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Separation with symmetric bilinear forms and symmetric selections of set-valued functions

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Abstract. The main result of this paper offers necessary and sufficient conditions for the separability of two given functions by symmetric bilinear forms. As a consequence of this result, we are able to characterize those functions that are pointwise suprema of symmetric bilinear functions and also those set-valued functions that are generated by symmetric linear operators.

The results obtained are in a strong relationship with those obtained for nonsymmetric bilinear forms in a previous paper of the author.

1. Introduction

The subject of this paper is to study the separability of two given functions by symmetric bilinear forms. The methods used and results obtained are parallel to those found in [3] concerning the separability with bilinear functions.

Let X be a locally convex Hausdorff topological vector space over the field of reals throughout this paper and denote by X^* its dual space. Let $R, P : X \times X \to \mathbb{R}$ be given functions with $R \le P$ on $X \times X$. The separability of P and R by a symmetric bilinear form means that there exists a symmetric bilinear function Q satisfying $R \le Q \le P$ on $X \times X$. A similar problem with nonsymmetric bilinear forms has been considered in [3]. The answer to the symmetric problem is analogous and contained in Corollary 1 below.

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Another problem, closely related, comes from the theory of second-order differentiation: Given a function $P : X \times X \to \mathbb{R}$ we look for the existence of a family $\mathcal{Q}$ of symmetric bilinear functions such that

$$P(x,y) = \sup\{Q(x,y) : Q \in \mathcal{Q}\} \qquad \text{for all} \quad x, y \in X.$$

Replacing the word "bilinear" by "linear", we can see that P can be represented in this form with the help of a family $\mathcal{Q}$ of additive functions if and only if P is subadditive and positively homogeneous. Analogously, we could think that exactly the symmetric bisublinear functions admit the above representation in terms of symmetric bilinear functions. In Corollary 2 below we give necessary and sufficient conditions for P to admit such a representation. This condition shows that the function P must satisfy a number of conditional inequalities; the bisublinearity of P turns out to be necessary but it is not sufficient. This result is also the key to prove that the second-order directional derivative of a differentiable function with Lipschitz derivative is the maximum of symmetric bilinear forms. In the last section we state the result that has been obtained recently in a joint work with V. Zeidan.

The third section deals with set-valued functions $F : X \to 2^{X^*}$ whose values are nonempty weak-$*$ compact sets in X^*. We investigate the existence of symmetric linear selections $A : X \to X^*$ of F. The *symmetry* of A means that $\langle A(x), y\rangle = \langle A(y), x\rangle$ for all $x, y \in X$. Similar questions were considered and sufficient conditions were obtained for the existence of nonsymmetric linear selections in [6], [7]. The necessary and sufficient condition for the nonsymmetric case has been obtained in [3] by the author. The main result of Section 3 offers a complete answer for this question. This section also deals with the representability of F in the form

$$F(x) = \{A(x) \,:\, A \in \mathcal{A}\},$$

where $\mathcal{A}$ is a family of continuous symmetric linear operators.

2. Separation with symmetric bilinear forms

The statement of the following Lemma is similar to that of [3].

Lemma 1. *Let X be a locally convex linear space and $Q : X \times X \to \mathbb{R}$ be a symmetric bilinear form. Then*

$$Q(x_1, y_1) + \cdots + Q(x_k, y_k) = 0 \tag{1}$$

holds true whenever $x_1, \dots, x_k, y_1, \dots, y_k \in X$ and

$$\langle x^*, x_1\rangle \langle x^*, y_1\rangle + \cdots + \langle x^*, x_k\rangle \langle x^*, y_k\rangle = 0 \qquad \text{for all} \quad x^* \in X^*. \tag{2}$$

Proof. Let $x_1, \dots, x_k, y_1, \dots, y_k \in X$ be fixed such that (2) is satisfied. These vectors span a finite dimensional subspace $\overline{X} \subset X$. Let $e_1, \dots, e_n$ be basis for $\overline{X}$. Then, as a consequence of the Hahn-Banach theorem, we can find a dual system $e_1^*, \dots, e_n^* \in X^*$ such that

$$\langle e_i^*, e_j \rangle = \delta_{i,j},$$

where δ is the Kronecker function. Now we obtain that

$$Q(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} Q(e_i, e_j) \langle e_i^*, x \rangle \langle e_j^*, y \rangle \qquad \text{for} \quad x, y \in \overline{X},$$

since this equation is trivially satisfied for $x, y \in \{e_1, \dots, e_n\}$. Interchanging x and y, and using the symmetry of Q, we get

$$\begin{aligned} Q(x,y) =& \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} Q(e_i, e_j) [\langle e_i^*, x \rangle \langle e_j^*, y \rangle + \langle e_i^*, y \rangle \langle e_j^*, x \rangle] \\ =& \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} Q(e_i, e_j) [\langle e_i^* + e_j^*, x \rangle \langle e_i^* + e_j^*, y \rangle \\ & \qquad\qquad - \langle e_i^* - e_j^*, x \rangle \langle e_i^* - e_j^*, y \rangle] \end{aligned}$$

for $x, y \in \overline{X}$. Applying this formula for $x = x_\ell$ and $y = y_\ell$, adding up the equations obtained for $\ell = 1, \dots, k$ and using (2) for $x^* = e_i^* + e_j^*$, $x^* = e_i^* - e_j^*$ $(i, j = 1, \dots, n)$, we get (1). $\square$

Having proved the above Lemma, we can formulate the main result of the paper.

Theorem 1. *Let $P : X \times Y \to \mathbb{R}$ be a positively bihomogeneous function. Then there exists a continuous symmetric bilinear function $Q : X \times X \to \mathbb{R}$ such that $Q \le P$ if and only if*

$$0 \le P(x_1, y_1) + \dots + P(x_k, y_k) \tag{3}$$

holds whenever $x_1, \dots, x_k, y_1, \dots, y_k \in X$ and (2) holds true. If, in addition, P is upper semicontinuous at the origin, and Q exists, then the continuity of Q can also be stated.

Proof. If Q exists, then applying Lemma 1, equation (3) follows directly. To prove the sufficiency of the condition, assume that (3) holds on the domain indicated. Let Z be the set of functions $z : X^* \to \mathbb{R}$ that have the representation

$$z(x^*) = \sum_{i=1}^{k} \langle x^*, x_i \rangle \langle x^*, y_i \rangle \qquad \text{for} \quad x^* \in X^* \tag{4}$$

with some elements $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$. Clearly, Z is a linear space. Define $\Phi : Z \to [-\infty, \infty[$ by

$$\Phi(z) := \inf\{P(x_1, y_1) + \cdots + P(x_k, y_k) : \\ \text{for } x_1, \ldots, x_k, y_1, \ldots, y_k \text{ satisfying (4)}\}.$$

A direct computation shows that Φ is a sublinear function. By (3), we have $\Phi(0) = 0$, therefore Φ must be real valued. Applying the Hahn-Banach theorem, we can find a linear function $\varphi : Z \to \mathbb{R}$ such that $\varphi \le \Phi$ holds on Z. Define the required function $Q : X \times X \to \mathbb{R}$ by

$$Q(x, y) = \varphi(z) \qquad \text{if} \quad z(x^*) = \langle x^*, x\rangle \langle x^*, y\rangle \qquad \text{for} \quad x^* \in X^*.$$

Then the symmetry of Q is obvious. Since φ is linear, hence the bilinearity of Q also follows easily.

If a bilinear Q exists, then the inequality $Q \le P$ yields

$$-P(-x, y) \le Q(x, y) \le P(x, y) \qquad \text{for} \quad x, y \in X.$$

Thus the usc property of P implies that Q is bounded in a neighborhood of the origin. Therefore it must be continuous as well. $\square$

Remark. The statement of Theorem 1 is analogous to that of Corollary 1 in [3], which states that a bilinear function $Q : X \times X \to \mathbb{R}$ with $Q \le P$ exists if and only if the inequality (3) holds whenever

$$\text{(5)} \quad \langle x^*, x_1\rangle \langle y^*, y_1\rangle + \cdots + \langle x^*, x_k\rangle \langle y^*, y_k\rangle = 0 \qquad \text{for all} \quad x^*, y^* \in X^*.$$

Clearly, the domain of (3) in Theorem 1 is larger set than the set of x_i's and y_i's described by (5). However, if the function P is symmetric, then the inequality (3) is equivalent on both domains. To prove this, one has to show that if (3) holds on the domain indicated in (5), then (3) also holds on the set described by (2). Let $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ satisfy (2). Putting $x^* := x^* + y^*$ and $x^* := x^* - y^*$ into (2) and subtracting the equations so obtained, we get

$$\langle x^*, x_1\rangle \langle y^*, y_1\rangle + \langle x^*, y_1\rangle \langle y^*, x_1\rangle + \cdots \\ + \langle x^*, x_k\rangle \langle y^*, y_k\rangle + \langle x^*, y_k\rangle \langle y^*, x_k\rangle = 0$$

for all $x^*, y^* \in X^*$. Hence, this relation implies

$$0 \le P(x_1, y_1) + P(y_1, x_1) + \cdots + P(x_k, y_k) + P(y_k, x_k).$$

By the symmetry of P, this equation is equivalent to (3), what was to be proved.

Now we list the most important corollaries of Theorem 1. The first result is about the symmetric bilinear separability of two given functions.

Corollary 1. *Let $P : X \times X \to \mathbb{R}$ and $R : X \times X \to [-\infty, \infty[$ be two positively bihomogeneous functions with $R \le P$ on $X \times X$. Then in order that there exist a symmetric bilinear function $Q : X \times X \to \mathbb{R}$ with $R \le Q \le P$, it is necessary and sufficient that*

$$R(u_1, v_1) + \cdots + R(u_n, v_n) \le P(x_1, y_1) + \cdots + P(x_k, y_k)$$

be valid whenever $u_1, \ldots, u_n, v_1, \ldots, v_n, x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ and

$$\langle x^*, u_1\rangle \langle x^*, v_1\rangle + \cdots + \langle x^*, u_n\rangle \langle x^*, v_n\rangle \\ = \langle x^*, x_1\rangle \langle x^*, y_1\rangle + \cdots + \langle x^*, x_k\rangle \langle x^*, y_k\rangle$$

holds for all $x^* \in X^*$.

Proof. A bilinear functional Q separates R and P if and only if

$$Q(x, y) \le \min\{P(x, y), -R(-x, y)\} =: \overline{P}(x, y) \qquad \text{for} \quad x, y \in X.$$

Thus a separating Q exists if and only if there is $Q \le \overline{P}$. Now the statement of the Corollary follows directly from Theorem 1. $\square$

Remark. This result is analogous to Corollary 3 in [3], where the separability with (nonsymmetric) bilinear functions is considered. One can also see, that taking $R \equiv -\infty$, the statement of this Corollary reduces to that of Theorem 1. Using this separation theorem, the so called extension version of Theorem 1 or Corollary 1 could also be formulated. An analogous result with (nonsymmetric) bilinear forms is stated in Theorem 1 of [3].

In the following Corollary we characterize those functions that can be obtained as pointwise suprema of symmetric bilinear functions.

Corollary 2. *Let $P : X \times X \to \mathbb{R}$ be an arbitrary function. Then the following three assertions are equivalent:*

(i) There exists a family $\mathcal{Q}$ of symmetric bilinear forms such that

$$P(x, y) = \max_{Q \in \mathcal{Q}} Q(x, y) \qquad \text{for all} \quad x, y \in X; \tag{6}$$

(ii) There exists a family $\mathcal{Q}$ of symmetric bilinear forms such that

$$P(x, y) = \sup_{Q \in \mathcal{Q}} Q(x, y) \qquad \text{for all} \quad x, y \in X; \tag{7}$$

(iii) The function P is positively bihomogeneous and, whenever the elements $x_0, x_1, \ldots, x_k, y_0, y_1, \ldots, y_k \in X$ satisfy

$$\langle x^*, x_0\rangle \langle x^*, y_0\rangle = \langle x^*, x_1\rangle \langle x^*, y_1\rangle + \cdots + \langle x^*, x_k\rangle \langle x^*, y_k\rangle \qquad (\forall x^* \in X^*), \tag{8}$$

then

$$P(x_0,y_0) \le P(x_1,y_1) + \cdots + P(x_k,y_k). \tag{9}$$

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. Assume now that (ii) is valid. The positive bihomogeneity of P follows immediately. Let $x_0, x_1, \dots, x_k,\ y_0, y_1, \dots, y_k \in X$ satisfy (8). We are going to deduce (9). Let $\varepsilon > 0$ be arbitrary. Then, by (ii), there exists a symmetric bilinear function $Q \in \mathcal{Q}$ such that

$$Q \le P \qquad \text{and} \qquad P(x_0,y_0) - \varepsilon \le Q(x_0,y_0).$$

In other words, Q separates the functions P and

$$R(x,y) := \begin{cases} \lambda\mu(P(x_0,y_0)-\varepsilon) & \text{if } (x,y) = (\lambda x_0, \mu y_0), \quad \lambda,\mu > 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Therefore, (8) and Corollary 1 yield that

$$P(x_0,y_0) - \varepsilon \le P(x_1,y_1) + \cdots + P(x_k,y_k).$$

Taking the limit $\varepsilon \to 0$, we obtain (9), what was to proved.

Assume now that (iii) holds. Let $\mathcal{Q}$ be the set of those symmetric bilinear forms Q that satisfy $Q \le P$ on $X \times X$. We are going to show that, for an arbitrary point (x_0,y_0) there exists $Q \in \mathcal{Q}$ such that $Q(x_0,y_0) = P(x_0,y_0)$. This equation then validates (6) at the point (x_0,y_0). Define $R : X \times X \to \mathbb{R}$ by

$$R(x,y) := \begin{cases} \lambda\mu P(x_0,y_0) & \text{if } (x,y) = (\lambda x_0, \mu y_0), \quad \lambda,\mu > 0 \\ -\infty & \text{otherwise.} \end{cases}$$

It follows from (iii) that the necessary and sufficient condition of Corollary 1 holds true for this R and for P. Hence, there exists a symmetric bilinear function Q such that $R \le Q \le P$, i.e.

$$Q \le P \qquad \text{and} \qquad P(x_0,y_0) \le Q(x_0,y_0).$$

Thus $Q \in \mathcal{Q}$ and $Q(x_0,y_0) = P(x_0,y_0)$. The proof is complete. □

Remark. The statement of this Corollary is related to that of Corollary 2 in [3]. The equivalence of (i) and (ii) is, however, an additional statement.

In the following examples we show that (iii) implies, among others, the bisublinearity of P.

Examples 1. Since, for all $x_1, x_2, y \in X$ and $x^* \in X^*$, we have

$$\langle x^*, x_1 + x_2 \rangle \langle x^*, y \rangle = \langle x^*, x_1 \rangle \langle x^*, y \rangle + \langle x^*, x_2 \rangle \langle x^*, y \rangle .$$

Hence, by (iii),

$$P(x_1 + x_2, y) \le P(x_1, y) + P(x_2, y),$$

which is the subadditivity of $P(\,.\,, y)$. Therefore P must be bisublinear.

2. Similarly, we get

$$\begin{aligned} P(x_1 + x_2 + x_3, y_1 + y_2 + y_3) &\le P(x_1, y_1 + y_2) + P(x_1 + x_2, y_3) \\ &+ P(x_3, y_2 + y_3) + P(x_2 + x_3, y_1) + P(x_2, y_2), \end{aligned}$$

which cannot be deduced from the bisublinearity of P.

3. The obvious relation

$$\langle x^*, x \rangle \langle x^*, y \rangle = \frac{1}{4} \langle x^*, x + y \rangle^2 - \frac{1}{4} \langle x^*, x - y \rangle^2$$

yields that

$$P(x, y) \le \frac{1}{4} P(x + y, x + y) + \frac{1}{4} P(x - y, y - x).$$

4. Given $x, y \in X$, one can easily establish that

$$\langle x^*, x + y \rangle^2 = 2 \langle x^*, x \rangle^2 + 2 \langle x^*, y \rangle^2 - \langle x^*, x - y \rangle^2 \qquad \text{for all} \quad x^* \in X^*.$$

Hence

$$P(x + y, x + y) \le 2P(x, x) + 2P(y, y) + P(x - y, y - x).$$

3. Symmetric linear selections of set-valued functions

Throughout this section assume that $F : X \to 2^{X^*}$ is a set-valued mapping with nonempty weak-$*$ compact convex values in X^*. Such a set-valued function is called upper semicontinuous at the origin if for any neighborhood V of the set $F(0) \subset X^*$ there exists a neighborhood U of $0 \in X$ such that $F(x) \subset V$ whenever $x \in U$.

Theorem 2. *Let F be a positively homogeneous set-valued function which is continuous at the origin in its first variable. Then F admits a continuous symmetric linear selection $A : X \to X^*$ if and only if the inclusion*

$$0 \in \langle F(x_1), y_1 \rangle + \cdots + \langle F(x_k), y_k \rangle \tag{10}$$

holds true whenever $x_1, \dots, x_k, y_1, \dots, y_k \in X$ and (2) is valid.

Proof. We are going to deduce this result from Theorem 1. Define the function $P : X \times X \to \mathbb{R}$ by

$$P(x, y) := \sup \langle F(x), y \rangle . \tag{11}$$

Then P is a positively bihomogeneous function which is continuous at the origin in its first variable.

To prove the necessity of (10), assume that A is a symmetric linear selection of F. Then $Q(x, y) = \langle Ax, y\rangle$ defines a symmetric bilinear form which satisfies $Q \leq P$. Applying Theorem 1, then (2) yields

$$\begin{aligned} &0 \leq P(x_1, y_1) + \cdots + P(x_k, y_k) \qquad \text{and} \\ &0 \leq P(x_1, -y_1) + \cdots + P(x_k, -y_k) \end{aligned} \tag{12}$$

whence (10) follows with the help of the formula

$$\langle F(x), y\rangle = [-P(x, -y), P(x, y)]. \tag{13}$$

Now we proceed to prove the sufficiency of the condition. Using the above argument, we see that (12) holds whenever (2) is true. Therefore, by Theorem 1, there exists a symmetric bilinear function that satisfies $Q \leq P$. The continuity of P at the origin in its first variable yields that Q is continuous in the first variable everywhere. By symmetry, Q is also continuous in its second variable. Fixing $x \in X$, define $a_x \in X^*$ by the equality $\langle a_x, y\rangle = Q(x, y)$. Then

$$\langle a_x, y\rangle \in \langle F(x), y\rangle \qquad \text{for} \quad y \in X.$$

Since $F(x)$ is convex and weak-$*$ compact, therefore $a_x \in F(x)$ (by the Hahn-Banach separation theorem). On the other hand, the mapping $x \to a_x$ is a linear mapping (using the linearity of Q in the first variable), hence the formula $Ax := a_x$ defines a linear map $A : X \to X^*$. The continuity of A follows from the usc of F easily. Thus A is the required selection of F. □

Remark. If F is a symmetric set-valued mapping, i.e. $\langle F(x), y\rangle = \langle F(y), x\rangle$ for all $x, y \in X$, then the domain of (10) described by (2) can be replaced by the domain described in (5). This follows by the argument used in the Remark after Theorem 1. The statement of the above Theorem is similar to that of Corollary 4 in [3], which was deduced from the extension version of this theorem for nonsymmetric selections. The extension version for symmetric mappings can be obtained in a similar way, therefore it is omitted here.

The next result describes those set valued mappings that can be generated by families of symmetric linear mappings.

Corollary 3. *For the set-valued map F, which is usc at the origin, the following three properties are equivalent:*

(i) There exists a family $\mathcal{A}$ of continuous symmetric linear operators such that

$$F(x) = \{Ax : A \in \mathcal{A}\} \qquad \text{for all} \quad x \in X. \tag{14}$$

(ii) There exists a family $\mathcal{A}$ of continuous symmetric linear operators such that

$$F(x) = \overline{\{Ax : A \in \mathcal{A}\}} \qquad \text{for all} \quad x \in X. \tag{15}$$

(iii) F is homogeneous, symmetric and, whenever $x_0, x_1, \dots, x_k, y_0, y_1, \dots, y_k \in X$ satisfy (8), then

$$\langle F(x_0), y_0\rangle \to \langle F(x_1), y_1\rangle + \dots + \langle F(x_k), y_k\rangle \tag{16}$$

holds true.

Proof. We are going to deduce the equivalence of (i),(ii) and (iii) from the equivalence of the analogous statements of Corollary 2 following the argument used in the proof of Theorem 2. Define $P : X \times X \to \mathbb{R}$ by (11).

Clearly, (i) implies (ii). Assume that (ii) is valid. Then F turns out to be homogeneous and symmetric at once and, consequently, P is bi-homogeneous. Let $\mathcal{Q}$ be the set of continuous symmetric bilinear forms $Q(x, y) = \langle Ax, y\rangle$, where $A \in \mathcal{A}$ is arbitrary. Then (ii) yields that the assertion (ii) of Corollary 2 holds also true. Thus we have (iii) of Corollary 2. Therefore, if (8) is satisfied, then

$$P(x_0, y_0) \le P(x_1, y_1) + \dots + P(x_k, y_k)$$

and

$$P(x_0, -y_0) \le P(x_1, -y_1) + \dots + P(x_k, -y_k)$$

whence, using (13), the inclusion (16) follows.

Conversely, assume that (iii) holds. Then (iii) of Corollary 2 is valid, too. Therefore we have that P is the maximum of symmetric bilinear forms that are (separately) continuous in both variables. We may assume that the set $\mathcal{Q}$ is convex (since in the proof of Corollary 2 it was constructed as the set of all symmetric bilinear forms that are below P).

To prove (14), let $x_0 \in X$ be fixed, and $x^* \in F(x_0)$ be arbitrary. Define $a_Q \in X^*$ by $a_Q(y) = Q(x_0, y)$. Then the set $H := \{a_Q : Q \in \mathcal{Q}\}$ is convex, weak-$*$ closed further

$$\langle x^*, y\rangle \le P(x_0, y) = \max_{Q \in \mathcal{Q}} a_Q(y) \qquad \text{for all} \quad y \in X.$$

Therefore, by the Hahn-Banach theorem again, $x^* \in H$, i.e. $x^* = a_{\overline{Q}}$ for some $\overline{Q} \in \mathcal{Q}$. Defining the operator $A : X \to X^*$ by the formula

$$\langle Ax, y\rangle = \overline{Q}(x, y), \qquad (\forall y \in X),$$

we can see that A is a continuous symmetric selection of F and $x^* = Ax_0$. Thus (12) is proved. $\square$

Remark. If (iii) is valid, then using the same relation as in Example 1, we obtain

$$\langle F(x_1 + x_2), y\rangle \subset \langle F(x_1), y\rangle + \langle F(x_2), y\rangle$$

for all $x_1, x_2, y \in X$. Hence

$$F(x_1 + x_2) \subset F(x_1) + F(x_2).$$

This property, together with the positive homogeneity, means that F is a sublinear set-valued function. Therefore, the sublinearity of F is necessary in order that condition (i) of Corollary 3 be valid. It is, however, far from being sufficient.

4. Application for $\mathcal{C}^{1,1}$ functions

Let X be a normed space and let $D \subset X$ be a nonempty open subset of X. The class of differentiable functions $f : D \to \mathbb{R}$ with Lipschitz derivative will be denoted by $\mathcal{C}^{1,1}$. (See [2], [1], [4].) The *second order generalized directional derivative* f'° of f at x_0 is defined by

$$f'^{\circ}(x_0; u, v) := \limsup_{\substack{x \to x_0 \\ \varepsilon \to 0+}} \frac{\langle \nabla f(x + \varepsilon u) - \nabla f(x), v\rangle}{\varepsilon} \qquad \text{for} \quad u, v \in X,$$

where ∇f stands for the gradient of f. Cominetti and Correa has established in [1, Prop. 1.3] a general result, which, for the $\mathcal{C}^{1,1}$ setting, reduces to the following identity:

$$f'^{\circ}(x_0; u, v) = \limsup_{\substack{x \to x_0 \\ \varepsilon, \delta \to 0+}} \frac{f(x + \varepsilon u + \delta v) - f(x + \varepsilon u) - f(x + \delta v) + f(x)}{\varepsilon\delta}$$

and hence, f'° turns out to be a symmetric bisublinear function.

The main result obtained in [4] is the following theorem.

Theorem 3. *Let $x_0, x_1, \dots, x_k$ and $y_0, y_1, \dots, y_k$ be elements of X such that (8) holds. Then, for all $z_0 \in D$,*

$$f'^{\circ}(z_0; x_0, y_0) \le \sum_{i=1}^{k} f'^{\circ}(z_0; x_i, y_i).$$

In other words, the function $P(u,v)=f'^{\circ}(x_0;u,v)$ satisfies the condition (iii) of Corollary 2. Thus, we immediately get that the set

$$\partial^2 f(x_0) := \{Q : X\times X\to\mathbb{R} \mid Q \text{ is continuous, symmetric, bilinear and } Q(u,v)\le f'^{\circ}(x_0;u,v)\ \forall u,v\in X\}.$$

is nonempty, moreover the formula

$$f'^{\circ}(x_0;u,v)=\max_{Q\in\partial^2 f(x_0)} Q(u,v)$$

holds. It follows from this result that $\partial^2 f(x_0)$ can be considered as generalized (set-valued) second order derivative for f, which seems to be analogous to the first order generalized subgradient introduced by Clarke. For the applications of these notions and ideas see the recent work of the author with V. Zeidan [4].

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