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Yet another generalization of Sylvester's theorem and its application

By SHANTA LAISHRAM (New Delhi), SUDHIR SINGH NGAIRANGBAM (Canchipur) and RANJIT SINGH MAIBAM (Canchipur)

Abstract. In this paper, we consider Sylvester's theorem on the largest prime divisor of a product of consecutive terms of an arithmetic progression, and prove another generalization of this theorem. As an application of this generalization, we provide an explicit method to find perfect powers in a product of terms of binary recurrence sequences and associated Lucas sequences whose indices come from consecutive terms of an arithmetic progression. In particular, we prove explicit results for Fibonacci, Jacobsthal, Mersenne and associated Lucas sequences.

1. Introduction

Let k be a positive integer. A well-known theorem of SYLVESTER [33] states that a product of k consecutive terms, each exceeding k, is divisible by a prime $> k$. In other words, for positive integers n, k with $n > k$,

$$P(n(n+1)\cdots(n+k-1)) > k, \tag{1}$$

where $P(m)$ denotes the greatest prime factor of a positive integer m with the convention $P(1) = 1$. The assumption $n > k$ is necessary, since the assertion is not valid at $n = 1$ for any k. Let n, d, k be positive integers. We assume from

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now onward that $\gcd(n,d)=1$ whenever n,d,k is given. For $d>1$, Shorey and Tijdeman [31] extended the result of Sylvester by showing that

$$P(n(n+d)\cdots(n+(k-1)d)>k \quad \text{for } k\geq 3 \quad \text{unless } (n,d,k)=(2,7,3). \tag{2}$$

Observe that $k\geq 3$ is necessary, since $n=1$, $d=2^r-1$, $r\geq 1$ give infinitely many counterexamples when $k=2$. Since a prime $>k$ can divide at most one term of $n+id$, $0\leq i<k$, we obtain from (1) and (2) that for positive n,d,k with $n>k$ if $d=1$ and $k\geq 3$ if $d>1$, there is a term $n+id$ with $0\leq i<k$ which is divisible by a prime $>k$ except when $(n,d,k)=(2,7,3)$. We consider a related question:

Question. Given positive integers n,d,k, does there exist an i with $0\leq i<k$ such that $n+id$ is odd and $P(n+id)>k$?

In [6], Bravo, Das, Guzman and Laishram answered this question for $d>1$ and $k\geq 6$ by proving the following result.

Theorem A. *Let $n\geq 1$, $d>1$ and $k\geq 6$ with $gcd(n,d)=1$. Then there is at least one i, $0\leq i<k$ with $P(n+id)>k$ and $n+id$ odd.*

We supplement this result and completely answer the above question, by proving

Theorem 1. *Let n,d,k be positive integers with $\gcd(n,d)=1$ and $n>k$ if $d=1$. Then there is an integer i, $0\leq i<k$ with $n+id$ odd and $P(n+id)>k$ unless*

$$\begin{aligned}
&k=1: n \text{ even};\\
&k=2: n=1,\ d=2^a-1 \text{ for } a\geq 0;\\
&k=3: n=1,\ d=\frac{1}{2}(3^a-1) \text{ for } a>0,\\
&\qquad n \text{ even},\ n+d=3^a \text{ for } a>0;\\
&k=4: n=1,\ d=\frac{1}{2}(3^a-1) \text{ for } a>0;\\
&k=5: n=\frac{1}{2}(3^{a+1}-5^b),\ d=\frac{1}{2}(5^b-3^a) \text{ with } 3^a<5^b<3^{a+1},\ a \text{ odd},\\
&\qquad n=\frac{3}{2}(5^b-3^{a-1}),\ d=\frac{1}{2}(3^a-5^b) \text{ with } 3^{a-1}<5^b<3^a,\ a \text{ odd}.
\end{aligned}$$

We can see easily that these exceptions are necessary. We prove Theorem 1 in Section 3. As an application of Theorem 1, we prove some results on the product of terms of a binary recurrence sequence being a perfect power.

Given $r, s \in \mathbb{Z}$ with $\gcd(r, s) = 1$, the binary recurrence sequences $U_n = U_n(r, s)$ and $V_n = V_n(r, s)$ given by

$$U_0 = 0, \quad U_1 = 1, \quad U_{n+2} = rU_{n+1} + sU_n, \quad \forall n \geq 0$$

and

$$V_0 = 2, \quad V_1 = r, \quad V_{n+2} = rV_{n+1} + sV_n, \quad \forall n \geq 0$$

are called Lucas sequences of the first kind and Lucas sequences of the second kind, respectively. U_n and V_n are given by explicit Binet formulas:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n,$$

where α and β are the roots of the characteristic equation $x^2 - rx - s = 0$ of the binary recurrence sequence. Some of the well-known Lucas sequences are:

- $U_n(1, 1)$: Fibonacci numbers F_n; $V_n(1, 1)$: Lucas numbers L_n.
- $U_n(2, 1)$: Pell numbers; $V_n(2, 1)$: companion Pell numbers or Pell–Lucas numbers.
- $U_n(1, 2)$: Jacobsthal numbers J_n; $V_n(1, 2)$: Jacobsthal–Lucas numbers $\mathfrak{J}_n$.
- $U_n(3, -2)$: Mersenne numbers $M_n = 2^n - 1$; $V_n(3, -2)$: Mersenne–Lucas numbers $\mathfrak{F}_n = 2^n + 1$, which include the Fermat numbers.

Given (r, s) with $\gcd(r, s) = 1$, the binary recurrence sequences $U_n(r, s)$ and $V_n(r, s)$ are said to be non-degenerate if $r^2 + 4s \neq 0$. From now on, we only consider non-degenerate binary recurrence sequences. Let $\mathcal{S}$ be a sequence of positive integers. Let $k \geq 1$ be an integer and $\mathcal{P}(k)$ be a function depending on k and the sequence $\mathcal{S}$. We consider equations

$$U_{n_1} U_{n_2} \cdots U_{n_k} = by^\ell \tag{3}$$

and

$$V_{n_1} V_{n_2} \cdots V_{n_k} = by^\ell \tag{4}$$

in positive integer variables, $k \geq 1$, $n_i \in \mathcal{S}$, $1 \leq i \leq k, b, y$, $\ell > 1$ with $n_1 < n_2 < \cdots < n_k$, and b is an ℓ^{th} power free positive integer with $P(b) \leq \mathcal{P}(k)$.

For a given b, it follows from results proved independently by Pethő [25] and Shorey and Stewart [30] that either one of equations (3) and (4) with $k = 1$ or $k = 2$ implies that n, d, y and m are bounded by an effectively computable number depending only on the sequence and b. In fact, the preceding assertion with b composed only of primes from a given finite set follows from the result of Pethő.

In [9], Bugeaud, Luca, Mignotte and Siksek considered equation (3) with b composed of fixed set of primes and $k < \ell$ with ℓ prime. In [22], Luca and Shorey considered (3) and (4) with $\mathcal{S} = \{n + id : i \geq 0\}$, $(n, d) = 1$ and $\mathcal{P}(k) \leq 2k$ to show that $\max\{n, d, k, b, y, \ell\}$ is bounded by an effectively computable number depending only on r, s and k. We refer to [9], [22] and [23] for more results in this direction.

On the lines of their ideas, we prove the following result (Theorem 2) as an application of Theorem 1.

Let p be a prime such that $p \mid U_n$(respectively V_n), but $p \nmid (r^2+4s)U_1 \cdots U_{n-1}$ (respectively $(r^2 + 4s)V_1 \cdots V_{n-1}$). Then p is said to be a primitive prime divisor of U_n(respectively V_n). It is easy to see that primitive divisors of V_n are precisely those primitive prime divisors of U_{2n}. It is known that primitive divisors always exist except for a finite number of n which are explicitly known (see [5]). In fact, primitive prime divisors always exist for U_n when $n > 30$ and for V_n when $n > 15$. Only odd prime powers $Q > 3$ for which U_Q does not have a primitive prime divisor for some sequence U_n are $5, 7, 13$ and the complete list of such sequences are given in the following table:

Q	(r, s)	U_Q
5	(1,1), (1,-2), (2,-11), (1,-3), (1,-4), (12,-55), (12,-377)	$U_5 = 5$ or $U_5 = \pm 1$
7	(1,-2), (1,-5)	$U_7 = 7$ or $U_7 = 1$
13	(1,-2)	$U_{13} = 45 = 3^2 \times 5$

Table 1

Only odd prime powers Q for which V_Q does not have a primitive prime divisor for some sequence V_n are 5 and 9 and the complete list of such sequences are given in the following table:

Q	$(r, s; V_Q)$
5	$(2, -3; 2), (5, -7; -5^2), (5, -18; -2^2 \cdot 5)$
9	(1,-2); -5)

Table 2

We note here that V_{2Q} has primitive divisors for each odd prime power $Q > 3$. Let

$$\mathcal{N}_Q^1 = \{(r, s) : U_Q(r, s) \text{ has no primitive prime divisor}\}$$

and

$$\mathcal{N}_Q^2 = \{(r, s) : V_Q(r, s) \text{ has no primitive prime divisor}\}.$$

Then $\mathcal{N}_Q^1 = \emptyset$ for $Q \geq 5$, $Q \notin \{5, 7, 13\}$ and $\mathcal{N}_Q^2 = \emptyset$ for $Q \geq 5$, Q$\notin \{5, 9\}$. Let

$$\mathcal{U}^{pow} := \mathcal{U}^{pow}(r, s) := \{(m, \ell) : U_m = y_1^\ell \text{ for some } y_1 > 1\},$$

$$\mathcal{V}^{pow} := \mathcal{V}^{pow}(r, s) := \{(m, \ell) : V_m = r_1 y_2^\ell \text{ for some } y_2 > 1 \text{ and } r_1 \text{ with } p|r_1 \Rightarrow p|r\}.$$

It follows from a result of Shorey and Stewart [30] that both $\mathcal{U}^{pow}$ and $\mathcal{V}^{pow}$ are finite and effectively computable depending only on r, s. As an application of Theorem 1, we prove the following result which gives finiteness of solutions of (3) and (4). In fact, our method gives a way for explicitly finding the solutions of (3) and (4) when $\mathcal{U}^{pow}$ and $\mathcal{V}^{pow}$ are given explicitly.

Theorem 2. *Suppose there is an integer i, $0 \leq i < k$ with a prime $Q \geq 5$, $Q \mid n_i$, $Q \nmid n_j$ for $1 \leq j \leq k$, $j \neq i$. Assume that*

$$2Q - 1 > \mathcal{P}(k) \quad \text{and} \quad p \not\equiv \pm 1 (\text{mod } 2Q) \text{ if } p|n_i.$$

(i) *Suppose $U_Q(r, s) \notin \mathcal{N}_Q^1$ if $Q \in \{5, 7, 13\}$. Then equation (3) implies $(Q^i, \ell) \in \mathcal{U}^{pow}$ for each $1 \leq i \leq \text{ord}_Q(n_i)$. In particular, if $(Q, \ell) \notin \mathcal{U}^{pow}$, then equation (3) has no solution.*

(ii) *Suppose $V_Q(r, s) \notin \mathcal{N}_Q^2$ if $Q = 5$. Further let n_i be an odd integer. Then equation (4) implies $(Q^i, \ell) \in \mathcal{V}^{pow}$ for each $1 \leq i \leq \text{ord}_Q(n_i)$. In particular, if $(Q, \ell) \notin \mathcal{V}^{pow}$, then equation (4) has no solution.*

We prove Theorem 2 in Section 4.

We now take $n_i = n + (i - 1)d$ for $1 \leq i \leq k$ for $(n, d) = 1$. In [22], Equation (3) with $b = 1$ was explicitly solved when $U_n = F_n$ and $U_n = \frac{x^n - 1}{x - 1}$. In [6] and [13, Theorem 6.1], equations (3) and (4) were explicitly solved for Pell and Pell–Lucas sequences with $\mathcal{P}(k) = f(k, d)$ given by

$$f(k, d) = \begin{cases} 2k, & \text{if } d > 1 \text{ or } d = 1,\ n > k, \\ k, & \text{if } d = 1 \text{ and } n \leq k. \end{cases} \tag{5}$$

Also equations of the form (3) and (4) for Balancing and Lucas Balancing numbers were considered in [13]. In this paper, we explicitly solve equations (3) and (4) with $\mathcal{P}(k) = f(k, d)$ given by (5) for the Fibonacci sequence F_n, Lucas sequence L_n, Jacobsthal sequence J_n, Jacobsthal–Lucas sequence $\mathfrak{J}_n$, Mersenne numbers M_n and Mersenne–Lucas numbers $\mathfrak{F}_n = 2^n + 1$. Let $U_n = F_n$ or J_n or M_n, and $V_n = L_n$ or $\mathfrak{J}_n$ or $\mathfrak{F}_n$. We prove

Theorem 3. *Let n, d, k, y, ℓ be positive integers with $\gcd(n, d) = 1$, $y > 1$, $\ell > 1$, b is ℓ^{th} power free and $P(b) \leq f(k, d)$ given by (5).*

(a) *The equation*

$$U_n U_{n+d} \cdots U_{n+(k-1)d} = by^\ell$$

(i) *has the only solution $F_6 = F_1F_6 = 2^3$, $F_{12} = F_1F_{12} = 12^2$ when $U_n = F_n$, the Fibonacci sequence;*

(ii) *has no solution when $U_n = J_n$, the Jacobsthal sequence;*

(iii) *has no solution when $U_n = M_n$, the Mersenne numbers.*

(b) *The equation*

$$V_n V_{n+d} \cdots V_{n+(k-1)d} = by^\ell$$

(i) *has the only solution $L_3 = L_1L_3 = 2^2$ when $V_n = L_n$, the Lucas sequence;*

(ii) *has no solution when $V_n = \mathfrak{J}_n$, the Jacobsthal–Lucas sequence;*

(iii) *has the only solution $\mathfrak{F}_3 = \mathfrak{F}_1\mathfrak{F}_3 = 3^2$ when $V_n = \mathfrak{F}_n$, the Mersenne–Lucas sequences.*

We prove Theorem 3 in Section 5. The preliminaries and lemmas for the proof of the above theorems are given in Section 2.

2. Notations and preliminaries

For any integer $n > 1$, we denote $\omega(n)$ the number of distinct prime divisors of n and we put $\omega(1) = 0$. For a non-zero integer n and a prime p, we write $\nu_p(n)$ for the highest power of p dividing n.

The binary recurrence sequence $U_n(r, s)$ and $V_n(r, s)$ are given by explicit Binet formulas $U_n(r, s) = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n(r, s) = \alpha^n + \beta^n$ $\forall n \geq 0$, where α and β are roots of the characteristic equation $x^2 - rx - s = 0$ which name such that $\alpha > \beta$ if $\alpha, \beta \in \mathbb{R}$, and $\text{Im}(\alpha) > 0$, $\text{Im}(\beta) < 0$ if α, β are complex. We list here α, β and the first few elements of the binary recurrence sequence we are considering.

Sequence	α	β	First few terms
Fibonacci sequence, F_n	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	0,1,1,2,3,5,8,13,...
Lucas sequence, L_n	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	2,1,3,4,7,11,18,29,...
Jacobsthal sequence, J_n	2	-1	0,1,1,3,5,11,21,43,...
Jacobsthal–Lucas sequence, $\mathfrak{J}_n$	2	-1	2,1,5,7,17,31,65,127,...
Mersenne sequence, M_n	2	1	0,1,3,7,15,31,63,127,...
Mersenne–Lucas sequence, $\mathfrak{F}_n$	2	1	2,3,5,9,17,33,65,129,...

Table 3

We now list some well-known properties for the binary recurrence sequences which will be used frequently.

Lemma 2.1. *For the sequences $(U_n)_{n=0}^{\infty}$ and $(V_n)_{n=0}^{\infty}$, we have*

(i) $U_{2n} = U_n V_n$;

(ii) $\gcd(U_m, U_n) = U_{\gcd(m,n)}$;

(iii) $\gcd(U_n, U_{mn}/U_n)$ *divides* m;

(iv) *for $n \geq 3$, a primitive prime divisor p of U_n is congruent to ± 1 modulo n;*

(v) *$n \geq 2$, a primitive prime divisor p of V_n is congruent to ± 1 modulo $2n$.*

As a consequence of Lemma 2.1, we have

Corollary 2.2. *Let q be an odd prime and $k > 0$ be any integer. Let p be an odd prime.*

(i) *Let $(r,s) \notin \mathcal{N}_q^1$. Then for $p \mid U_{q^k}$, we have $p \equiv \pm 1$ modulo $2q$. In particular, $p \geq 2q-1$.*

(ii) *Let $q \in \{5,7\}$ and $(r,s) \in \mathcal{N}_q^1$. Then if $p \mid U_{q^k}$ with $k > 1$, we have either $p = q$ or $p \equiv \pm 1$ modulo $2q^2$. In particular, $p = q$ or $p \geq 2q^2 - 1$.*

(iii) *Let $q = 13$ and $(r,s) = (1,-2)$. Then if $p \mid U_{13^k}$ with $k > 1$, we have either $p \in \{3,5\}$ or $p \equiv \pm 1$ modulo $2 \cdot 13^2$.*

(iv) *Let $(r,s) \notin \mathcal{N}_{q^i}^2$ for $1 \leq i \leq k$. Then for $p \mid V_{q^k}$ and $p \nmid r$, we have $p \equiv \pm 1$ modulo $2q$. In particular, $p \geq 2q-1$ if $p \nmid r$.*

(v) *Let $q = 5$ and $(r,s) \in \mathcal{N}_5^2$. Then if $p \mid V_{5^k}$ with $k > 1$, we have either $p \in \{2,5\}$ or $p \equiv \pm 1$ modulo $2 \cdot 5^2$. In particular, $p \in \{2,5\}$ or $p \geq 101$.*

(vi) *Let $q > 3$. For $p \mid V_{2q^k}$ and $p \nmid 2rs(r^2+2s)$, we have $p \equiv \pm 1$ modulo $4q$. In particular, $p \geq 4q-1$ if $p \nmid 2rs(r^2+2s)$.*

PROOF. By Lemma 2.1 (ii), $\gcd(U_{q^k}, U_n) = U_1 = 1$ if $q \nmid n$, and $\gcd(U_{q^k}, U_{q^b}) = U_{q^b}$ for every $1 \leq b < k$. Hence $p \mid U_{q^k}$ implies either p is a primitive prime divisor of U_{q^k} or a primitive prime divisor of U_{q^b} for some $1 \leq b < k$ if $(r,s) \notin \mathcal{N}_q^1$. Hence for $(r,s) \notin \mathcal{N}_q^1$, we have $p \equiv \pm 1$ modulo q^i for some $1 \leq i \leq k$ implying $p \equiv \pm 1$ modulo $2q^i$, since $p \pm 1$ is even. Thus for $(r,s) \notin \mathcal{N}_q^1$, we have $p \equiv \pm 1$ modulo $2q^i$ implying $p \equiv \pm 1$ modulo $2q$. This proves the assertion (i). The assertions (ii) and (iii) follow by using Table 1.

For positive integers m and n, let $g = \gcd(m,n)$. Then we observe from Lemma 2.1 (i) and (ii) that

$$\gcd(V_m, V_n) \mid \gcd\left(\frac{U_{2m}}{U_g}, \frac{U_{2n}}{U_g}\right) = \frac{U_{2\gcd(m,n)}}{U_g} = \frac{U_{2g}}{U_g} = V_g.$$

Thus if $p|V_{q^k}$ is not a primitive prime divisor of V_{q^k}, then $p|V_1$ or $p|V_{q^i}$ for some $1 \leq i < k$.

Let $(r,s) \notin \mathcal{N}_{q^i}^2$ for $1 \leq i \leq k$. Then each of V_{q^i} has a primitive prime divisor. Let $p \mid V_{q^k}$. Then either p is a primitive prime divisor of V_{q^k} or $p|V_1 = r$ or p is a primitive prime divisor of V_{q^i} for some $1 \leq i < k$. Hence the assertion (iv) follows from Lemma 2.1 (v). The assertion (v) follows by using Table 2.

For the assertion (vi), we observe that if $p|V_{2q^k}$ is not a primitive prime divisor of V_{2q^k}, then $p|V_1$ or $p|V_2$ or $p|V_{q^i}$ or $p|V_{2q^i}$ for some $1 \leq i < k$. From the equality $V_{2n} = V_n^2 + 2(-s)^n$ and $V_{q^i}|V_{q^k}$ if $i \leq k$, we obtain that if $p|V_{2q^k}$ and $p|V_{q^i}$ for some $1 \leq i \leq k$, then $p|2s$. Thus every prime divisor of V_{2q^k} is either a primitive prime divisor of V_{2q^i} for some $1 \leq i \leq k$ or $p|V_1 = r$ or $p|V_2 = r^2 + 2s$ or $p|2s$. Hence the assertion (vi) follows from Lemma 2.1 (v). □

In the next lemma, we derive some algebraic properties for the sequences.

Lemma 2.3. *Let $m \mid n$ and $\frac{n}{m}$ is odd. If a prime $p \mid \gcd\left(V_m, \frac{V_n}{V_m}\right)$, then $p \mid \frac{n}{m}$.*

PROOF. Recall that U_n and V_n are given by $U_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n = \alpha^n + \beta^n$, where α and β are the roots of the characteristic equation $x^2 - rx - s = 0$ of the sequence. Let $p|V_m$. Then $\alpha^m \equiv -\beta^m \pmod{p}$. If $n = mk$ with k odd, we have

$$\begin{aligned}\frac{V_n}{V_m} = \frac{(\alpha^m)^k + (\beta^m)^k}{\alpha^m + \beta^m} &= (\alpha^m)^{k-1} - (\alpha^m)^{k-2}(\beta^m) + \cdots (\beta^m)^{k-1} \\ &\equiv k\alpha^{m(k-1)} \equiv k\beta^{m(k-1)} (\mathrm{mod}\ p)\end{aligned}$$

Consequently, $p^2 \mid k^2(\alpha\beta)^{m(k-1)}$, which implies that $p \mid k$, as desired. □

The following result is an easy consequence of Lemma 2.3.

Corollary 2.4. *Suppose that $m \mid n$ and $\frac{n}{m}$ is odd. If $P(\frac{n}{m}) < p$ for any odd prime p dividing V_m, then* $\gcd\left(V_m, \frac{V_n}{V_m}\right) = 1$.

The following result is on the Nagell–Ljunggren equation, see [2].

Lemma 2.5. *The solutions of the Nagell–Ljunggren equation*

$$\frac{x^n - 1}{x - 1} = y^q \text{ in integers } |x| > 1,\ |y| > 1,\ n > 2,\ q \geq 2$$

other than

$$\frac{3^5 - 1}{3 - 1} = 11^2, \quad \frac{7^4 - 1}{7 - 1} = 20^2, \quad \frac{18^3 - 1}{18 - 1} = 7^3 \quad \text{and} \quad \frac{(-19)^3 - 1}{(-19) - 1} = 7^3$$

satisfy that

- *$q \geq 3$ is odd;*
- *the least prime divisor p of n satisfies $p \geq 5$;*
- *$|x| \geq 10^4$ and x has a prime divisor $p \equiv 1 (\mathrm{mod}\, q)$.*

Next we consider a Diophantine equation.

Lemma 2.6. *The equation $2^n \pm 1 = 3^\alpha y^\ell$ in positive integers $y > 1$, $\ell > 1$ and $\alpha \geq 0$ has the only solution given by $2^3 + 1 = 3^2$.*

Proof. It follows from [1, Corollary 1.4] that there is no solution for the given equation when n is even. Thus n is odd, and we have $2^n + 1 = 3^\alpha y^\ell$ by taking modulo 3. Suppose $\alpha = 1$. Then

$$y^\ell = \frac{2^n + 1}{2 + 1} = \frac{(-2)^n - 1}{(-2) - 1},$$

which has no solution by Lemma 2.5. Thus $\alpha \geq 2$. Then $2^n + 1 \equiv 0$ modulo 9, implying $3|n$. Write $n = 3^e n_1$ with $3 \nmid n_1$. Then $3||(2^{n_1} + 1)$ and

$$\gcd\left(2^{n_1} + 1, \frac{2^n + 1}{2^{n_1} + 1}\right)|3^e = 1, \quad \text{implying } \frac{2^{n_1} + 1}{2 + 1} = \frac{(-2)^{n_1} - 1}{(-2) - 1} = y_1^\ell$$

for some y_1. By Lemma 2.5, this gives $n_1 = y_1 = 1$ or $n = 3^e$. Then $2^{3^e} + 1 = 3^\alpha y^\ell$. There is no solution with $e = 1$ as $y > 1$. Thus $e \geq 2$, implying $19|(2^{3^2} + 1)|(2^n + 1)$. However, $19^2 \nmid (2^{3^e} + 1)$, and hence there are no other solution of the equation. □

We now state some results on the almost perfect powers in recurrence sequences.

Lemma 2.7. *Let n, y, ℓ, u, v be positive integers with $\ell > 1$. Then*

(i) $F_n = y^\ell \Rightarrow n = 1, 2, 6, 12$.
(ii) $F_n = 2^u y^\ell \Rightarrow F_3 = 2 \cdot 1^\ell$, $F_6 = 2^3$, $F_{12} = 2^4 \cdot 3^2$.
(iii) $F_n = 3^u y^\ell \Rightarrow F_4 = 3 \cdot 1^\ell$, $F_{12} = 3^2 \times 2^4$.
(iv) $F_n = 2^u 3^v y^\ell \Rightarrow F_{12} = 3^2 \times 2^4$.
(v) $L_n = y^\ell \Rightarrow L_1 = 1^\ell$, $L_3 = 2^2$.
(vi) $L_n = 2^u y^\ell \Rightarrow L_3 = 2^2$, $L_6 = 2 \times 3^2$.
(vii) $L_n = 3^u y^\ell \Rightarrow L_2 = 3 \cdot 1^\ell$.
(viii) $L_n = 2^u 3^v y^\ell \Rightarrow L_6 = 2 \times 3^2$.
(ix) $J_n = y^\ell \Rightarrow J_1 = J_2 = 1^\ell$.
(x) $J_n = 3^u y^\ell \Rightarrow J_3 = 3 \cdot 1^\ell$.
(xi) $\mathfrak{J}_n = y^\ell \Rightarrow \mathfrak{J}_1 = 1^\ell$.
(xii) $\mathfrak{J}_n = 3^u y^\ell \Rightarrow$ *no solution.*
(xiii) $M_n = y^\ell \Rightarrow M_1 = 1^\ell$.
(xiv) $M_n = 3^u y^\ell \Rightarrow M_2 = 3 \cdot 1^\ell$.
(xv) $\mathfrak{F}_n = 2^n + 1 = y^\ell \Rightarrow \mathfrak{F}_3 = 3^2$.
(xvi) $\mathfrak{F}_n = 2^n + 1 = 3^u y^\ell \Rightarrow \mathfrak{F}_1 = 3 \cdot 1^\ell$, $\mathfrak{F}_3 = 3^2 \cdot 1^\ell$.

Proof. The results (i) and (v) are due to [8]. The results (ii), (iii), (vi) are in [7], (vii) is contained in [10], and (iv) follows from [9, Theorem 4].

For the assertion (viii), we first observe that $2||n$, since $3|L_n$, and also $3|n$, since $2|L_n$. We have a solution at $n = 6$. Suppose $n = 2 \cdot 3^z$ for some $z > 1$. Then $L_{18}|L_n$. Since $107||L_{18}$ and using Lemma 2.3, we obtain that $\nu_{107}(L_n) = \nu_{107}(L_{18}) = 1$, implying the equation has no solution. Thus $P(n) = Q > 3$. Write $n = 2Q^e n_2$, where $Q \nmid n_2$ and $P(n_2) < Q$. By Corollary 2.2 (vi) and $r = s = 1$, we have that $p|L_{2Q^e}$ implies $p \in \{2, 3\}$ or $p \geq 4Q - 1 > P(n_2)$. This, together with $2^u 3^v y^\ell = L_n = L_{2Q^e} \frac{L_n}{L_{2Q^e}}$ and Lemma 2.3, implies $L_{2Q^e} = 2^{u_1} 3^{v_1} y_1^\ell$ for some $u_1, v_1, y_1 > 1$. Since $3 \nmid 2Q^e$, we have $2 \nmid L_{2Q^e}$, and hence $L_{2Q^e} = 3^{v_1} y_1^\ell$. By (vii), this is not possible, and hence the assertion (viii) is valid.

For the remaining assertions, we need to consider the equation $2^n \pm 1 = 3^\alpha y^\ell$ for some integers n, α, y, ℓ. By Lemma 2.6, we note that the only solution of this equation is $2^3 + 1 = 3^2$. Hence the assertions follow. □

Now we state a result due to Laishram and Shorey [20, Lemma 4].

Lemma 2.8. *Let $\delta \in \{1, -1\}$. The solutions of*

(i) $2^x - 3^y 5^z = \delta$,

(ii) $3^x - 2^y 5^z = \delta$,

(iii) $5^x - 2^y 3^z = \delta$

in integers $x > 0$, $y > 0$, $z > 0$ are given by

$$(x, y, z, \delta) = \begin{cases} (4, 1, 1, 1), & \text{for (i)}, \\ (4, 4, 1, 1), (2, 1, 1, -1), & \text{for (ii)}, \\ (2, 3, 1, 1), (1, 1, 1, -1), & \text{for (iii)}, \end{cases}$$

respectively.

The following result is contained in [21, Theorem 3].

Lemma 2.9. *Let $k \geq 2$ and n odd with $n > 2k$. Then*

$$P\left(\prod_{i=0}^{k-1}(n + 2i)\right) > 3.5k$$

unless $(n, k) \in \{(5, 2), (7, 2), (25, 2), (243, 2), (9, 4), (13, 5), (17, 6), (15, 7), (21, 8), (19, 9)\}$.

We also need the following result on intervals containing primes, see [6, Lemma 2.8].

Lemma 2.10. *Let $x > 10$ be an integer. Then the interval $(2x/3, x]$ contains a prime.*

3. Proof of Theorem 1

For positive integers n, d, k, recall that $\Delta = \Delta(n, d, k) = n(n + d) \cdots (n + (k - 1)d)$.

Let $k = 2$. Here $\Delta = n(n+d)$. If n is even, then $n+d$ is odd and $P(n+d) > 2$. If n is odd and $n > 1$, then $P(n) > 2$. If $n = 1$, then $P(1+d) > 2$ unless $d = 2^r - 1$ for some integer $r \geq 1$.

For $k = 3$, $\Delta = n(n + d)(n + 2d)$. If $n = 1$ and d is even, then $1 + d$, $1 + 2d$ are odd integers. So at least one of them will have a prime factor greater than 3. If d is odd, then $1 + 2d$ is odd and $P(1 + 2d) > 3$ for $d \neq \frac{1}{2}(3^r - 1)$ for some integer $r \geq 1$. If $n > 1$ is odd, then $n + 2d$ is also odd. So either $P(n) > 3$ or $P(n + 2d) > 3$. If n is even, then $n + d$ is odd. Hence $P(n + d) > k$ unless $d = 3^r - n$ for some $r \geq 1$.

For $k = 4$, $\Delta = n(n+d)(n+2d)(n+3d)$. If n is even, then $n+d$ and $n+3d$ are distinct odd numbers > 1, and so at least one of them will have a prime factor greater than 4. If $n > 1$ is odd, then n and $n+2d$ are distinct odd integers, and hence one of them will have a prime factor greater than 4. If $n = 1$ and d is an odd integer, then $P(1+2d) > 4$ except when $1+2d$ is a power of 3, i.e., when $d = \frac{1}{2}(3^r - 1)$ for some $r \geq 1$.

For $k = 5$, $\Delta = n(n+d)(n+2d)(n+3d)(n+4d)$. Let d be even. Then $n+d$, $n+2d$, $n+3d$, $n+4d$ all are distinct odd integers and since 3 can divide at most two terms and 5 can divide at most 1 term, there will be at least term which have a prime divisor > 5. Thus we take d odd. Let n be odd. Then $n, n+2d, n+4d$ are distinct odd integers and 3 and 5 can divide at most one term each among them. Hence one of the term will have a prime divisor > 5 except when $n = 1$ and

$$1 + 2d = 3^a \quad \text{and} \quad 1 + 4d = 5^b \Rightarrow 5^b - 2 \cdot 3^a = -1$$
$$\text{or } 1 + 2d = 5^b \quad \text{and} \quad 1 + 4d = 3^a \Rightarrow 3^a - 2 \cdot 5^b = -1.$$

By Lemma 2.8, we get the solution $5^1 - 2 \cdot 3 = 1$ in the first case, which gives $d = 1$, and the solution $3^2 - 2 \cdot 5 = -1$ in the latter case, which gives $d = 2$, which is not possible since d is odd. Thus $(n, d) = (1, 1)$ is one of the exceptional case.

Let n be even. Then $n+d$ and $n+3d$ are odd terms and both have prime divisors ≤ 5 only when

$$n + d = 3^a \quad \text{and} \quad n + 3d = 5^b \qquad \text{or} \qquad n + d = 5^b \quad \text{and} \quad n + 3d = 3^a.$$

The first case gives $n = \frac{1}{2}(3^{a+1} - 5^b)$, $d = \frac{1}{2}(5^b - 3^a)$ with a odd as n is even, and the latter case gives $n = \frac{3}{2}(5^b - 3^{a-1})$, $d = \frac{1}{2}(3^a - 5^b)$ with a odd as n is even.

Let $k \geq 6$. For $d > 1$, we have the assertion from Theorem A. Hence we now consider $d = 1$. For $n + k \leq 11$, we check that the assertion of Theorem 1 is true. Hence we suppose that $n + k \geq 12$. Let $n \leq 2k$. Then $\frac{2}{3}(n+k-1) \geq n$ if $n < 2k - 1$, and $\left\lceil \frac{2}{3}(n+k-1) \right\rceil = n$ if $n \in \{2k-1, 2k\}$. By Lemma 2.10 and $n + k \geq 12$, the interval $(\frac{2}{3}(n+k-1), n+k-1]$ contains a prime which is of the form $n+i$ for some some i, $0 \leq i < k$. Then $n+i$ is odd and further $n+i > k$, since $n > k$, implying the assertion of the Theorem.

Thus $n > 2k$. Then the odd terms of among $\{n, n+d, \ldots, n+(k-1)d\}$ are given by

$$n, n+2, \ldots, n+2\left[\frac{k-1}{2}\right] \text{ if } n \text{ is odd},$$
$$n+1, n+3, \ldots, n+1+2\left[\frac{k-2}{2}\right] \text{ if } n \text{ is even},$$

and hence there are at least $1+[\frac{k-2}{2}]=[\frac{k}{2}]$ consecutive odd terms. Since $k\geq 6$, $[\frac{k}{2}]\geq 3$. Let $S=\{(5,2),(7,2),(25,2),(243,2),(9,4),(13,5),(17,6),(15,7),(21,8),(19,9)\}$. For $(n,[\frac{k-1}{2}]+1)\in S$ if n is odd, and $(n+1,[\frac{k}{2}])\in S$ if n is even, we check that here is an odd term $n+i$ for which $P(n+i)>k$. Thus we now suppose that $(n,[\frac{k-1}{2}]+1)\notin S$ if n is odd and $(n+1,[\frac{k}{2}])\notin S$ if n is even. Then by Lemma 2.9, the greatest prime factor of these consecutive odd terms is at least $3.5[\frac{k}{2}]\geq 3.5(\frac{k-1}{2})>k$, and hence there is an odd term $n+i$ for which $P(n+i)>k$. □

4. Proof of Theorem 2

(i) Write $n_i=Q^et$ with $Q\nmid t$. For any b with $1\leq b\leq e$, put $Q_b=Q^b$, and rewrite equation (3) as

$$U_{Q_b}\frac{U_{n_i}}{U_{Q_b}}\prod_{j\neq i}U_{n_j}=by^\ell.$$

By Corollary 2.2 (i), we see that $p|U_{Q_b}\Rightarrow p\equiv\pm 1$ modulo $2Q$, and hence $p\nmid Qt$ by our assumption. Further, $p\geq 2Q-1>\mathcal{P}(k)\geq P(b)$. We now show that U_{Q_b} is coprime to the other factors on the left-hand side of the above equation.

If $p\mid\left(U_{Q_b},\frac{U_{n_i}}{U_{Q_b}}\right)$, then $p|\frac{n_i}{Q_b}=Q^{e-b}t$. This with $p\nmid Qt$ for $p|U_{Q_b}$ implies $\gcd\left(U_{Q_b},\frac{U_{n_i}}{U_{Q_b}}\right)=1$.

Also $\gcd(U_{Q_b},U_{n_j})=U_{\gcd(Q_b,n_j)}=U_1=1$ for $j\neq i$ by our assumption. Hence $U_{Q_b}=y_b^\ell$ for some $y_b|y$. Thus $Q_b\in\mathcal{U}^\ell$ for each $1\leq b\leq e$. In particular, if $Q^e\notin\mathcal{U}^\ell$, then equation (3) does not have solution.

(ii) Let n_i be an odd integer. We write $n_i=Q^et$ with $Q\nmid t$. For any b with $1\leq b\leq e$, put $Q_b=Q^b$, and we rewrite equation (4) as

$$V_{Q_b}\frac{V_{n_i}}{V_{Q_b}}\prod_{j\neq i}V_{n_j}=by^\ell.$$

Let $p\mid V_{Q_b}$, $p\nmid r$. From Corollary 2.2 (ii), we have $p\equiv\pm 1$ modulo $2Q$. In particular, $p\nmid Qt$ and $p\geq 2Q-1>\mathcal{P}(k)\geq P(b)$. We show that the common prime divisors of V_{Q_b} and the other factors on the left-hand side of the above equation are prime divisors of r.

We have from $\gcd(V_{Q_b},V_{n_j})\mid V_1=r$. Also from $\gcd\left(V_{Q_b},\frac{V_{n_i}}{V_{Q_b}}\right)|\frac{n_i}{Q_b}$ and $p\nmid Qt$ for $p\nmid r$ that $\gcd\left(V_{Q_b},\frac{V_{n_i}}{V_{Q_b}}\right)=r'$, where $p|r'$ implies $p|r$. Hence we get

$V_{Q_b} = r_b y_b^\ell$ for some integer $y_b | y$ and $\prod_{p|r_b} p \mid \prod_{p|r} p$. Thus $Q^b = Q_b \in \mathcal{V}^\ell$ for each $1 \le b \le e$. In particular, if $Q^e \notin \mathcal{V}^\ell$, then equation (4) does not have solution. □

5. Proof of Theorem 3

For the sequences we consider in Theorem 3, observe that $\mathcal{N}_q^1 = \mathcal{N}_q^2 = \emptyset$ for primes and powers of primes ≥ 5 except for the Fibonacci sequence (F_n) where F_5 has no primitive prime divisor. By Corollary 2.2 (ii), we have that $p|F_{5^e}$ with $e \ge 2$ implies either $p = 5$ or $p \equiv \pm 1 (\mathrm{mod}\ 50)$, which gives $p \ge 101$.

(a) From Lemma 2.7, we can assume that $k \ge 2$. Let $k = 2$. Then equation (3) becomes $U_n U_{n+d} = by^\ell$ with $P(b) \le 3$. Since $(U_n, U_{n+d}) = 1$, we have $U_n = b_1 y_1^\ell$ and $U_{n+d} = b_2 y_2^\ell$ for some b_1, b_2 with $P(b_1 b_2) \le 3$. By Lemma 2.7 (i)–(iv) and using $(n, n+d) = 1$, we get $n = 1$, and further $n + d \in \{2, 3, 4, 6, 12\}, \{3\}, \{2\}$ according as $U_n = F_n$, J_n or M_n, respectively. We check for solutions given by these values.

We now take $k \ge 3$. Let $d = 1$ and $n \le k$. First we take all pairs (n, k) with $n + k \le 11$, and check for the solutions of (3), and we find that there are no solutions. Thus we take $n + k > 11$. Then $n + k - 1 > 10$. Let $Q = P(n(n+1)\cdots(n+k-1))$. Since $n + k - 1 > \frac{2(n+k-1)}{3} \ge n$, we obtain from Lemma 2.10 that $Q = n + i_0 > \frac{2(n+k-1)}{3} \ge \frac{20}{3}$ or $Q \ge 7$. Further, $2Q - 1 > \frac{4(n+k-1)}{3} - 1 \ge k$, since $n \ge 1$ and $k \ge 3$. All the assumptions of Theorem 2 are satisfied, since $Q \nmid n + i$ for $i \ne i_0$, and hence by Theorem 2 and Lemma 2.7, we find that there are no solutions.

Therefore, we take either $d = 1$, $n > k$ or $d > 1$. We check that there are no solutions when $(n, d, k) = (2, 7, 3)$, and hence assume that $(n, d, k) \ne (2, 7, 3)$. Let $Q = P(n(n+d)\cdots(n+(k-1)d))$. By (1) and (2), we have $Q > k$, and hence $Q \ge 5$. Since a prime $> k$ divide at most one term of $n(n+d)\cdots(n+(k-1)d)$, the assumptions of Theorem 2 are satisfied, and hence there are no solutions for (3), except possibly when $U_n = F_n$ and $Q = 5$. So we consider $U_n = F_n$ and assume that $Q = 5$. Then $k \le 4$. Observe from $(n + id, n + jd) | ((i - j)$ that $\gcd(F_{n+id}, F_{n+jd}) | F_{(i-j)} \le F_3 = 2$. Since $Q \le 5$, at least one of the terms is divisible by m with $m \in \{9, 10, 15, 16, 24\}$. Choose $m \in \{9, 10, 15, 16, 24\}$ smallest such that $m | (n + id)$ for some $0 \le i < k$. Let p be a primitive prime divisor of F_m. Observe that all primitive divisors > 7 and divide to first power in F_m for $m \in$

$\{9, 10, 15, 16, 24\}$. Then from $\gcd\left(F_m, \frac{F_{mt}}{F_m}\right) | t$ and $\gcd(F_{n+id}, F_{n+jd}) | F_{(i-j)} \leq 2$, we find that

$$\nu_p\left(\prod_{i=0}^{k-1} F_{n+id}\right) = \nu_p(F_m) = 1,$$

and hence there is no solution for (3). □

(b) By Lemma 2.7, we may assume that $k \geq 2$. Let $k = 2$. Then equation (4) becomes

$$V_n V_{n+d} = by^\ell \text{ with } P(b) \leq 3.$$

Since $(V_n, V_{n+d}) | V_1 = r$, we have $V_n = b_1 y_1^\ell$ and $V_{n+d} = b_2 y_2^\ell$ for some b_1, b_2 with $P(b_1 b_2) \leq 3$. By Lemma 2.7 (v)–(viii), (xi), (xii), (xv), (xvi) and using $(n, n+d) = 1$, we get $n = 1$, and further $n + d \in \{3, 6\}$ if $V_n = L_n$, and $n + d = 3$ if $V_n = \mathfrak{F}_n$, giving the solutions $L_1 L_3 = 2^2$, $L_1 L_6 = 2 \cdot 3^2$ and $\mathfrak{F}_1 \mathfrak{F}_3 = 3^3$.

We now take $k \geq 3$. Let $d = 1$ and $n \leq k$. First we take all pairs (n, k) with $n + k \leq 11$ and check for the solutions of (4), and we find that there are no solutions. Thus we take $n + k > 11$. Then $n + k - 1 > 10$. Let $Q = P(n(n + 1) \cdots (n + k - 1))$. Since $n + k - 1 > \frac{2(n+k-1)}{3} \geq n$, we obtain from Lemma 2.10 that $Q = n + i_0 > \frac{2(n+k-1)}{3} \geq \frac{20}{3}$ or $Q \geq 7$. Further, $2Q - 1 > \frac{4(n+k-1)}{3} - 1 \geq k$, since $n \geq 1$ and $k \geq 3$. All the assumptions of Theorem 2 are satisfied, since $Q \nmid n + i$ for $i \neq i_0$ and also $n + i_0 = Q$ is odd. Hence by Theorem 2 (ii) and Lemma 2.7, we find that there are no solutions.

Therefore, we take either $d = 1$, $n > k$ or $d > 1$. We check that there are no solutions when $(n, d, k) = (2, 7, 3)$, and hence assume that $(n, d, k) \neq (2, 7, 3)$. The assertion for $k \geq 6$ follows from Theorem 2 (ii) and Theorem 1. Thus we now take $k \in \{3, 4, 5\}$. Further by Theorem 2 (ii) and Theorem 1, we may restrict those pairs (n, d, k) listed as exceptions in Theorem 1. Let $k \in \{3, 4\}$. There is a term $n + i_0 d = 3^a$ for some $i_0 \in \{1, 2\}$. We may assume that $a \geq 2$ as otherwise $n + i_0 d = 3$, and we check that for such n and d, there are no solutions. From $\gcd(V_{n+id}, V_{3^a}) | V_1 = 1$, we have $V_{3^a} = r_1 b_1 y_1^\ell$ for some r_1, b_1, y_1 with $P(b_1) \leq f(k, d), y_1 | y$ and $p | r_1$ implies $p | r$. Let p be a primitive root of V_9. We find that $p > 7 \geq P(r_1 b_1)$ and $\nu_p(V_9) = 1$. On the other hand, we observe from Lemma 2.3 that $p \nmid \frac{V_{3^a}}{V_9}$, since $p > 7$. Hence $\nu_p(V_{3^a}) = \nu_p(V_9) = 1$, implying $V_{3^a} = r_1 b_1 y_1^\ell$ has solution. Thus the original equation has no solution.

Let $k = 5$. There is a term $n + i_0 d = 5^b$ for some $i_0 \in \{1, 3\}$. From $\gcd(V_{n+id}, V_{5^b}) | V_1 = 1$, we have $V_{5^b} = r_1 b_1 y_1^\ell$ for some r_1, b_1, y_1 with $P(b_1) \leq 10, y_1 | y$ and $p | r_1$ implies $p | r$. Let p be a primitive root of V_5. We find that $p \geq 11 > P(r_1 b_1)$ and $\nu_p(V_5) = 1$. On the other hand, we observe from Lemma 2.3

that $p \nmid \frac{V_{5^b}}{V_5}$, since $p > 5$. Hence $\nu_p(V_{5^b}) = \nu_p(V_5) = 1$, implying $V_{5^b} = r_1 b_1 y_1^\ell$ has solution. Thus the assertion of Theorem 3(b) follows. □

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SHANTA LAISHRAM
STAT-MATH UNIT
INDIA STATISTICAL INSTITUTE
7, S. J. S. SANSANWAL MARG
NEW DELHI, 110016
INDIA

E-mail: shanta@isid.ac.in
URL: http://www.isid.ac.in/~shanta

RANJIT SINGH MAIBAM
DEPARTMENT OF MATHEMATICS
MANIPUR UNIVERSITY
CANCHIPUR, MANIPUR, 795003
INDIA

E-mail: mranjit@manipuruniv.ac.in

SUDHIR SINGH NGAIRANGBAM
DEPARTMENT OF MATHEMATICS
MANIPUR UNIVERSITY
CANCHIPUR, MANIPUR, 795003
INDIA

E-mail: sudhirng@manipuruniv.ac.in

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