iy^j} = 2(3\Theta^2\Theta_{y^i}\Theta_{y^j} + \Theta^3\Theta_{y^iy^j})$. It follows that

$$(\Theta^2)_{x^ix^j}x^ix^j = 2\Theta^2[3(\Theta_{y^i}x^i)^2 + \Theta\Theta_{y^iy^j}x^ix^j]. \tag{5.12}$$

This, together with (5.7) and (5.10), yields

$$\begin{aligned}(\Theta^2)_{x^ix^j}x^ix^j + 2(\Theta^2)_{x^i}x^i &= 2\Theta^2[3(\Theta_{y^i}x^i)^2 + \Theta\Theta_{y^iy^j}x^ix^j] + 4\Theta^2\Theta_{y^i}x^i \\ &= 4\Theta^2\left[\frac{\Theta}{\alpha(1-|x|^2)}-1\right] + 6\Theta^2\left[\frac{\Theta}{\alpha(1-|x|^2)}-1\right]^2 + 2\Theta^3\frac{|x|^2|y|^2-\langle x,y\rangle^2}{\alpha^3(1-|x|^2)^3}. \end{aligned}\tag{5.13}$$

Taking $k = 2$ in Theorem 1.2 and using (5.13), we obtain the following:

**Proposition 5.2.** *Let $f$ be a function defined by*

$$f(t,s) = 4\Phi^2\left(\frac{\Phi}{\Psi} - 1\right) + 6\Phi^2\left(\frac{\Phi}{\Psi} - 1\right)^2 + 2(2t - s^2)\frac{\Phi^3}{\Psi^3},$$

*where*

$$\Psi := \sqrt{1 - 2t - s^2}, \qquad \Phi := \frac{\Psi + s}{1 - 2t}. \tag{5.14}$$

*Then the spherically symmetric Finsler metric given by*

$$F(x,y) = |y|\sqrt{f\left(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|}\right)}$$

*is dually flat on $\mathbb{B}^n$.*

Now we discuss the case $k = 3$ in Theorem 1.2. From (5.4), (5.10) and Lemma 2.1, we get

$$\begin{aligned}\Theta_{y^iy^jy^k}x^ix^jx^k = (\Theta_{y^iy^j}x^ix^j)_{y^k}x^k &= \left[\frac{|x|^2|y|^2 - \langle x,y\rangle^2}{\alpha^3(1-|x|^2)^3}\right]_{y^k}x^k \\ &= \frac{|x|^2|y|^2 - \langle x,y\rangle^2}{(1-|x|^2)^3}\left(\frac{1}{\alpha^3}\right)_{y^k}x^k. \end{aligned}\tag{5.15}$$

Taking $n=5$ in (3.2), we obtain

$$\begin{aligned}(\Theta^2)_{x^ix^jx^k} &= \frac{2}{5}(\Theta^5)_{y^iy^jy^k}\\ &= 2\Theta^2\Big[12\Theta_{y^i}\Theta_{y^j}\Theta_{y^k}+4\Theta\Theta_{y^i}\Theta_{y^jy^k}(i\to j\to k\to i)+\Theta^2\Theta_{y^iy^jy^k}\Big],\end{aligned}$$

where $i\to j\to k\to i$ denotes cyclic permutation. It follows that

$$\begin{aligned}&(\Theta^2)_{x^ix^jx^k}x^ix^jx^k\\ &= 2\Theta^2\Big[12(\Theta_{y^i}x^i)^3+12\Theta(\Theta_{y^i}x^i)(\Theta_{y^jy^k}x^jx^k)+\Theta^2\Theta_{y^iy^jy^k}x^ix^jx^k\Big]\\ &= 2\Theta^2\Big[12(\Theta_{y^i}x^i)^3+12\Theta(\Theta_{y^i}x^i)(\Theta_{y^jy^k}x^jx^k)-\frac{3\Theta^2\langle x,y\rangle}{\alpha^2(1-|x|^2)^2}(\Theta_{y^ky^l}x^kx^l)\Big]. \quad (5.16)\end{aligned}$$

Applying (5.16) together with (5.12) and (5.7), we find that

$$\begin{aligned}&(\Theta^2)_{x^ix^jx^k}x^ix^jx^k+3(\Theta^2)_{x^ix^j}x^ix^j\\ &\quad= 2\Theta^2\Big[12(\Theta_{y^i}x^i)^3+12\Theta(\Theta_{y^i}x^i)(\Theta_{y^jy^k}x^jx^k)-\frac{3\Theta^2\langle x,y\rangle}{\alpha^2(1-|x|^2)^2}(\Theta_{y^ky^l}x^kx^l)\Big]\\ &\qquad+6\Theta^2[3(\Theta_{y^i}x^i)^2+\Theta\Theta_{y^iy^j}x^ix^j]\\ &\quad= 2\Theta^2\bigg\{12\Big[\frac{\Theta}{\alpha(1-|x|^2)}-1\Big]^3+12\Theta\Big[\frac{\Theta}{\alpha(1-|x|^2)}-1\Big]\frac{|x|^2|y|^2-\langle x,y\rangle^2}{\alpha^3(1-|x|^2)^3}\\ &\qquad-\frac{3\Theta^2\langle x,y\rangle}{\alpha^2(1-|x|^2)^2}\frac{|x|^2|y|^2-\langle x,y\rangle^2}{\alpha^3(1-|x|^2)^3}\bigg\}\\ &\qquad+6\Theta^2\bigg\{3\Big[\frac{\Theta}{\alpha(1-|x|^2)}-1\Big]^2+\Theta\frac{|x|^2|y|^2-\langle x,y\rangle^2}{\alpha^3(1-|x|^2)^3}\bigg\}. \quad (5.17)\end{aligned}$$

Taking $k=3$ in Theorem 1.2 and using (5.17), we have the following:

**Proposition 5.3.** *Let $f$ be a function defined by*

$$\begin{aligned}f(t,s) &= 2\Phi^2\bigg[12\Big(\frac{\Phi}{\Psi}-1\Big)^3+12\Phi\Big(\frac{\Phi}{\Psi}-1\Big)\frac{2t-s^2}{\Psi^3}-\frac{3s\Phi^2(2t-s^2)}{\Psi^5}\bigg]\\ &\quad+6\Phi^2\bigg[3\Big(\frac{\Phi}{\Psi}-1\Big)^2+\Phi\frac{2t-s^2}{\Psi^3}\bigg],\end{aligned}$$

*where* $\Psi$ *and* $\Phi$ *are given in* (5.14). *Then the spherically symmetric Finsler metric given by*

$$F(x,y) = |y|\sqrt{f\left(\frac{|x|^2}{2}, \frac{\langle x, y\rangle}{|y|}\right)}$$

*is dually flat on* $\mathbb{B}^n$.

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