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Multivariate stochastic integrals with respect to independently scattered random measures on δ-rings

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Abstract. In this paper we construct general vector-valued infinitely-divisible independently scattered random measures with values in $\mathbb{R}^m$ and their corresponding stochastic integrals. Moreover, given such a random measure, the class of all integrable matrix-valued deterministic functions is characterized in terms of certain characteristics of the random measure. In addition, a general construction principle is presented.

1. Introduction

Various stochastic processes and random fields are built by integrating a family of deterministic functions with respect to an infinitely-divisible random measure (e.g. a noise). One of the first and most prominent examples is the fractional Brownian motion. This was extended to the so-called fractional stable motion by replacing the Gaussian random measure by a symmetric α-stable ($S\alpha S$) random measure, see [20] for details.

Based on $S\alpha S$ random measures, a vast class of stochastic processes and random fields has been constructed. See, e.g., [1], [2], [6], [20] and [21] to name a few. All these processes and fields are univariate and have $S\alpha S$ marginal distributions by construction. The general theory of arbitrary infinitely-divisible independently scattered random measures (ISRMs) and the class of integrable functions was carried out in [17].

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Surprisingly enough, and except the Gaussian case, much less is known in the multivariate case. On the one hand, there is merely an ad hoc construction of a multivariate $S\alpha S$ random measure in [14]. On the other hand, the works in [7] (see Chapter III.6) and [15] certainly treat the multivariate case, however, they extend the results in [17] just in parts. The purpose of this paper is to carefully develop an honest theory of general infinitely-divisible ISRMs on δ-rings and their corresponding integrals for matrix-valued deterministic functions. Our approach follows along the lines of [17]. However, since we construct vector-valued measures, some univariate methods using monotonocity no longer apply.

In a subsequent paper [12], our methods will be used to construct an $\mathbb{R}^m$-valued ISRM with operator-stable marginals.

The paper is organized as follows. We start with some notation and useful preliminaries about infinitely-divisible distributions and δ-rings in Section 2. We then characterize all infinitely-divisible $\mathbb{R}^m$-valued random measures in Section 3, already suggesting a complex-valued point of view and proposing a useful construction principle in Theorem 3.4. Finally, in Section 4, the integrators provided by Section 3 are used to define the corresponding stochastic integral for matrix-valued functions. Here we will characterize the class of integrable functions (w.r.t. to a given random measure) and clarify the intimate relation between the real-valued and complex-valued perspective as announced before.

2. Preliminaries

Let $\mathrm{L}(\mathbb{K}^m)$ denote the set of all linear operators on $\mathbb{K}^m$, represented as $m \times m$ matrices with entries from $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Furthermore, let $\|\cdot\|$ be the Euclidian norm on $\mathbb{R}^m$ with inner product $\langle \cdot, \cdot \rangle$, while the identity operator on $\mathbb{R}^m$ is denoted by I_m. Then it is well-known (as the *Lévy–Khintchine Formula*, see [16, Theorem 3.1.11]) that $\varphi = \exp(\psi)$ with $\psi : \mathbb{R}^m \to \mathbb{C}$ is the Fourier transform (or characteristic function) of an infinitely-divisible (i.d.) distribution on $\mathbb{R}^m$, if and only if ψ can be represented as

$$\psi(t) = \mathrm{i}\langle \gamma, t \rangle - \frac{1}{2}\langle Qt, t \rangle + \int_{\mathbb{R}^m} \left(\mathrm{e}^{\mathrm{i}\langle t, x \rangle} - 1 - \frac{\mathrm{i}\langle t, x \rangle}{1 + \|x\|^2} \right) \phi(dx), \quad t \in \mathbb{R}^m$$

for a *shift* $\gamma \in \mathbb{R}^m$, some *normal component* $Q \in \mathrm{L}(\mathbb{R}^m)$ which is symmetric and positive semi-definite and a Lévy measure ϕ, i.e., ϕ is a measure on $\mathbb{R}^m$ with $\phi(\{0\}) = 0$ and $\int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \, \phi(dx) < \infty$. For the distribution μ with $\widehat{\mu} = \varphi$, we write $\mu \sim [\gamma, Q, \phi]$, as γ, Q and ϕ are uniquely determined by μ. ψ is the only continuous function with $\psi(0) = 0$ and $\widehat{\mu} = \exp(\psi)$, subsequently referred to as the *log-characteristic function* of μ.

Lemma 2.1. *Let (μ_n) be a sequence of i.d. distributions on $\mathbb{R}^m$. Then $\mu_n \sim [\gamma_n, Q_n, \phi_n]$ converges weakly to the point measure in zero ε_0 if and only if $\gamma_n \to 0$, $Q_n \to 0$ and*

$$\int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \phi_n(dx) \to 0 \quad (n \to \infty). \tag{2.1}$$

PROOF. By [16, Theorem 3.1.16], it obviously remains to check that (2.1) is equivalent to $\phi_n(A) \to 0$ for all Borel sets A which are bounded away from zero together with

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\{x:0<\|x\|<\varepsilon\}} \langle t, x\rangle^2\, \phi_n(dx) = 0 \quad \text{for all } t \in \mathbb{R}^m.$$

Therefore, by distinguishing the sign of each component, we can decompose $\mathbb{R}^m$ into sets M_j $(j = 1, \ldots, 2m)$ such that $\|x\|^2 \leq \|x\|_1^2 = \langle t_j, x\rangle^2$ for all $x \in M_j$ and suitable $t_j \in \{-1, 1\}^m$, where $\|\cdot\|_1$ is the 1-norm on $\mathbb{R}^m$. □

Throughout this paper, let S be any non-empty set. Then a family of sets $\mathcal{S} \subset \mathcal{P}(S) := \{A : A \subset S\}$ is called a *δ-ring* (on S) if it is a ring (i.e., closed under union and difference together with $\emptyset \in \mathcal{S}$) such that there is a sequence $(S_n) \subset \mathcal{S}$ with $\cup_{n=1}^{\infty} S_n = S$, and which is also additionally closed under countably many intersections. Using the properties of a ring, the sequence (S_n) can assumed to be increasing as well as disjoint, depending on the respective occurrence. Note that any δ-ring $\mathcal{S}$ with $S \in \mathcal{S}$ is a σ-algebra. The next result is also elementary, but helpful, where $\sigma(\mathcal{S})$ denotes the σ-algebra on S that is generated by $\mathcal{S}$.

Lemma 2.2. *Let $\mathcal{S}$ be a δ-ring on S. Then $A \cap M \in \mathcal{S}$ for all $A \in \mathcal{S}$ and $M \in \sigma(\mathcal{S})$.*

PROOF. Observe that $\mathcal{S} \subset \mathcal{D} := \{M \in \sigma(\mathcal{S}) \,|\, \forall A \in \mathcal{S} : A \cap M \in \mathcal{S}\}$. Then we just have to check that $\mathcal{D}$ is already a σ-algebra on S. Since $A \cap M^c = A \setminus (A \setminus M)$, it follows that $M^c \in \mathcal{D}$, whenever $M \in \mathcal{D}$ is true. Analogously, for $A \in \mathcal{S}$ arbitrary and any sequence $(M_n) \subset \mathcal{D}$, we see that $A \cap (\cup_{n=1}^{\infty} M_n)^c = \cap_{n=1}^{\infty} A \setminus (A \cap M_n) \in \mathcal{S}$ holds true. Hence $\mathcal{D}$ is closed under countably many unions. □

We now want to consider vector-valued set functions with domain $\mathcal{S}$. For our purpose it is sufficient to assume that V is a Banach space (with norm $\|\cdot\|_V$). Then we call $T : \mathcal{S} \to V$ *additive* if $T(\emptyset) = 0$ and $T(A_1 \cup \cdots \cup A_k) = T(A_1) + \cdots + T(A_k)$ for any $k \in \mathbb{N}$ and disjoint sets $A_1, \ldots, A_k \in \mathcal{S}$. Furthermore, if

$$T\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} T(A_n) \quad \text{w.r.t. } \|\cdot\|_V$$

holds for any disjoint sequence $(A_n) \subset \mathcal{S}$ with $\cup_{n=1}^{\infty} A_n \in \mathcal{S}$, then T is called *σ-additive*. Finally σ-additive set functions on σ-algebras are called *vector measures*. As we claim $T(A) \in V$ for every $A \in \mathcal{S}$, one can use standard arguments (see [10, Theorem 1.36], for example) to show that an additive set function $T : \mathcal{S} \to V$ is σ-additive if and only if

$$V\text{-}\lim_{n\to\infty} T(A_n) = 0 \quad \text{for all } (A_n) \subset \mathcal{S} \text{ with } A_n \downarrow \emptyset. \tag{2.2}$$

In this context, we distinguish the previous definition from the term *pre-measure*, i.e., those σ-additive set functions on $\mathcal{S}$ that take values in $[0, \infty]$. Yet, given any set function $T : \mathcal{S} \to V$, the *total variation* $|T|$ (of T) connects these concepts, i.e., for any $A \in \mathcal{S}$, we define

$$|T|(A) := \sup\left\{\sum_{j=1}^{n} \|T(A_j)\|_V \mid n \in \mathbb{N} \text{ and } A_1, \ldots, A_n \in \mathcal{S} \text{ disjoint with } A_j \subset A\right\}.$$

Theorem 2.3. *Let $T : \mathcal{S} \to V$ be a σ-additive set function. Then $|T|$ is a pre-measure. Additionally, if V is finite-dimensional, then $|T|$ is $[0, \infty)$-valued, i.e., a finite pre-measure.*

Proof. As in [4, III 1, Lemma 6], we get that $|T|$ is additive, although $\mathcal{S}$ is just a (δ-)ring. Using this and the arguments in the proof of [4, III 4, Lemma 7], it follows that $|T|$ is even σ-additive. Finally, if $V = \mathbb{R}^n$ (without loss of generality), we can assume that $n = 1$ by equivalence of norms and by considering the component functions of T which inherit the σ-additivity. Now, due to (2.2) and the closure of $\mathcal{S}$ under countably many intersections, we can argue as in [13, XI, Theorem 8] to obtain the assertion. □

Remark 2.4. In view of the quoted proofs, we observe that if $T : \mathcal{S} \to [0, \infty)$ is *σ-subadditive*, then its total variation $|T|$ is at least still a pre-measure (in general with values in $[0, \infty]$).

Unfortunately, it is impossible to formulate the *Hahn–Jordan decomposition* on δ-rings. But for the case $V = \mathbb{R}$, we can at least consider the *positive variation*

$T^+ : \mathcal{S} \to [0,\infty)$ and the *negative variation* $T^- : \mathcal{S} \to [0,\infty)$ of the σ-additive set function T, defined by $T^{\pm}(A) := \frac{1}{2}(|T|(A) \pm T(A))$, respectively. Then it is clear that T^+ and T^- are finite pre-measures with $T = T^+ - T^-$ as well as $|T| = T^+ + T^-$. Although it was formulated for σ-algebras in [4, III 1, Theorem 8], we see that the following representations hold for every $A \in \mathcal{S}$:

$$T^+(A) = \sup\{T(B) : B \in \mathcal{S} \text{ with } B \subset A\} \tag{2.3}$$

and

$$T^-(A) = -\inf\{T(B) : B \in \mathcal{S} \text{ with } B \subset A\}. \tag{2.4}$$

3. Infinitely-divisible random measures

In this section we define and analyze ISRMs with values in $\mathbb{K}^m$ defined on δ-rings. Hence if we denote by $L^0(\Omega, \mathbb{K}^m)$ the set of all $\mathbb{K}^m$-valued random vectors defined on any abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a mapping $M : \mathcal{S} \to L^0(\Omega, \mathbb{K}^m)$ is shortly called an independently scattered random measure (on $\mathcal{S}$ with values in $\mathbb{K}^m$), if the following conditions hold:

(RM_1) For every finite choice $A_1, \ldots, A_k$ of disjoint sets in $\mathcal{S}$, the random vectors $M(A_1), \ldots, M(A_k)$ are stochastically independent.

(RM_2) For every sequence $(A_n) \subset \mathcal{S}$ of disjoint sets with $\cup_{n=1}^{\infty} A_n \in \mathcal{S}$, we have

$$M(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n) \quad \text{almost surely (a.s.).}$$

By introducing the mapping $\Xi_{(m)}(z) := (\operatorname{Re} z, \operatorname{Im} z) \in \mathbb{R}^{2m}$ for $z \in \mathbb{C}^m$, condition (RM_1) here means independence of $\Xi(M(A_1)), \ldots, \Xi(M(A_k))$. Furthermore, and with an analogous extension for $\mathbb{K} = \mathbb{C}$, we call such an ISRM *infinitely-divisible*, if this true for (the distribution of) every random vector $M(A), A \in \mathcal{S}$. In this case, we get the following characterization, where we first consider $\mathbb{K} = \mathbb{R}$:

Theorem 3.1. *Let M be an i.d. ISRM on $\mathcal{S}$ with values in $\mathbb{R}^m$, where $M(A) \sim [\gamma_A, Q_A, \phi_A]$ for every $A \in \mathcal{S}$. Then we have:*

(a) *The mapping $\mathcal{S} \ni A \mapsto \gamma_A \in \mathbb{R}^m$ is σ-additive.*

(b) *The mapping $\mathcal{S} \ni A \mapsto Q_A \in L(\mathbb{R}^m)$ is σ-additive.*

(c) *The mapping $\mathcal{S} \ni A \mapsto \phi_A(B)$ is a finite pre-measure for every fixed Borel set B which is bounded away from zero.*

Conversely, for every family of triplets $([\gamma_A, Q_A, \phi_A])_{A\in\mathcal{S}}$ *that satisfies* (a)–(c) *there exists an i.d. ISRM* M *(on some suitable probability space) with* $M(A) \sim [\gamma_A, Q_A, \phi_A]$ *for every* $A \in \mathcal{S}$*. Furthermore, the finite-dimensional distributions of* M *are uniquely determined by the latter property.*

Proof. Assume first that M is an infinitely-divisible ISRM. Since $M(\emptyset) = 0$ a.s., the additivity of the mappings in (a)–(c) can be easily deduced from the Lévy–Khintchine Formula and its uniqueness statement by using (RM_1) and (RM_2) for only finitely many sets. Then it is even clear that $\phi_{A_1\cup\cdots\cup A_k}$ equals the measure $\phi_{A_1} + \cdots + \phi_{A_k}$. Now let $(B_n) \subset \mathcal{S}$ be a sequence with $(B_n) \downarrow \emptyset$ and define $C_1 = \emptyset$, $C_n = B_{n-1} \setminus B_n$ (for $n \geq 2$) to observe that

$$M(B_1) = M(\cup_{k=1}^{\infty} C_n) = \lim_{k\to\infty} (M(B_1) - M(B_k)),$$

which leads to $M(B_k) \to 0$ a.s. Then (a) and (b) follow by Theorem 2.1 together with (2.2). Similarly, using [16, Theorem 3.1.16], we obtain (c).

Concerning the second part, denote by $\Theta(A, \cdot)$ the log-characteristic function of the i.d. distribution on $\mathbb{R}^m$ with triplet $[\gamma_A, Q_A, \phi_A]$ for $A \in \mathcal{S}$. Moreover, for any $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{S}$, we define

$$\psi_{A_1,\ldots,A_n}(t) := \sum_{J\subset\{1,\ldots,n\}} \Theta\left(\mathcal{Z}_J^{(n)}, \sum\nolimits_{j\in J} t_j\right),$$

where $t = (t_1, \ldots, t_n) \in \mathbb{R}^{n\cdot m}$ and

$$\mathcal{Z}_J^{(n)} := \mathcal{Z}_J(A_1, \ldots, A_n) := \begin{cases} \emptyset, & \text{if } J = \emptyset \\ \left[\bigcap_{j\in J} A_j \setminus \bigcup_{l\in J^c} A_l\right], & \text{if } J \neq \emptyset \end{cases} \in \mathcal{S}.$$

Then, with [8, Lemma 3.5.9], for example, it easy to see that $\exp(\psi_{A_1,\ldots,A_n}(\cdot))$ is not only continuous, but also positive semi-definite in the sense of Bochner's theorem, as this is true for the functions $\exp(\Theta(A, \cdot))$ already. Then by the theorem itself, we obtain the existence of a distribution $\mu_{A_1,\ldots,A_n}$ on $\mathbb{R}^{n\cdot m}$ whose Fourier transform is given by $\exp(\psi_{A_1,\ldots,A_n}(\cdot))$, in particular, we have $\mu_A \sim [\gamma_A, Q_A, \phi_A]$ for all $A \in \mathcal{S}$. Then on the one hand, we can check that

$$\mathcal{Z}_J(A_1, \ldots, A_{n+1}) \cup \mathcal{Z}_{J\cup\{n+1\}}(A_1, \ldots, A_{n+1}) = \mathcal{Z}_J(A_1, \ldots, A_n)$$

for $A_1, \ldots, A_{n+1} \in \mathcal{S}$ and every $J \in \mathcal{P}(\{1, \ldots, n\})\setminus\emptyset$, where the union is disjoint. On the other hand, (c) implies for all $B_1, B_2 \in \mathcal{S}$ disjoint and $t \in \mathbb{R}^m$ that

$\Theta(B_1 \cup B_2, t) = \Theta(B_1, t) + \Theta(B_2, t)$. Hence, for $t_1, \ldots, t_n \in \mathbb{R}^m$ arbitrary, we get with $t_{n+1} := 0$ that

$$\begin{aligned}\psi_{A_1,\ldots,A_{n+1}}(t_1,\ldots,t_n,0) &= \sum_{J\subset\{1,\ldots,n+1\}} \Theta(\mathcal{Z}_J^{(n+1)}, \sum_{j\in J} t_j) \\ &= \sum_{J\subset\{1,\ldots,n\}} \left[\Theta(\mathcal{Z}_J^{(n+1)}, \sum_{j\in J} t_j) + \Theta(\mathcal{Z}_{J\cup\{n+1\}}^{(n+1)}, \sum_{j\in J} t_j)\right] \\ &= \sum_{\substack{J\subset\{1,\ldots,n\},\\ J\neq\emptyset}} \left[\Theta(\mathcal{Z}_J^{(n+1)}, \sum_{j\in J} t_j) + \Theta(\mathcal{Z}_{J\cup\{n+1\}}^{(n+1)}, \sum_{j\in J} t_j)\right] \\ &= \psi_{A_1,\ldots,A_n}(t_1,\ldots,t_n).\end{aligned}$$

Overall, this mostly proves that the considered system is projective and by Kolmogorov's consistency theorem, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a family $M = \{M(A) : A \in \mathcal{S}\}$ of random vectors with marginal distributions $\mathcal{L}((M(A_1), \ldots, M(A_n))) = \mu_{A_1,\ldots,A_n}$. For $A_1, \ldots, A_n \in \mathcal{S}$ disjoint, we have that $\mathcal{Z}_J^{(n)} = A_j$ if $J = \{j\}$, and $\mathcal{Z}_J^{(n)} = \emptyset$ else, which yields that (RM_1) is fulfilled. For (RM_2) we first fix $A_1, A_2 \in \mathcal{S}$ arbitrary and write

$$\widehat{\mathcal{L}}(M(A_1 \cup A_2) - M(A_1) - M(A_2))(t) = \widehat{\mu}_{A_1\cup A_2, A_1, A_2}(t, -t, -t), \quad t \in \mathbb{R}^m$$

to see that M is finitely additive, as the right-hand side equals 1 by construction. Thus for a sequence like given in (RM_2), it suffices to show that

$$M(\cup_{j=1}^{\infty} A_j) - M(\cup_{j=1}^{k} A_j) = M(\cup_{j=k+1}^{\infty} A_j) \xrightarrow[(k\to\infty)]{\mathbb{P}} 0$$

by a straightforward multivariate extension of the the three-series-theorem (see [3, Theorem 9.7.1]) and by what we have shown before. If we let $B_k := \cup_{j=k+1}^{\infty} A_j$ with $B_k \downarrow \emptyset$, it follows by (a) and (b) that $\gamma_{B_k} \to 0$ as well as that $Q_{B_k} \to 0$. Provided that

$$\lim_{k\to\infty} \int\limits_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \phi_{B_k}(dx) = 0, \tag{3.1}$$

the assertion would follow via Theorem 2.1. Fix $\varepsilon > 0$ and choose $\delta > 0$ sufficiently small such that

$$\int\limits_{\{x:\|x\|<\delta\}} \min\{1, \|x\|^2\}\, \phi_{B_k}(dx) \le \int\limits_{\{x:\|x\|<\delta\}} \min\{1, \|x\|^2\}\, \phi_{B_1}(dx) < \varepsilon, \quad k \in \mathbb{N}$$

in face of $\phi_{k+1} \leq \phi_k$ (see above), such that (3.1) follows by (c) again. Finally, for uniqueness we fix $A_1, A_2 \in \mathcal{S}$ and observe that $\langle t_1, M(A_1)\rangle + \langle t_2, M(A_2)\rangle$ equals

$$\langle t_1, M(A_1 \setminus A_2)\rangle + \langle t_1 + t_2, M(A_1 \cap A_2)\rangle + \langle t_2, M(A_2 \setminus A_1)\rangle,$$

where the three last-mentioned random variables are independent due to (RM_1). Now the statement can be deduced easily. □

Let us remark that the previous theorem as well as the following ones are similar to the corresponding, but univariate results in [17].

Theorem 3.2. *Let M be an i.d. ISRM as before, then there exists a σ-finite measure λ_M on $\sigma(\mathcal{S})$, called control measure of M, which is uniquely determined by*

$$\lambda_M(A) = |\gamma|_A + \mathrm{tr}(Q_A) + \int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \phi_A(dx), \quad A \in \mathcal{S}, \tag{3.2}$$

where $|\gamma|_A := |\gamma|(A)$. Furthermore, for any sequence $(A_n) \subset \mathcal{S}$, we have:

(i) *$\lambda_M(A_n) \to 0$ implies $M(A_n) \to 0$ in probability.*

(ii) *If $M(A'_n) \to 0$ in probability for every sequence $(A'_n) \subset \mathcal{S}$ with $A'_n \subset A_n$, then it follows that $\lambda_M(A_n) \to 0$.*

Proof. We have to show that (3.2) defines a finite pre-measure on $\mathcal{S}$, then λ_M would be its unique extension on $\sigma(\mathcal{S})$: non-negativity is obvious. Moreover, $|\gamma|$ is finite by Theorem 2.3 and Theorem 3.1 (a). The mapping $A \mapsto \mathrm{tr}(Q_A)$ preserves the σ-additivity in Theorem 3.1 (b) by continuity of the trace-mapping $\mathrm{tr}(\cdot)$. Finally, we could already show that $A \mapsto \phi_A$ is additive, thus, as before, it remains to show that

$$\int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \phi_{B_n}(dx) \to 0 \tag{3.3}$$

for any sequence $(B_n) \subset \mathcal{S}$ with $B_n \downarrow \emptyset$. Actually, the previous proof even revealed that $M(B_n) \to 0$ a.s., such that (3.3) follows by (2.1).

Now, if $\lambda_M(A_n) \to 0$ for a sequence as above, the same holds for each of the corresponding expressions in (3.2), which allows us to use Theorem 2.1 again. Because of $\|\gamma_{A_n}\| \leq |\gamma|_{A_n}$ and since $\mathrm{tr}(Q_{A_n}) \to 0$ implies $Q_{A_n} \to 0$, we get $M(A_n) \to 0$ in probability. Conversely, the proof of $\lambda_M(A_n) \to 0$ reduces to the verification of $|\gamma|_{A_n} \to 0$ after using similar arguments as before and especially the assumption that $M(A_n) \to 0$ in probability. Consider the component functions $\gamma^{(1)}, \dots, \gamma^{(m)}$ and fix some $\varepsilon > 0$ and $j \in \{1, \dots, m\}$, where Theorem 3.1 (a) and the combination of (2.3)–(2.4) guarantee the existence of sequences $(A_{n,i})_n \subset \mathcal{S}$ with $A_{n,i} \subset A_n$ for $i = 1, 2$ with

$$|\gamma^{(j)}|_{A_n} \leq \gamma^{(j)}_{A_{n,1}} - \gamma^{(j)}_{A_{n,2}} + \varepsilon, \quad n \in \mathbb{N}.$$

Now one can use the given assumption together with Theorem 2.1 again to see that $\gamma^{(j)}_{A_{n,i}} \to 0$ for $i = 1, 2$, which yields $|\gamma^{(j)}|_{A_n} \to 0$ and therefore the assertion of (ii), see the proof of Theorem 2.3. □

Next, we want to extend [17, Lemma 2.3], which yields a construction principle for ISRMs in Theorem 3.4 (b) below: Given measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, a mapping $\kappa : \Omega_1 \times \mathcal{A}_2 \to [0, \infty]$ is called a *simultaneous σ-finite transition function from* Ω_1 *to* Ω_2, if the following conditions hold:

(i) $\omega_1 \mapsto \kappa(\omega_1, A_2)$ is $\mathcal{A}_1$-$\mathcal{B}([0, \infty])$-measurable for every $A_2 \in \mathcal{A}_2$.

(ii) $A_2 \mapsto \kappa(\omega_1, A_2)$ is a measure on $(\Omega_2, \mathcal{A}_2)$ for every $\omega_1 \in \Omega_1$. Moreover, there exist sequences $(A_{2,n}) \subset \mathcal{A}_2$ and $(r_n) \subset [0, \infty)$ such that

$$\bigcup_{n=1}^{\infty} A_{2,n} = \Omega_2 \quad \text{and} \quad \forall\, n \in \mathbb{N}\ \forall\, \omega_1 \in \Omega_1 : \kappa(\omega_1, A_{2,n}) \leq r_n. \tag{3.4}$$

Furthermore, if $\kappa(\omega_1, \cdot)$ is a probability measure for every $\omega_1 \in \Omega$, we say that κ is *Markovian*.

Proposition 3.3. *Let $(\Omega_1, \mathcal{A}_1, \nu)$ be a σ-finite measure space, and κ a simultaneous σ-finite transition function from Ω_1 to Ω_2. Then there exists a unique σ-finite measure $\nu \odot \kappa$ on the product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ with the property*

$$(\nu \odot \kappa)(A_1 \times A_2) = \int_{A_1} \kappa(\omega_1, A_2)\, \nu(d\omega_1) \quad \text{for all } A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2.$$

Moreover, we have

$$\int_{\Omega_1 \times \Omega_2} f(x)\, (\nu \odot \kappa)(dx) = \int_{\Omega_1} \int_{\Omega_2} f(\omega_1, \omega_2)\, \kappa(\omega_1, d\omega_2)\, \nu(d\omega_1) \tag{3.5}$$

for every measurable $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ that is non-negative or integrable w.r.t. $\mu \odot \kappa$.

Proof. Choose $(A_{1,n}) \subset \mathcal{A}_1$ disjoint with $\cup_{n=1}^{\infty} A_{1,n} = \Omega_1$ and $\nu(A_{1,n}) < \infty$ for all $n \in \mathbb{N}$. Let $\nu^{(n)}(\cdot) := \nu(\cdot \cap A_{1,n})$. Similarly, $\kappa^{(n)}(\omega_1, \cdot) := \kappa(\omega_1, \cdot \cap A_{2,n})$ is a finite transition function with $(A_{2,n})$ from (3.4) for every $\omega_1 \in \Omega_1$ and $n \in \mathbb{N}$. As the assertion is well-known for ν and κ being finite (see 14.23 and 14.29 in [10]), one easily checks that it is enough to define

$$(\nu \odot \kappa)(C) := \int_{\Omega_1} \int_{\Omega_2} \mathbb{1}_C(\omega_1, \omega_2)\, \kappa(\omega_1, d\omega_2)\, \nu(d\omega_1), \quad C \in \mathcal{A}_1 \otimes \mathcal{A}_2.$$

More precisely, we can consider $C_n := A_{1,\pi_1(n)} \times A_{2,\pi_2(n)}$ with a suitable mapping $\pi = (\pi_1, \pi_2) : \mathbb{N} \to \mathbb{N}^2$ which is one-to-one. Then $(\nu \odot \kappa)(\cdot \cap C_n)$ is finite under the given assumption on κ, and moreover equals $\nu^{(\pi_1(n))} \odot \kappa^{(\pi_2(n))}$ for every $n \in \mathbb{N}$. □

Theorem 3.4. *Let $\mathcal{S}$ be a δ-ring as above and consider the σ-algebra $\sigma(\mathcal{S})$.*

(i) *For every i.d. ISRM M on $\mathcal{S}$ with values in $\mathbb{R}^m$, there exists a simultaneous σ-finite transition function ρ_M from S to $\mathbb{R}^m$ with $(\lambda_M \odot \rho_M)(A \times B) = \phi_A(B)$ for every $A \in S$ and $B \in \mathcal{B}(\mathbb{R}^m)$, where ϕ_A is the Lévy measure of $M(A)$. Here ρ_M is uniquely determined λ_M-almost everywhere (a.e.) and can be chosen such that*

$$\int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \rho_M(s, dx) \le 1 \quad \textit{for every } s \in S. \tag{3.6}$$

(ii) *Conversely, let λ be a measure on S which is finite on $\mathcal{S}$, and ρ a transition function from S to $\mathbb{R}^m$ fulfilling* (3.6), *i.e., being simultaneous σ-finite. Then there exists an ISRM M with $\lambda = \lambda_M$ and $\rho = \rho_M$ (in the previous sense).*

Proof. Let $(S_n) \subset \mathcal{S}$ be a disjoint sequence that exhausts S. Then, as in the proof of Theorem 3.2, we see that $Q_0^*(A, B) := \int_B \min\{1, \|x\|^2\}\, \phi_A(dx)$ is a finite pre-measure on $\mathcal{S}$ for any fixed Borel set $B \subset \mathbb{R}^m$, and we denote its unique extension towards a σ-finite measure on $\sigma(\mathcal{S})$ by $Q_0(\cdot, B)$. Hence for $A \in \sigma(\mathcal{S})$ and $(B_k) \subset \mathcal{B}(\mathbb{R}^m)$ disjoint, we observe by Theorem 2.2 that

$$Q_0\left(A, \bigcup_{k=1}^{\infty} B_k\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} Q_0^*(A \cap S_n, B_k) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} Q_0^*(A \cap S_n, B_k) = \sum_{k=1}^{\infty} Q_0(A, B_k).$$

Consequently, the assumptions of [17, Proposition 2.4] are fulfilled, and by a slight refinement (in particular $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are isomorphic as measurable spaces), we get the existence of a Markovian transition function κ from S to $\mathbb{R}^m$ such that $Q_0(A, B) = (\lambda_0 \odot \kappa)(A \times B)$ for every $A \in \sigma(\mathcal{S})$ and $B \in \mathcal{B}(\mathbb{R}^m)$, where $\lambda_0(\cdot) := Q_0(\cdot, \mathbb{R}^m) \le \lambda_M(\cdot)$. Let τ_0 be a λ_M-derivative of λ_0 with $\tau_0(s) \le 1$ for every $s \in S$, and set

$$\rho_M(s, dx) := \tau_0(s) \cdot \min\{1, \|x\|^2\}^{-1} \cdot \mathbb{1}_{\mathbb{R}^m \setminus \{0\}}(x) \kappa(s, dx), \quad s \in S.$$

This shows (3.6). Moreover, the following calculation, which is valid for every $A \in \mathcal{S}, B \in \mathcal{B}(\mathbb{R}^m)$ and which benefits from the simplicity of the integrand, yields

$$\begin{aligned}\int_A \rho_M(s, B)\, \lambda_M(ds) &= \int_A \int_{B\setminus\{0\}} (\min\{1, \|x\|^2\})^{-1}\, \kappa(s, dx)\, \lambda_0(ds) \\ &= \int_{A\times(B\setminus\{0\})} (\min\{1, \|x\|^2\})^{-1}\, (\lambda_0 \odot \kappa)(ds, dx) \\ &= \int_{B\setminus\{0\}} (\min\{1, \|x\|^2\})^{-1}\, Q_0^*(A, dx) = \phi_A(B).\end{aligned}$$

The uniqueness of ρ_M follows by the Radon–Nikodým theorem after countably many unions of null sets by considering the generator $\{M_1 \times \cdots \times M_m : M_j \in \mathcal{M}\}$ of $\mathcal{B}(\mathbb{R}^m)$ with

$$\mathcal{M} := \{\{0\} \cup (-\infty, q_1] \cup [q_2, \infty) : q_1 \in \mathbb{Q}_{<0}, q_2 \in \mathbb{Q}_{>0}\}.$$

Conversely, the assumption in (ii) ensures that $\phi_A(B) := \int_A \rho(s, B)\, \lambda(ds)$ with

$$\int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \phi_A(dx) = \int_A \int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \rho(s, dx)\, \lambda(ds) \le \lambda(A)$$

is a Lévy measure on $\mathbb{R}^m$ for every $A \in \mathcal{S}$, whereas the total variation of

$$\mathcal{S} \ni A \mapsto \gamma_A := \left(\lambda(A) - \int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \phi_A(dx)\right) e_1$$

is given by the non-negative expression in brackets for every $A \in \mathcal{S}$ (notice (3.6) again). Here e_j generally denotes the j-th unit vector. Now we can obviously use Theorem 3.1 for the triplets $[\gamma_A, 0, \phi_A]$ to obtain the assertion. □

Proposition 3.5. *Let M be an $\mathbb{R}^m$-valued and i.d. ISRM on $\mathcal{S}$, where $M(A) \sim [\gamma_A, Q_A, \phi_A]$ for every $A \in \mathcal{S}$.*

(i) *There are $\sigma(\mathcal{S})$-measurable mappings $\alpha_M : S \to \mathbb{R}^m$ and $\beta_M : S \to L(\mathbb{R}^m)$ such that the following integrals exist (component-wise) with*

$$\int_A \alpha_M(s)\,\lambda_M(ds) = \gamma_A, \qquad \int_A \beta_M(s)\,\lambda_M(ds) = Q_A, \tag{3.7}$$

for every $A \in \mathcal{S}$. α_M and β_M are uniquely determined λ_M-a.e. by (3.7).

(ii) *$\beta_M(s)$ is symmetric and positive semi-definite λ_M-a.e.*

(iii) *The mapping*

$$\mathbb{R}^m \ni t \mapsto \int_A K_M(t,s)\,\lambda_M(ds) \tag{3.8}$$

is the log-characteristic function of $M(A)$ for every $A \in \mathcal{S}$, where the function $K_M : \mathbb{R}^m \times S \to \mathbb{C}$ is defined by

$$K_M(t,s) = \mathrm{i}\langle \alpha_M(s), t\rangle - \frac{1}{2}\langle \beta_M(s)t, t\rangle + \int_{\mathbb{R}^m} \left(\mathrm{e}^{\mathrm{i}\langle t,x\rangle} - 1 - \frac{\mathrm{i}\langle t,x\rangle}{1+\|x\|^2} \right) \rho_M(s,dx). \tag{3.9}$$

Proof. (i) We start with a general observation: Consider $T : \mathcal{S} \to \mathbb{R}$ σ-additive, then $|T|$ can be uniquely extended to a σ-finite measure $\widehat{|T|}$, where we assume that $\widehat{|T|} \ll \lambda_M$. Hence the same holds for the extensions $\widehat{T^+}$ of T^+ and $\widehat{T^-}$ of T^- such that the Radon–Nikodým theorem provides measurable, $[0,\infty]$-valued mappings $f^\pm$ with $\widehat{T^\pm}(A) = \int_A f^\pm(s)\,\lambda_M(ds)$ for $A \in \sigma(\mathcal{S})$. Consider $(S_n) \subset \mathcal{S}$ disjoint with $\cup_{n=1}^\infty S_n = S$. Then $f^+ \mathbb{1}_{S_n}$ and $f^- \mathbb{1}_{S_n}$ are finite λ_M-a.e. Hence there are λ_M-null sets N^+ and N^- such that $f^+\mathbb{1}_{N^+}$ and $f^+\mathbb{1}_{N^+}$ are finite, preserving the integral relation above instead of $f^\pm$, respectively. Then $f := f^+\mathbb{1}_{N^+} - f^+\mathbb{1}_{N^+}$ is λ_M-integrable over every set $A \in \mathcal{S}$ with value $T(A)$. Thus the mappings α_M and β_M can be obtained by using the previous method for each of its components, where $|Q| \le \lambda_M$ (on $\mathcal{S}$) and therefore $\widehat{|Q|} \ll \lambda_M$, which can be shown similarly as in the proof of Theorem 3.2.

(ii) In view of Theorem 2.2, we observe that $A \mapsto \langle Q_{A\cap S_n} x, x\rangle$ is a finite measure on $\sigma(\mathcal{S})$, while the Cauchy–Schwarz inequality yields that this measure is also absolutely continuous w.r.t λ_M. At the same time, we know by (i) that $\langle \beta_M(\cdot)x, x\rangle \mathbb{1}_{S_n}(\cdot)$ is a corresponding λ_M-derivative which has to be non-negative λ_M-a.e. due to the Radon–Nikodým theorem. Therefore, we have $\langle \beta_M(\cdot)x, x\rangle \ge 0$ except a λ_M-null set and for all $x \in \mathbb{Q}^m$, which finally means that $\beta_M(\cdot)$ is positive semi-definite λ_M-a.e. by continuity of the inner product. The symmetry follows if we consider the components $Q^{i,j}$ of Q. In particular, we see that $A \mapsto$

$(Q^{i,j}_{A\cap S_n} - Q^{j,i}_{A\cap S_n})$ equals the zero measure on $\sigma(\mathcal{S})$ for every $n \in \mathbb{N}$ as $Q_{A\cap S_n}$ is symmetric.

(iii) The λ_M-integrability of $K_M(t,\cdot)$ and (3.8) are almost obvious (see (i) and remember that $M(A) \sim [\gamma_A, Q_A, \phi_A]$). Using Theorem 3.4 and (3.5), it is easy to see that the following integral

$$\int_A \int_{\mathbb{R}^m} h(t,x)\, \rho_M(s,dx)\, \lambda_M(ds)$$
$$= \int_{S\times\mathbb{R}^m} h(t,x) 1\!\!1_A(s)\, (\lambda_M \odot \rho_M)(ds,dx) = \int_{\mathbb{R}^m} h(t,x)\, \phi_A(dx),$$

where the last step is similar as before and $h(t,x)$ denotes the integrand used in the definition of K_M. □

Remark 3.6. In view of (3.8) and the uniqueness of the Lévy–Khintchine Formula, we write $M \sim (\lambda_M, K_M)$. And in the case of $\alpha_M = \beta_M = 0$, we may even write $M \sim (\lambda_M, \rho_M)$, respectively. Observe that the latter case applies to Theorem 3.4 (ii) as long as (3.6) holds with equality.

Example 3.7. (a) Consider a σ-finite measure space (S, Σ, ν) and assume that $\mu \sim [\gamma', Q', \phi']$ is an i.d. distribution on $\mathbb{R}^m$ with log-characteristic function ψ and not being the point measure at zero. Then $\mathcal{S}_\nu := \{A \in \Sigma : \nu(A) < \infty\}$ is a δ-ring with $\sigma(\mathcal{S}_\nu) = \Sigma$, which can be verified easily with the aid of (S_n). Hence, according to Theorem 3.1, there exists an i.d. ISRM M with $M(A) \sim [\nu(A)\cdot\gamma', \nu(A)\cdot Q', \nu(A)\cdot\phi']$ for every $A \in \mathcal{S}_\nu$, and we say that M is *generated by ν and μ*. Moreover, with

$$C_\mu := \|\gamma'\| + \operatorname{tr}(Q') + \int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\, \phi'(dx) \quad \in (0,\infty)$$

we get that $\lambda_M(\cdot) = C_\mu \cdot \nu(\cdot)$, while $\rho_M(\cdot) = C_\mu^{-1}\cdot\phi'(\cdot)$ and $K_M(\cdot) = C_\mu^{-1}\cdot\psi(\cdot)$ are both constant in $s \in S$. Therefore, it is convenient to write $M \sim (\nu, \mu)$ and one can check by the construction in Theorem 3.1 that $M(A_1)$ and $M(A_2)$ are independent if and only if $\nu(A_1 \cap A_2) = 0$. Furthermore, independence of $M(A_1), \ldots, M(A_n)$ is equivalent to pairwise independence.

(b) In [5] an $\mathbb{R}$-valued ISRM M_α is constructed such that the log-characteristic function of $M_\alpha(A)$ is given by

$$\mathbb{R} \ni t \mapsto -\int_A |t|^{\alpha(s)}\, ds \tag{3.10}$$

for every Borel set $A \subset \mathbb{R}$ with finite Lebesgue measure. Here $\alpha : \mathbb{R} \to [a, b]$ is a measurable function with $0 < a \le b < 2$, and M is called an $\alpha(s)$-*multistable random measure*. On the one hand, Theorem 3.1 says that M_α is uniquely determined by (3.10), on the other hand, M can be recovered by our approach and (3.8): Denote by $\rho_\alpha(s, \cdot)$ for every $s \in \mathbb{R}$ the Borel measure with Lebesgue density $x \mapsto \theta(s)\,|x|^{-\alpha(s)-1}$, where $\theta(s) := \frac{\alpha(s)}{4}(2 - \alpha(s)) \in [c_1, c_2]$ for all $s \in \mathbb{R}$ and suitable $0 < c_1 \le c_2 < \infty$ by the assumption on $\alpha(s)$, i.e., (3.6) is fulfilled with equality. Similarly and as in [20], there exists a measurable function $\eta : \mathbb{R} \to [c_3, c_4] \subset (0, \infty)$ such that

$$\eta(s) \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1 + x^2} \right) |x|^{-\alpha(s)-1}\, dx = -|t|^{\alpha(s)}$$

for every $s, t \in \mathbb{R}$. Finally, let $\lambda_\alpha(\cdot)$ be the Borel measure with Lebesgue density $s \mapsto (\theta(s)\eta(s))^{-1}$ and apply Theorem 3.4, which means $M_\alpha \sim (\lambda_\alpha, \rho_\alpha)$ by Theorem 3.6.

Remark 3.8. If we identify $\mathcal{B}(\mathbb{C}^m)$ and $\mathcal{B}(\mathbb{R}^{2m})$ by means of Ξ, we can observe that the relation between i.d. random measures with values in $\mathbb{C}^m$ and $\mathbb{R}^{2m}$, respectively, is one-to-one. Generally, for any $\mathbb{C}^m$-valued ISRM M, we say that $\Xi(M)$ is its *real associated* ISRM.

Of course, we can (and will do) interpret every $\mathbb{R}^m$-valued i.d. ISRM M as such a one with values in $\mathbb{C}^m$, having no imaginary parts, which leads to $\Xi(t) := (t, 0)$ for every $t \in \mathbb{R}^m$. Hence, in this case, we understand Ξ as a mapping with domain $\mathbb{R}^m$. Furthermore, we then see that $\Xi(M)(A) \sim [\tilde{\gamma}_A, \tilde{Q}_A, \tilde{\phi}_A]$ with

$$\tilde{\gamma}_A = (\gamma_A, 0),\, \tilde{Q}_A = \begin{pmatrix} Q_A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\phi}_A = \Xi(\phi_A), \quad A \in \mathcal{S}.$$

Similarly, this works for the objects in Theorem 3.5 and one immediately checks that $\lambda_{\Xi(M)} = \lambda_M$. At the same time, it can be computed that $\rho_{\Xi(M)}(s, A) = \rho_M(s, \Xi^{-1}(A))$ for any $A \in \mathcal{B}(\mathbb{R}^{2m})$, together with $K_M(s, t_1) = K_{\Xi(M)}(s, t)$ for all $s \in S$ and $t = (t_1, t_2) \in \mathbb{R}^{2m}$.

4. Integrals with respect to ISRMs

Let M be a $\mathbb{K}^m$-valued ISRM on a δ-ring $\mathcal{S}$, where we assume that M is i.d. Then a matrix-valued mapping $f : S \to \mathrm{L}(\mathbb{K}^m)$ is called $\mathcal{S}$-*simple* if f can be represented by $f = \sum_{j=1}^n R_j \mathbb{1}_{A_j}$ with $R_1, \ldots, R_n \in \mathrm{L}(\mathbb{K}^m)$ and $A_1, \ldots, A_n \in \mathcal{S}$

disjoint. In this case, we define the stochastic integral of $f\mathbb{1}_A$ w.r.t M by

$$I_M(f\,\mathbb{1}_A) := I(f\,\mathbb{1}_A) := \int_A f\,dM := \int_A f(s)\,M(ds) := \sum_{j=1}^{n} R_j\,M(A \cap A_j). \quad (4.1)$$

Note that, in view of Theorem 2.2, the mentioned truncation is valid for every $A \in \sigma(\mathcal{S})$ and that the stochastic integral is well-defined a.s. by (RM_2). Write $I_M(f)$ and so on for $A = S$.

Definition 4.1. Let $f : S \to \mathrm{L}(\mathbb{K}^m)$ be $\sigma(\mathcal{S})$-$\mathcal{B}(\mathrm{L}(\mathbb{K}^m))$-measurable.

(a) f is called *M-integrable* if there exists a sequence (f_n) of $\mathcal{S}$-simple functions such that the following conditions hold:
  (I_1) $f_n \to f$ pointwise $\lambda_M/\lambda_{\Xi(M)}$-a.e. for $\mathbb{K} = \mathbb{R}/\mathbb{C}$.
  (I_2) The sequence $I(f_n\mathbb{1}_A)$ converges in probability for every $A \in \sigma(\mathcal{S})$ and we refer to this limit as $I_M(f\mathbb{1}_A)$ or any synonymous notation from (4.1), respectively.

(b) Consider $\mathbb{K} = \mathbb{C}$. If we relax (I_2) in such a way that we merely want either the sequences $\mathrm{Re}\, I(f_n\mathbb{1}_A)$ or the sequences $\mathrm{Im}\, I(f_n\mathbb{1}_A)$ to converge for every $A \in \sigma(\mathcal{S})$, then f is called *partially M-integrable (in the real/imaginary sense).*

Finally, we define

$$\mathcal{I}_{(p)}(M) := \{f : (S, \sigma(\mathcal{S})) \to (\mathrm{L}(\mathbb{K}^m), \mathcal{B}(\mathrm{L}(\mathbb{K}^m))) \mid f \text{ is (partially) } M\text{-integrable}\}.$$

Remark 4.2. (i) The previous definition coincides with (4.1) for simple f, whereas the notation in (I_2) will be justified by Theorem 4.4 (a).

(ii) If the imaginary parts of f and M vanish, we get back the case $\mathbb{K} = \mathbb{R}$.

(iii) The two types of partial integrability differ only in the consideration of f and $-\mathrm{i}f$. Hence we restrict to partial integrability in the real sense and write $\mathrm{Re}\, I_M(f\mathbb{1}_A)$ for the corresponding limit in (b), even if $I_M(f\mathbb{1}_A)$ may not exist in accordance to (a). However, we have $\mathcal{I}(M) \subset \mathcal{I}_{(p)}(M)$, generally with non-equality.

Now we state some useful properties, starting with the linearity which illuminates the notation *(stochastic) integral.* Throughout and for accuracy, we should identify random vectors that are identical a.s. Also notice that * denotes the adjoint operator in the Hermitian sense.

Proposition 4.3. *Let M be as before. Then we have:*

(a) *$\mathcal{I}(M)$ is a $\mathbb{K}$-vector space and the mapping $\mathcal{I}(M) \ni f \mapsto I_M(f)$ is linear a.s.*

(b) *$f \in \mathcal{I}(M)$ implies that for every $Q \in L(\mathbb{K}^m)$ the function $Q \cdot f$, defined by $(Q \cdot f)(s) = Qf(s)$, also belongs to $\mathcal{I}(M)$ with $I_M(Q \cdot f) = QI_M(f)$ a.s.*

Both statements hold accordingly for $\mathcal{I}_p(M)$ with $\mathbb{K} = \mathbb{R}$.

Proof. The linearity in (a) is obvious for simple functions when considering a common partition $A_1, \ldots, A_n \in \mathcal{S}$. For general $f, g \in \mathcal{I}(M)$ (with $\mathcal{S}$-simple approximating sequences (f_n) and (g_n)) this property can be extended, since $h_n := \alpha f_n + \alpha_2 g_n$ approximates $h := \alpha_1 f + \alpha_2 g$ properly for any $\alpha_1, \alpha_2 \in \mathbb{K}$. Merely note in the case of $\mathbb{K} = \mathbb{C}$ that, for any $A \in \sigma(\mathcal{S})$, we can write

$$\operatorname{Re} I_M(h_n \mathbb{1}_A) = x_1 \operatorname{Re}(f_n \mathbb{1}_A) - y_1 \operatorname{Im}(f_n \mathbb{1}_A) + x_2 \operatorname{Re}(g_n \mathbb{1}_A) - y_2 \operatorname{Im}(g_n \mathbb{1}_A),$$

if $\alpha_i = x_i + \mathrm{i} y_i$; similarly for the imaginary parts. In particular we get $h \in \mathcal{I}(M)$ by additivity of the stochastic limit which implies that $\mathcal{I}(M)$ is a vector space. Part (b) and the additional statement for $\mathcal{I}_p(M)$ can be proven quite similarly. □

For the time being we consider the case $\mathbb{K} = \mathbb{R}$. Recall from (3.2) and (3.9) the definition of λ_M and K_M, respectively.

Theorem 4.4. *Let M be as before.*

(a) *If $f \in \mathcal{I}(M)$, then $I_M(f\mathbb{1}_A)$ is i.d. for any $A \in \sigma(\mathcal{S})$ and its log-characteristic function is given by*

$$\mathbb{R}^m \ni t \mapsto \int_A K_M(f(s)^* t, s)\, \lambda_M(ds). \tag{4.2}$$

Particularly the integral in (4.2) exists and $I_M(f\mathbb{1}_A)$ is well-defined a.s.

(b) *If $f_1, \ldots, f_n \in \mathcal{I}(M)$, then we have for any $t_1, \ldots, t_n \in \mathbb{R}^m$:*

$$\mathbb{E}\left(e^{\mathrm{i} \sum_{j=1}^n \langle I(f_j), t_j \rangle}\right) = \exp\left(\int_S K_M\left(\sum_{j=1}^n f_j(s)^* t_j, s\right) \lambda_M(ds)\right).$$

(c) *For $f, f_1, f_2, \cdots \in \mathcal{I}(M)$ we have that $I_M(f_n) \to I_M(f)$ in probability is equivalent to*

$$\int_{\mathbb{R}^m} K_M((f_n(s) - f(s))^* t, s)\, \lambda_M(ds) \to 0, \quad t \in \mathbb{R}^m.$$

(d) *Let* $f_1, f_2 \in \mathcal{I}(M)$ *such that* $\|f_1(s)\| \cdot \|f_2(s)\| = 0$ *holds* λ_M*-a.e. Then* $I_M(f_1)$ *and* $I_M(f_2)$ *are independent.*

Proof. For simple f, one checks that $I_M(f \mathbb{1}_A)$ is i.d. (see [16, Proposition 3.1.21]) for every $A \in \sigma(\mathcal{S})$, while $K_M(0, \cdot) = 0$ and (3.8) yield that its characteristic function is given by (4.2). Note that $t \mapsto K_M(t, s)$ is the log-characteristic function of the distribution with triplet $[\alpha_M(s), \beta_M(s), \rho_M(s)]$, i.e., is continuous for every $s \in S$. On the one hand, this merely shows that the integral function in (4.2) is really *the* log-characteristic function of $I_M(f)$. On the other hand, it allows us to perform a simple multivariate extension of [17, Proposition 2.6], which states that (4.2) and the previous implication concerning the log-characteristic function also hold for general $f \in \mathcal{I}(M)$, namely the limit in (I_2). This limit preserves the infinite divisibility, and since the right-hand side in (4.2) does not depend on the choice of approximating functions (f_n), we see that $I_M(f \mathbb{1}_A)$ is uniquely determined a.s. after consideration of $(f_n - f_n')$, provided that (f_n') also approximates f properly. This immediately yields (a). The proof of (b) will be covered by the one in Theorem 4.12 (b), while part (c) is a direct conclusion of (a), the linearity and [16, Lemma 3.1.10]. Finally, for (d) we show that $\|f_1(s)\| \cdot \|f_2(s)\| = 0$ expect a potential λ_M-null set implies the independence of $I_M(f_1)$ and $I_M(f_2)$. Define $A_i := \{s : f_i(s) \neq 0\}$ $(i = 1, 2)$ and observe that $M(A) = 0$ a.s. for every $A \subset (A_1 \cap A_2)$ by assumption and the use of Theorem 3.2 (ii). Now if $(f_{n,i})$ is an approximating sequence of simple functions for f_i, we see that this also applies to $f_{n,i}\mathbb{1}_{A_i}$ and that $I_M(f_{n,i}\mathbb{1}_{A_i}) = I_M(f_{n,i}\mathbb{1}_{A_i \setminus (A_1 \cap A_2)})$ a.s. In view of (RM_1), this gives the assertion. □

In the following, we are going to characterize the class $\mathcal{I}(M)$ for a given ISRM M in terms of its control measure λ_M and the related function K_M. Also recall the definition of α_M, β_M and ρ_M in Theorem 3.4, as well as in Theorem 3.5. Then we define $U_M : \mathrm{L}(\mathbb{R}^m) \times S \to \mathbb{R}^m$ by

$$(R, s) \mapsto R\,\alpha_M(s) + \int_{\mathbb{R}^m} \left(\frac{R\,x}{1 + \|R\,x\|^2} - \frac{R\,x}{1 + \|x\|^2} \right) ,\rho_M(s, dx)$$

as well as $V_M : \mathrm{L}(\mathbb{R}^m) \times S \to \mathbb{R}_+$ by

$$(R, s) \mapsto \int_{\mathbb{R}^m} \min\{1, \|R\,x\|^2\}\, \rho_M(s, dx).$$

Recall that these functions are multivariate extensions of those in [17] and a simple calculation shows that

$$\left\| \frac{R\,x}{1 + \|R\,x\|^2} - \frac{R\,x}{1 + \|x\|^2} \right\| \le \max\{2, \|R\| + \|R\|^3\}\, \min\{1, \|x\|^2\} \tag{4.3}$$

holds for all $R \in \mathrm{L}(\mathbb{R}^m)$ and $x \in \mathbb{R}^m$. Similarly and with the help of the Cauchy–Schwarz inequality, we see that

$$\left| \frac{\langle t, y \rangle}{1 + \|y\|^2} - \sin\langle t, y \rangle \right| \leq (1 + \|t\| + \|t\|^2) \min\{1, \|y\|^2\}, \quad t, y \in \mathbb{R}^m. \tag{4.4}$$

Observe that, in view of (4.3), U_M exists. At this point, we generally note that (deterministic) integrals w.r.t. vector-valued or matrix-valued integrals are meant component-wise (compare Theorem 3.5). The following proposition is the first step in the promised characterization of $\mathcal{I}(M)$ and also provides the Lévy–Khintchine triplet of the i.d. random vector $I_M(f)$. But in contrast to the univariate case considered in [17], in our situation the arguments are more involved.

Proposition 4.5. *Consider $f \in \mathcal{I}(M)$. Then the following integrals exist:*

$$\gamma_f := \int_S U_M(f(s), s)\, \lambda_M(ds), \qquad Q_f := \int_S f(s)\beta_m(s) f(s)^* \, \lambda_M(ds),$$

and

$$\phi_f(A) := (\lambda_M \odot \rho_M)(\{(s, x) \in S \times \mathbb{R}^m : f(s)x \in A \setminus \{0\}\}), \quad A \in \mathcal{B}(\mathbb{R}^m)$$

defines a Lévy measure. Moreover, we have $I_M(f) \sim [\gamma_f, Q_f, \phi_f]$.

Proof. The given assumption and Theorem 4.4 (a) ensure the existence of

$$\int_S K_M(f(s)^* t, s)\, \lambda_M(ds) \tag{4.5}$$

for every $t \in \mathbb{R}^m$ as well as the continuity of

$$\mathbb{R}^m \ni t \mapsto \int_S \operatorname{Re} K_M(f(s)^* t, s)\, \lambda_M(ds). \tag{4.6}$$

Let us emphasize that both statements will suffice to perform the present proof. First, Theorem 3.5 (b) permits the following decomposition for every $t \in \mathbb{R}^m$, and the use of (3.5) combined with the definition of ϕ_f yields

$$\int_S \operatorname{Re} K_M(f(s)^*t,s)\,\lambda_M(ds)$$
$$=-\int_S \frac{1}{2}\langle\beta_M(s)f(s)^*t, f(s)^*t\rangle\lambda_M(ds) - \int_S\int_{\mathbb{R}^m}(1-\cos\langle f(s)^*t,x\rangle)\rho_M(s,dx)\,\lambda_M(ds)$$
$$=-\int_S \frac{1}{2}\langle f(s)\beta_M(s)f(s)^*t,t\rangle\,\lambda_M(ds) - \int_{\mathbb{R}^m}(1-\cos\langle t,x\rangle)\phi_f(dx).$$

Now let $C(s) := f(s)\beta_M(s)f(s)^*$ with $C(s) = (C^{i,j}(s))_{i,j=1,\dots,m}$, and first consider $t = e_i$ to check the λ_M-integrability of the diagonal components $C^{i,i}$. Repeat this argument for $t = e_i + e_j$ for the λ_M-integrability of $C^{i,j} + C^{j,i}$, which finally gives the existence of Q_f due to the symmetry in Theorem 3.5 (b). Here we should also note that Q_f is symmetric and positive semi-definite since β_M is (at least λ_M-a.e.). In particular, we know that

$$\int_{\mathbb{R}^m}(1-\cos\langle t,x\rangle)\phi_f(dx) = -\frac{1}{2}\langle Q_f t,t\rangle - \int_S \operatorname{Re} K(f(s)^*t,s)\lambda_M(ds), \quad t\in\mathbb{R}^m. \tag{4.7}$$

Hence the left-hand side is continuous in t according to (4.6), i.e., ϕ_f is a Lévy measure if we include $\phi_f(\{0\}) = 0$ and perform similar steps as done in the proof of [18, Theorem 3.3.10]. Then we can argue as above that this implies the λ_M-integrability of $V_M(f(\cdot),\cdot)$. For the existence of γ_f, it finally suffices to show that $\langle t, U_M(f(\cdot),\cdot)\rangle$ is λ_M-integrable for every $t\in\mathbb{R}^m$. Observe that we have the decomposition

$$\langle t, U(f(s),s)\rangle = \operatorname{Im} K_M(f(s)^*t,s) + \int_{\mathbb{R}^m}\left(\frac{\langle t, f(s)x\rangle}{1+\|f(s)x\|^2} - \sin\langle t, f(s)x\rangle\right)\rho_M(s,dx)$$

for every $s\in S, t\in\mathbb{R}^m$ in view of (4.3) and (4.4). Furthermore, (4.4) implies that

$$\int_S |\langle t, U(f(s),s)\rangle|\,\lambda_M(ds) \le \int_S |K(f(s)^*t,s)|\,\lambda_M(ds) + C(t)\int_S V(f(s),s)\,\lambda_M(ds)$$

with $C(t) := 1+\|t\|+\|t\|^2$. In view of what we have shown before, this gives the existence of γ_f. Now it is easy to see that $I_M(f)\sim[\gamma_f, Q_f, \phi_f]$. □

Lemma 4.6. *Let $f : S \to L(\mathbb{R}^m)$ be measurable. Then the inequality*

$$\|U(f(s)\mathbb{1}_A(s), s)\| \leq \|U(f(s), s)\|\mathbb{1}_A(s) + 2V(f(s), s)$$

holds for every $A \in \sigma(\mathcal{S})$ and $s \in S$.

Proof. With a little abuse of notation apply (4.3) to $\tilde{R} := \mathbb{1}_A(s)I_m$ and $\tilde{x} := f(s)x$. Then some simple calculations provide the desired conclusion. □

The previous Lemma can be regarded as a multivariate alternative for [17, Lemma 2.8], whereas the following one uses some ideas from the proof of [18, Theorem 3.2.2].

Lemma 4.7. *For $f \in \mathcal{I}(M)$, let $(f_n)_{n\in\mathbb{N}}$ be a corresponding sequence of simple functions. Then for any $\varepsilon_1, \varepsilon_2 > 0$, there exists an $\zeta = \zeta(\varepsilon_1, \varepsilon_2)$ such that*

$$\forall n \geq \zeta \ \ \forall A \in \sigma(\mathcal{S}) \qquad \mathbb{P}(\|I(f\mathbb{1}_A) - I(f_n\mathbb{1}_A)\| \geq \varepsilon_1) \leq \varepsilon_2.$$

Proof. Let $g_n := f - f_n$. Then by linearity, Theorem 4.5 and Theorem 2.1, we have that

$$\gamma_{g_n}(A) := \int_A U(g_n(s), s)\, \lambda_M(ds) \to 0, \quad A \in \sigma(\mathcal{S}). \tag{4.8}$$

This convergence is even uniform in A. To prove this, we define the measure

$$\lambda_M^*(E) := \sum_{l=1}^{\infty} 2^{-l} \frac{\lambda_M(E \cap S_l)}{1 + \lambda_M(S_l)}, \quad E \in \sigma(\mathcal{S}),$$

where $(S_l) \subset \mathcal{S}$ is a disjoint exhaustion of S again. Then $A \mapsto \gamma_{g_n}(A)$ defines a vector measure with $\gamma_{g_n} \ll \lambda_M \ll \lambda_M^*$, i.e., the components $\gamma_{g_n}^{(k)}$ are signed measures with $\gamma_{g_n}^{(k)} \ll \lambda_M^*$ for every $n \in \mathbb{N}$ and $k = 1, \ldots, m$. Thus we can apply the Hahn–Saks–Vitali Theorem (see [19, Proposition C.3]): For every $\varepsilon > 0$, there are $\delta_1, \ldots, \delta_m > 0$ fulfilling the implications

$$\forall A \in \sigma(\mathcal{S}) \qquad \left(\lambda_M^*(A) \leq \delta_k \Longrightarrow \sup_{n\in\mathbb{N}} |\gamma_{g_n}^{(k)}(A)| \leq \varepsilon\right)$$

for $k = 1, \ldots, m$. Hence there exists a $C > 0$ such that the following assertion holds likewise with $\delta := \min\{\delta_1, \ldots, \delta_m\}$:

$$\forall A \in \sigma(\mathcal{S}) \qquad \left(\lambda_M^*(A) \leq \delta \Longrightarrow \sup_{n\in\mathbb{N}} \|\gamma_{g_n}(A)\| \leq C\,\varepsilon\right). \tag{4.9}$$

Using dominated convergence, we have that $U_M(\cdot, s)$ is continuous for each $s \in S$ and therefore that $U_M(g_n(s), s) \to 0$ λ_M-a.e. Proceeding with Egorov's Theorem (note that λ_M^* is finite), there exists a measurable set D' such that the previous convergence is uniformly on D' with $\lambda_M^*(S \setminus D') \le \delta/2$. Finally, we use (S_l) and Theorem 2.2 to verify that the same is true on an appropriate set D belonging to $\mathcal{S}$ with $\lambda_M^*(S \setminus D) \le \delta$. Especially, we have $\lambda_M(D) < \infty$ as well as the following estimation for every $A \in \sigma(\mathcal{S})$ and $n \in \mathbb{N}$ due to (4.9):

$$\begin{aligned}\|\gamma_{g_n}(A)\| &\le C\,\varepsilon + \sup_{s \in A \cap D} \|U(g_n(s), s)\| \cdot \lambda_M(A \cap D)\\ &\le C\,\varepsilon + \sup_{s \in D} \|U(g_n(s), s)\| \cdot \lambda_M(D),\end{aligned}$$

which obviously means that (4.8) holds uniformly. Moreover, for $\mathbb{R}^m$-valued random vectors X and Y, we can define $d(X, Y) := \int \min\{1, \|X - Y\|\}\, d\mathbb{P}$ and know that d is a metric whose induced convergence is equivalent to that in probability (when identifying random vectors which are equal a.s., see the proof of [10, Theorem 6.7]). We now show for $X_n(A) := I_M(g_n \mathbb{1}_A) - \gamma_{g_n}(A)$ that

$$c_n := \sup_{A \in \sigma(\mathcal{S})} d(X_n(A), 0) \in [0, 2], \quad n \in \mathbb{N}$$

converges to zero. For this purpose, we choose $A_n \in \sigma(\mathcal{S})$ such that $c_n \le d(X_n(A_n), 0) + 1/n$. At the same time, we have

$$I_M(g_n) = X_n(A_n) + I_M(g_n \mathbb{1}_{A_n^c}) + \gamma_{g_n}(A_n) =: X_n(A_n) + Y_n \to 0$$

in probability (see above). This also implies $X_n(A_n) \to 0$ by Theorem 2.1 and monotonicity. For instance, and provided that $X_n(A_n) \sim [0, Q_n, \phi_n]$ as well as $Y_n \sim [\tilde{\gamma}_n, \tilde{Q}_n, \tilde{\phi}_n]$, we obtain

$$0 \le \langle Q_n t, t\rangle \le \langle Q_n t, t\rangle + \langle \tilde{Q}_n t, t\rangle = \langle (Q_n + \tilde{Q}_n)t, t\rangle \to 0, \quad t \in \mathbb{R}^m,$$

since $Q_n + \tilde{Q}_n$ equals the Gaussian component of $I_M(g_n)$ by independence of $X_n(A_n)$ and Y_n (see Theorem 4.4 (d) and Proposition 3.1.21 in [16]). Hence $c_n \to 0$. Furthermore, we see that $d(I_M(g_n \mathbb{1}_A), 0) \le d(X_n(A), 0) + \|\gamma_{g_n}(A)\|$ holds for every $A \in \sigma(\mathcal{S})$, and $n \in \mathbb{N}$ due to the fact that $[0, \infty) \ni x \mapsto \min\{1, x\}$ is subadditive. By what we have seen before, this shows that $d(I_M(g_n \mathbb{1}_A), 0)$ converges to 0 uniformly in $A \in \sigma(\mathcal{S})$. Finally, let $0 < \varepsilon_1 \le 1$ arbitrary ($\varepsilon_1 > 1$ obvious), then we obtain the assertion by reading this convergence together with

$$\mathbb{P}(\|I(f\mathbb{1}_A) - I(f_n \mathbb{1}_A)\| \ge \varepsilon_1) = \mathbb{P}(\|I(g_n \mathbb{1}_A)\| \ge \varepsilon_1) \le \varepsilon_1^{-1} \sup_{A \in \sigma(\mathcal{S})} d(I(g_n \mathbb{1}_A), 0),$$

where we used that $\mathbb{P}(\|X\| \ge \varepsilon_1) \le d(X, 0)/\varepsilon_1$ (for any random vector X). □

Theorem 4.8. *Let $f : S \to L(\mathbb{R}^m)$ be $\sigma(\mathcal{S})$-$\mathcal{B}(L(\mathbb{R}^m))$-measurable. Then the following statements are equivalent:*

(I) $f \in \mathcal{I}(M)$.

(II) *The integrals γ_f as well as Q_f exist and ϕ_f is a Lévy measure.*

(III) *The integral in (4.5) exists for every $t \in \mathbb{R}^m$ and the mapping in (4.7) is continuous.*

Proof. In view of what we pointed out before, especially in the proof of Theorem 4.5, it obviously suffices to show that (II) implies (I). Throughout the proof, let $(S'_n) \subset \mathcal{S}$ be an increasing sequence whose union is S and write $f(s) = (f^{i,j}(s))_{i,j=1,\dots,m}$ for every $s \in S$.

First step: We define $S_n := S'_n \cap \{s : |f^{i,j}(s)| < n \text{ for all } 1 \le i,j \le m\} \in \mathcal{S}$ with $S_n \uparrow S$, and thereafter the sequence (f_n) of $\mathcal{S}$-simple functions (see Theorem 2.2) via

$$f_n^{i,j}(s) := \mathbb{1}_{S_n}(s) \cdot \begin{cases} \frac{l}{n}, & \text{if } \frac{l}{n} \le f^{i,j}(s) < \frac{l+1}{n} \text{ for } l = 0,\dots,n^2-1, \\ -\frac{l}{n}, & \text{if } -\frac{l+1}{n} < f^{i,j}(s) \le -\frac{l}{n} \text{ for } l = 0,\dots,n^2-1, \\ 0, & \text{if } |f^{i,j}(s)| \ge n. \end{cases}$$

Hence we see that $f_n \to f$ pointwise with $|f_n^{i,j}(s)| \le |f^{i,j}(s)|$ for every $s \in S$, whereas $|f_n^{i,j}(s) - f^{i,j}(s)| \le 1/n$ merely holds for $s \in S_n$. Moreover, there exist $C_1, C_2 > 0$ such that $\|f_n(s)\| \le C_1 \|f(s)\|$ for all $s \in S$, and $\|f_n(s) - f(s)\|$ is bounded by C_2/n as long as $s \in S_n$. Particularly, we obtain for all $j \ge n$ and $s \in S$:

$$\|f_n(s) - f_j(s)\| \le C_1 \|f(s)\| \mathbb{1}_{S_j \setminus S_n}(s) + 2C_2 \mathbb{1}_{S_n}(s). \tag{4.10}$$

Second step: Next, we show that $g^{(k)} := f\mathbb{1}_{S_k} \in \mathcal{I}(M)$ for $k \in \mathbb{N}$ arbitrary by means of the $\mathcal{S}$-simple sequence $(g_n^{(k)})_n$ which is defined via $g_n^{(k)} := f_n \mathbb{1}_{S_k}$. Obviously, we have $g_n^{(k)} \to g^{(k)}$ pointwise, and with $C := 2C_1$ one confirms that

$$\|g_n^{(k)}(s) - g_j^{(k)}(s)\| \le C\, \mathbb{1}_{S_k}(s) \tag{4.11}$$

is true for all $j \ge n \ge k$ and $s \in S$ due to (4.10). In view of Theorem 4.1, it suffices to show that $(I_M(g_n^{(k)} \mathbb{1}_A))_n$ converges in probability. For this purpose, we now fix an arbitrary sequence $n_1 < j_1 < n_2 < \cdots$ of increasing natural numbers and prove that the convergences

$$\int_S U_M\left((g_{n_l}^{(k)}(s) - g_{j_l}^{(k)}(s))\mathbb{1}_A(s), s\right) \lambda_M(ds) \to 0, \tag{4.12}$$

$$\int_A (g_{n_l}^{(k)}(s) - g_{j_l}^{(k)}(s))\, \beta_M(s)\, (g_{n_l}^{(k)}(s) - g_{j_l}^{(k)}(s))^* \,\lambda_M(ds) \to 0, \tag{4.13}$$

$$\int_S V_M\left((g_{n_l}^{(k)}(s) - g_{j_l}^{(k)}(s))\mathbb{1}_A(s), s\right) \lambda_M(ds) \to 0 \tag{4.14}$$

hold for $l \to \infty$, respectively. By continuity of $U_M(\cdot, s)$ and $V_M(\cdot, s)$, it is first clear that the integrands in (4.12)–(4.14) converge to zero for every $s \in S$. Then the assertion follows by dominated convergence in each case: For (4.13), use (4.11) and observe that $\|\beta_M(s)\|\mathbb{1}_{A\cap S_k}(s)$ is λ_M-integrable. On the other hand, we see that the integrand in (4.14) is dominated by $V_M(C\mathbb{1}_{A\cap S_k}(s)I_m, s)$ (here and below at least for l sufficiently large), whereas (3.5) and Theorem 3.4 provide the following steps that have been performed similarly before:

$$\begin{aligned}
&\int_S V_M(C\mathbb{1}_{A\cap S_k}(s)I_m, s)\,\lambda_M(ds) \\
&\le (1 + C^2) \int_{S\times\mathbb{R}^m} \min\{1, \|x\|^2\}\mathbb{1}_{A\cap S_k}(s)\,(\lambda_M \odot \rho_M)(ds, dx) \\
&= (1 + C^2) \int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\,\phi_{A\cap S_k}(dx) < \infty.
\end{aligned}$$

Using (4.3), we can argue likewise that the integrand in (4.12) is dominated by

$$s \mapsto \left(C\|\alpha_m(s)\| + C' \int_{\mathbb{R}^m} \min\{1, \|x\|^2\}\,\rho_M(s, dx) \right) \mathbb{1}_{A\cap S_k}(s)$$

with $C' := \max\{2, C + C^3\}$, and that the mapping we mentioned recently is λ_M-integrable. Finally, suppose that $(I_M(g_n^{(k)}\mathbb{1}_A))_n$ would not converge in probability, then it would not be Cauchy either (in view and in the sense of [10, Corollary 6.15]). Hence we obtain a sequence $n_1 < j_1 < n_2 < \cdots$ as above such that $I_M(g_{n_l}^{(k)}\mathbb{1}_A) - I_M(g_{j_l}^{(k)}\mathbb{1}_A) = I_M((g_{n_l}^{(k)} - g_{n_l}^{(k)})\mathbb{1}_A)$ neither converges in probability to zero nor in distribution. By Theorem 4.5, and in view of Theorem 2.1 together with (4.12)–(4.14), this gives the contradiction.

Third step: For $A \in \sigma(\mathcal{S})$ arbitrary, we further conclude that there is an increasing sequence (j_l^A) of natural numbers which fulfils the following implication for every $l \in \mathbb{N}$:

$$k_1, k_2 \ge j_l^A \Longrightarrow \mathbb{P}\left(\|I(g^{(k_1)}\mathbb{1}_A) - I(g^{(k_2)}\mathbb{1}_A)\| \ge 1/l\right) \le 1/l. \tag{4.15}$$

Similarly to the previous step, this is again equivalent to the following assertions:

$$\int_S U_M\left((g^{(l_k)}(s)-g^{(n_k)}(s))\mathbb{1}_A(s),s\right)\lambda_M(ds)\to 0, \tag{4.16}$$

$$\int_A \left(g^{(l_k)}(s)-g^{(n_k)}(s)\right)\,\beta_M(s)\,\left(g^{(l_k)}(s)-g^{(n_k)}(s)\right)^*\lambda_M(ds)\to 0, \tag{4.17}$$

$$\int_S V_M\left((g^{(l_k)}(s)-g^{(n_k)}(s))\mathbb{1}_A(s),s\right)\lambda_M(ds)\to 0 \tag{4.18}$$

for $k\to\infty$, respectively, and with any fixed sequence $n_1<l_1<n_2<\cdots$ as before. In virtue of $(S_{l_k}\setminus S_{n_k})\subset(S\setminus S_k)\downarrow\emptyset$, we only have to find λ_M-integrable functions again which dominate the previous integrands. Concerning (4.17) and (4.18), this is obvious, as we assume the existence of Q_f and the λ_M-integrability of $V_M(f(\cdot),\cdot)$. For (4.16) we use Theorem 4.6, and then again, the assumption on $V_M(f(\cdot),\cdot)$ as well as the one on $U_M(f(\cdot),\cdot)$.

Fourth step: Inductively, Theorem 4.7 provides a sequence (ζ_k) of increasing natural numbers such that

$$\forall A\in\sigma(\mathcal{S})\ \ \forall k\in\mathbb{N}\qquad \mathbb{P}\left(\|I(g^{(k)}\mathbb{1}_A)-I(g^{(k)}_{\zeta_k}\mathbb{1}_A)\|\ge 1/k\right)\le 1/k. \tag{4.19}$$

Then we replace the sequence (f_k) from the first step by $f_k:=g^{(k)}_{\zeta_k}$ and realize that $f_k\to f$ pointwise again. Let $A\in\sigma(\mathcal{S})$ as well as $\varepsilon_1,\varepsilon_2>0$ be arbitrary. Then the following calculation yields that $(I_M(f_k\mathbb{1}_A))$ is a Cauchy sequence w.r.t. convergence in probability. In fact, we choose a $K_0\in\mathbb{N}$ such that $K_0^{-1}\le\min\{\varepsilon_1,\varepsilon_2\}/3$ and set $K:=\max\{K_0,j^A_{K_0}\}$. Then for any $k_1,k_2\ge K$, we get using (4.15) and (4.19) that

$$\begin{aligned}
&\mathbb{P}\left(\|I(f_{k_1}\mathbb{1}_A)-I(f_{k_2}\mathbb{1}_A)\|\ge\varepsilon_1\right)\\
&\le\mathbb{P}\left(\|I(g^{(k_1)}_{\zeta_{k_1}}\mathbb{1}_A)-I(g^{(k_1)}\mathbb{1}_A)\|\ge K_0^{-1}\right)+\mathbb{P}\left(\|I(g^{(k_1)}\mathbb{1}_A)-I(g^{(k_2)}\mathbb{1}_A)\|\ge K_0^{-1}\right)\\
&\qquad+\mathbb{P}\left(\|I(g^{(k_2)}\mathbb{1}_A)-I(g^{(k_2)}_{\zeta_{k_2}}\mathbb{1}_A)\|\ge K_0^{-1}\right)\\
&\le\mathbb{P}\left(\|I(g^{(k_1)}_{\zeta_{k_1}}\mathbb{1}_A)-I(g^{(k_1)}\mathbb{1}_A)\|\ge k_1^{-1}\right)+\mathbb{P}\left(\|I(g^{(k_1)}\mathbb{1}_A)-I(g^{(k_2)}\mathbb{1}_A)\|\ge K_0^{-1}\right)\\
&\qquad+\mathbb{P}\left(\|I(g^{(k_2)}\mathbb{1}_A)-I(g^{(k_2)}_{\zeta_{k_2}}\mathbb{1}_A)\|\ge k_2^{-1}\right)\\
&\le k_1^{-1}+K_0^{-1}+k_2^{-1}\le\varepsilon_2,
\end{aligned}$$

and the proof is complete. □

With $f_j = \mathbb{1}_{A_j} I_m$ and the following result, which extends the conclusion in [9], we see that the infinite divisibility of an ISRM implicitly extends to its finite dimensional distributions.

Corollary 4.9. *For $f_1, \ldots, f_n \in \mathcal{I}(M)$, the random vector $(I_M(f_1), \ldots, I_M(f_n))$ has an i.d. distribution.*

Proof. Denote the characteristic function of $(I_M(f_1), \ldots, I_M(f_n))$ by φ and fix some arbitrary $l \in \mathbb{N}$. Then it suffices to show that the function $\varphi^{1/l}$, which we should not understand in any logarithmic sense (see Theorem 4.4 (b) instead), also describes a characteristic function on $\mathbb{R}^{n \cdot m}$. Thus if $M(A) \sim [\gamma_A, Q_A, \phi_A]$, we see that M' with $M'(A) \sim [l^{-1}\gamma_A, l^{-1}Q_A, l^{-1}\phi_A]$ (for every $A \in \mathcal{S}$) is also a valid ISRM according to Theorem 3.1. Then Theorem 4.8 leads to $\mathcal{I}(M) = \mathcal{I}(M')$ such that $(I_{M'}(f_1), \ldots, I_{M'}(f_n))$ has the characteristic function $\varphi^{1/l}$. □

Remark 4.10. Sometimes it might be more natural to consider vector-valued integrands $f_0 : S \to \mathbb{R}^m$ and hence to obtain one-dimensional stochastic integrals. Actually, this would require to modify (4.1) and to use inner products of the form $\langle R_j, M(A \cap A_j)\rangle$ accordingly.

Our approach includes this idea, if we use $f : S \to \mathrm{L}(\mathbb{R}^m)$, where the first row of $f(s)$ equals $f_0(s)^T$ for any $s \in S$ and all other rows are zero. Finally, a projection on the first coordinate of $I(f)$ gives the desired result.

For the rest of this paper, we briefly want to study the close relation between $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, which can be clarified by introducing the *(partially) associated mapping of* f, namely $\tilde{f}, \tilde{f}_p : S \to \mathrm{L}(\mathbb{R}^{2m})$ by

$$\tilde{f}(s) := \begin{pmatrix} \operatorname{Re} f(s) & -\operatorname{Im} f(s) \\ \operatorname{Im} f(s) & \operatorname{Re} f(s) \end{pmatrix} \quad \text{and} \quad \tilde{f}_p(s) := \begin{pmatrix} \operatorname{Re} f(s) & -\operatorname{Im} f(s) \\ 0 & 0 \end{pmatrix},$$

where $f : S \to \mathrm{L}(\mathbb{C}^m)$ is arbitrary. More precisely and with regard to Theorem 3.8, we get the following observation in which we assume M to be a $\mathbb{C}^m$-valued i.d. ISRM.

Proposition 4.11. *For $f : S \to L(\mathbb{C}^m)$ we have that f is M-integrable if and only if $\tilde{f}$ is $\Xi(M)$-integrable, and in this case $\Xi(I_M(f\mathbb{1}_A)) = I_{\Xi(M)}(\tilde{f}\mathbb{1}_A)$ a.s. for every $A \in \sigma(\mathcal{S})$. Similarly, f is partially M-integrable if and only if $\tilde{f}_p$ is $\Xi(M)$-integrable, and in this case $\Xi(\operatorname{Re} I_M(f\mathbb{1}_A)) = I_{\Xi(M)}(\tilde{f}_p\mathbb{1}_A)$ a.s. for every $A \in \sigma(\mathcal{S})$.*

Proof. This follows by a simple calculation using (4.1) and passing through the limit. □

On the one hand, this immediately allows us to apply Theorem 4.8 and Theorem 4.9 accordingly. On the other hand, it shows that the complex-valued perspective mostly simplifies the description of several problems that actually have a real origin. We derive the following.

Corollary 4.12. *Let M be as before, particularly $\mathbb{C}^m$-valued.*

(a) *If $f \in \mathcal{I}(M)$, then $I_M(f\mathbb{1}_A)$ is well-defined and i.d. for every $A \in \sigma(\mathcal{S})$, whereas the log-characteristic function of $\Xi(I_M(f\mathbb{1}_A))$ is given by*

$$\mathbb{R}^{2m} \ni t \mapsto \int_A K_{\Xi(M)}(\tilde{f}(s)^* t, s)\, \lambda_{\Xi(M)}(ds) = \int_A K_{\Xi(M)}(\Xi(f(s)^* z), s)\, \lambda_{\Xi(M)}(ds)$$

with $z := \Xi^{-1}(t) \in \mathbb{C}^m$.

(b) *If $f_1, \dots, f_n \in \mathcal{I}(M)$, then we have for any $t_1, \dots, t_n \in \mathbb{R}^{2m}$,*

$$\mathbb{E}\left(e^{\mathrm{i}\sum_{j=1}^n \langle \Xi(I(f_j)), t_j\rangle}\right) = \exp\left(\int_S K_{\Xi(M)}\left(\sum_{j=1}^n \tilde{f}_j(s)^* t_j, s\right) \lambda_{\Xi(M)}(ds)\right).$$

(c) *For $f, f_1, f_2, \dots \in \mathcal{I}(M)$, we have that $I_M(f_n) \to I_M(f)$ in probability is equivalent to*

$$\int_{\mathbb{R}^m} K_{\Xi(M)}((\tilde{f}_n(s) - \tilde{f}(s))^* t, s)\, \lambda_{\Xi(M)}(ds) \to 0, \quad t \in \mathbb{R}^{2m}.$$

(d) *Let $f_1, f_2 \in \mathcal{I}(M)$ such that $\|\tilde{f}_1(s)\| \cdot \|\tilde{f}_2(s)\| = 0$ holds $\lambda_{\Xi(M)}$-a.e. Then $I_M(f_1)$ and $I_M(f_2)$ are independent.*

Proof. In view of Theorem 4.11, part (a) follows by Theorem 4.4 and the claimed equality can be checked immediately. And since, by linearity, $I_M(f_n) \to I_M(f)$ is equivalent to $\Xi(I_M(f_n - f)) \to 0$ in probability, this gives (c) again. Moreover, Theorem 4.11 says that the assertion in (d) is equivalent to the independence of $I_{\Xi(M)}(\tilde{f}_1)$ and $I_{\Xi(M)}(\tilde{f}_2)$ such that the proof reduces to the case $\mathbb{K} = \mathbb{R}$. Finally, we write $t_j = (t_{j,1}, t_{j,2})$ as well as $t_{j,i} = Q_{j,i} e$ with $e = (1, .., 1) \in \mathbb{R}^m$ and $Q_{j,i} \in \mathrm{L}(\mathbb{R}^m)$ suitable. Then for $R_j := \frac{1}{2}(R_{j,1} + R_{j,2}), Q_j := \frac{1}{2}(R_{j,1} - R_{j,2})$ and $V_j := R_j - \mathrm{i}Q_j \in \mathrm{L}(\mathbb{C}^m)$, we observe, similar to [20, Proposition 6.2.1], that

$$\sum_{j=1}^{n}\langle\Xi(I_M(f_j)),t_j\rangle=\sum_{j=1}^{n}\langle R_{j,1}^*(\operatorname{Re}I_M(f_j))+R_{j,2}^*(\operatorname{Im}I_M(f_j)),e\rangle$$
$$=\sum_{j=1}^{n}\langle R_j^*(\operatorname{Re}I_M(f_j))-Q_j^*(\operatorname{Im}I_M(f_j))+Q_j^*(\operatorname{Re}I_M(f_j))+R_j^*(\operatorname{Im}I_M(f_j)),e\rangle$$
$$=\sum_{j=1}^{n}\langle\operatorname{Re}V_j^*I_M(f_j)+\operatorname{Im}V_j^*I_M(f_j),e\rangle=\left\langle\Xi\left(I_M\left(\sum_{j=1}^{n}V_j^*\cdot f_j\right)\right),\begin{pmatrix}e\\e\end{pmatrix}\right\rangle$$

by both parts of Theorem 4.3. Verify the identity

$$\left(\sum_{j=1}^{n}V_j^*f_j(s)\right)^*(e+\mathrm{i}e)=\sum_{j=1}^{n}f_j(s)^*(t_{j,1}+\mathrm{i}t_{j,2})=\sum_{j=1}^{n}\tilde{f}_j(s)^*t_j,\quad s\in S$$

to see that (b) follows by (a). □

Remark 4.13. We also observe that $\Xi(f(s)^*t_1)$ equals $\tilde{f}_p(s)^*t$ for every $t=(t_1,t_2)\in\mathbb{R}^{2m}$. Then the properties for the partial case (see Theorem 4.1) can be formulated and proved similarly, which is therefore left to the reader. We merely note that the following key relation holds for any $f_1,\dots,f_n\in\mathcal{I}_p(M)$ and $t_1,\dots,t_n\in\mathbb{R}^m$:

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\sum_{j=1}^{n}\langle\operatorname{Re}I(f_j),t_j\rangle}\right)=\exp\left(\int_S K_{\Xi(M)}\left(\Xi\left(\sum_{j=1}^{n}f_j(s)^*t_j\right),s\right)\lambda_{\Xi(M)}(ds)\right).$$

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