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On the weighted sum of consecutive values of an additive representation function

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Abstract. Let $\mathbb{N}$ be the set of nonnegative integers. For any set $A \subset \mathbb{N}$, let $R_A(n)$ denote the number of solutions of the equation $n = a+b$ with $a, b \in A$. Recently, Kiss and Sándor established some relations between $|\lambda_0 R_A(n) + \lambda_1 R_A(n-1) + \cdots + \lambda_d R_A(n-d)|$ and $|\{m : m \le n, \lambda_0\chi_A(m) + \lambda_1\chi_A(m-1) + \cdots + \lambda_d\chi_A(m-d) \ne 0\}|$, where $\chi_A(k) = 1$ if $k \in A$, otherwise $\chi_A(k) = 0$. In this paper, we improve one of the results of Kiss and Sándor to the best possible up to a constant factor.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For any set $A \subset \mathbb{N}$, let $R_A(n)$ denote the number of solutions of the equation $n = a + b$ with $a, b \in A$. Let $\chi_A(k) = 1$ if $k \in A$, otherwise $\chi_A(k) = 0$. Erdős, Sárközy and Sós (see [1], [2], [3]) obtained many properties on the magnitude of $R_A(n)$. Recently, Kiss and Sándor [9] generalized some results of [3]. For $\underline{\lambda} = (\lambda_0, \dots, \lambda_d) \in \mathbb{Z}^{d+1}$, let

$$B(A, \underline{\lambda}, n) = \left|\{m : m \le n, \sum_{i=0}^{d} \lambda_i \chi_A(m-i) \ne 0\}\right|.$$

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Kiss and Sándor [9] obtained several related results, two of which are as follows:

Theorem A ([9, Kiss and Sándor])**.** *Let* $\lambda_0, \dots, \lambda_d$ *be arbitrary integers and* $A \subset \mathbb{N}$*. Then*

$$\limsup_{n\to\infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \geq \frac{|\sum_{i=0}^{d} \lambda_i|}{2(d+1)^2} \limsup_{n\to\infty} \left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} \right)^2 .$$

Theorem B ([9, Kiss and Sándor])**.** *Let* $\sum_{i=0}^{d} \lambda_i > 0$*. Then for every positive integer* N*, there exists a set* $A \subset \mathbb{N}$ *such that*

$$\limsup_{n\to\infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \leq 4 \sum_{i=0}^{d} |\lambda_i| \limsup_{n\to\infty} \left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} \right)^2$$

and

$$\limsup_{n\to\infty} \frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} \geq N.$$

The inequality in Theorem B is trivial if

$$\limsup_{n\to\infty} \frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} = \infty.$$

With a minor change of the proof of [9, Theorem 1], one may obtain the following result.

Theorem C. *Let* λ_i $(0 \leq i \leq d)$ *be arbitrary integers and* $A \subset \mathbb{N}$*. Then*

$$\limsup_{n\to\infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \geq \frac{|\sum_{i=0}^{d} \lambda_i|}{2v^2} \limsup_{n\to\infty} \left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} \right)^2 ,$$

where $v = |\{w : \lambda_w \neq 0, 0 \leq w \leq d\}|$*.*

In this paper, we prove the following results.

Theorem 1. *Let* λ_i $(0 \leq i \leq d)$ *be arbitrary integers. Then for every positive integer* N*, there exists a set* $A \subset \mathbb{N}$ *such that*

$$\limsup_{n\to\infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \leq \left(1 + \frac{1}{N+1} \right) \frac{2\sum_{i=0}^{d} |\lambda_i|}{v^2} \limsup_{n\to\infty} \left(\frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} \right)^2$$

and

$$\limsup_{n\to\infty} \frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} = v(N+1),$$

where $v = |\{w : \lambda_w \neq 0, 0 \leq w \leq d\}|$*.*

We have the following corollary immediately.

Corollary 1. *Let $\lambda_i \neq 0$ $(0 \le i \le d)$. Then for every positive integer N, there exists a set $A \subset \mathbb{N}$ such that*

$$\limsup_{n\to\infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \le \left(1 + \frac{1}{N+1}\right) \frac{2\sum_{i=0}^{d} |\lambda_i|}{(d+1)^2} \limsup_{n\to\infty} \left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}} \right)^2$$

and

$$\limsup_{n\to\infty} \frac{B(A,\underline{\lambda},n)}{\sqrt{n}} = (d+1)(N+1).$$

Comparing Theorem A and Corollary 1, both Theorem A and Corollary 1 are the best possible up to constant factors. Comparing Theorem C and Theorem 1, both Theorem C and Theorem 1 are the best possible up to constant factors.

2. Proof of Theorem 1

Let $p_1, p_2, \dots$ be primes with $p_1 > 16N^2(d+1)^2$ and $p_i > p_{i-1}^6$ $(i = 2, 3, \dots)$ and let

$$M_i = p_i^2 + p_i + 1.$$

By Singer's Theorem (see [11]), for each M_i, there is a Sidon set $S_i \subseteq [1, M_i]$ with $|S_i| = p_i$. Since S_i is a Sidon set, there is at most one pair $s, s' \in S_i$ with $s - s' = a$ for each integer a. Now we remove all pairs $s, s' \in S_i$ with $0 < s - s' \le \sqrt{p_i}$ and all $s \in S_i$ with $s \le \sqrt{p_i}$ or $s \ge M_i - \sqrt{p_i}$. The remaining set is T_i. Then T_i is a Sidon set with $T_i \subseteq (\sqrt{p_i}, M_i - \sqrt{p_i})$ and

$$p_i - 2\sqrt{p_i} - 2 = |S_i| - 2\sqrt{p_i} - 2 \le |T_i| \le |S_i| = p_i$$

such that $t - t' > \sqrt{p_i}$ for any $t, t' \in T_i$ with $t > t'$. Let

$$A = \bigcup_{k=1}^{\infty} \left(T_k \cup (T_k + (d+1)) \cup \cdots \cup (T_k + N(d+1))\right).$$

For $k < l$, if $t_k \in T_k$ and $t_l \in T_l$, then

$$\begin{aligned} t_l - t_k &> \sqrt{p_l} - (M_k - \sqrt{p_k}) > p_{l-1}^3 - (p_k^2 + p_k + 1 - \sqrt{p_k}) \\ &> p_k^3 - p_k^2 - p_k > p_k > 16N^2(d+1)^2. \end{aligned}$$

It follows that

$$(T_k + i(d+1)) \cap (T_l + j(d+1)) = \emptyset$$

for all $k \neq l$ and $0 \le i, j \le N$.

Lemma 1. *For any positive integer m, we have $R_A(m) \le 2(N+1)^2 + 2(N+1)$.*

Proof. It is clear that $R_A(m) = 0$ for $m \le 2\sqrt{p_1}$. For every positive integer $m \ge 2\sqrt{p_1}$, there exists a positive integer s such that $\sqrt{p_s} \le m < \sqrt{p_{s+1}}$. Noting that $T_i \subseteq (\sqrt{p_i}, M_i - \sqrt{p_i})$, we have

$$R_A(m) \le \sum_{k=1}^{s}\sum_{l=1}^{s}\sum_{i=0}^{N}\sum_{j=0}^{N} |\{(t,t') : t + i(d+1) + t' + j(d+1) = m, t \in T_k, t' \in T_l\}|.$$

If $s = 1$, then by T_1 being a Sidon set, we have

$$R_A(m) \le \sum_{i=0}^{N}\sum_{j=0}^{N} |\{(t,t') : t + i(d+1) + t' + j(d+1) = m, t, t' \in T_1\}| \le 2(N+1)^2.$$

If $s \ge 2$, $k, l \le s-1$ and $t \in T_k, t' \in T_l$, then for any $0 \le i, j \le N$, we have

$$\begin{aligned} t + i(d+1) + t' + j(d+1) &\le 2(M_{s-1} - \sqrt{p_{s-1}}) + 2N(d+1) \\ &< 2M_{s-1} < 4p_{s-1}^2 < \sqrt{p_s} \le m. \end{aligned}$$

It follows that

$$\begin{aligned} R_A(m) \le& \sum_{i=0}^{N}\sum_{j=0}^{N} |\,\{(t,t') : t + i(d+1) + t' + j(d+1) = m, t \in T_s, t' \in T_s\}\,| \\ &+ 2\sum_{l=1}^{s-1}\sum_{i=0}^{N}\sum_{j=0}^{N} |\,\{(t,t') : t + i(d+1) + t' + j(d+1) = m, t \in T_s, t' \in T_l\}\,| \\ \le& \, 2(N+1)\sum_{l=1}^{s-1}\sum_{u=0}^{2N} |\,\{(t,t') : t + t' + u(d+1) = m, t \in T_s, t' \in T_l\}\,| \\ &+ 2(N+1)^2. \end{aligned} \tag{1}$$

Suppose that (t, t', u) and $(t_1, t_1', u_1) (t \ge t_1)$ are two distinct solutions of the equation

$$x + y + z(d+1) = m, \quad x \in T_s,\ y \in \bigcup_{l=1}^{s-1} T_l,\ 0 \le z \le 2N, \tag{2}$$

then

$$t + t' + u(d+1) = t_1 + t_1' + u_1(d+1).$$

Thus

$$0 \le t - t_1 = t_1' - t' + (u_1 - u)(d+1) \le M_{s-1} - \sqrt{p_{s-1}} + 2N(d+1)$$
$$< M_{s-1} < 2p_{s-1}^2 < \sqrt{p_s}.$$

By the definition of T_s, we have $t - t_1 = 0$. So $t' + u(d+1) = t_1' + u_1(d+1)$. Thus $|t' - t_1'| = |u_1 - u|(d+1) \le 2N(d+1) < \sqrt{p_i}$ for any $i \ge 1$. It follows from the definition of T_i that t' and t_1' cannot belong to the same T_i. Noting that $T_i \subseteq (\sqrt{p_i}, M_i - \sqrt{p_i})$ and

$$\sqrt{p_{i+1}} - (M_i - \sqrt{p_i}) > p_i^3 - 2p_i^2 \ge p_i > 2N(d+1) \ge |t' - t_1'|,$$

we know that t' and t_1' cannot belong to the different T_i. We obtain a contradiction. Hence equation (2) has at most one solution. It follows from (1) that

$$R_A(m) \le 2(N+1) + 2(N+1)^2.$$

This completes the proof of Lemma 1. □

Lemma 2.
$$\limsup_{n\to\infty} \frac{B(A, \underline{\lambda}, n)}{\sqrt{n}} = v(N+1).$$

Proof. Let $W = \{w : \lambda_w \ne 0, w = 0, \dots, d\}$. Since $a - a' > d$ for $a', a \in A$ with $a' < a$, it follows that $\sum_{i=0}^{d} \lambda_i \chi_A(n-i) \ne 0$ if and only if $n = a + w$, where $a \in A$ and $w \in W$. For any $x \ge 1$, let $A(x)$, $S_i(x)$, $T_i(x)$ etc., denote the counting functions of A, S_i, T_i etc., respectively. Recall that

$$B(A, \underline{\lambda}, n) = \left| \{m : m \le n, \sum_{i=0}^{d} \lambda_i \chi_A(m-i) \ne 0\} \right|,$$

so we have

$$B(A, \underline{\lambda}, n) = \left| \{m : m \le n, m \in A + W\} \right|. \tag{3}$$

It follows that

$$B(A, \underline{\lambda}, n) \le |W| A(n) = vA(n). \tag{4}$$

For every positive integer $n \ge \sqrt{p_2}$, there exists an integer $u \ge 2$ such that $\sqrt{p_u} \le n < \sqrt{p_{u+1}}$. Thus

$$A(n) \le (N+1) \sum_{i=1}^{u} T_i(n) \le (N+1) \sum_{i=1}^{u} S_i(n) = (N+1) \left(S_u(n) + \sum_{i=1}^{u-1} p_i \right).$$

Noting that

$$\sum_{i=1}^{u-1} p_i \le (u-1) p_{u-1} = o(\sqrt{p_u}) = o(\sqrt{n}),$$

we have

$$A(n) \le (N+1)(S_u(n) + o(\sqrt{n})).$$

It is well known that $S_u(n) \le \sqrt{n} + o(\sqrt{n})$ (see [4]). It follows that

$$A(n) \le (N+1)\sqrt{n}\ (1+o(1)). \tag{5}$$

By (4) and (5), we have

$$\limsup_{n\to\infty} \frac{B(A,\underline{\lambda},n)}{\sqrt{n}} \le \limsup_{n\to\infty} \frac{vA(n)}{\sqrt{n}} \le v(N+1). \tag{6}$$

Let $n_k = M_k + (N+1)(d+1)$. Then

$$\begin{aligned} A(n_k) &= (N+1)\sum_{i=1}^{k} |T_i| = (N+1)\sum_{i=1}^{k} p_i(1+o(1)) \\ &= (N+1)p_k(1+o(1)) = (N+1)\sqrt{n_k}(1+o(1)). \end{aligned}$$

By $p_{k+1} > p_k^6$ and $p_k \ge p_1 > 16N^2(d+1)^2$, we have $n_k < \sqrt{p_{k+1}}$. Since

$$\sqrt{p_{k+1}} > n_k > M_k - \sqrt{p_k} + N(d+1) + d,$$

it follows that $A \cap [n_k - d, n_k] = \emptyset$. Thus, by (3), we have

$$B(A,\underline{\lambda},n_k) = A(n_k)|W| = v(N+1)\sqrt{n_k}\ (1+o(1)).$$

It follows that

$$\lim_{k\to\infty} \frac{B(A,\underline{\lambda},n_k)}{\sqrt{n_k}} = v(N+1). \tag{7}$$

By (6) and (7),

$$\limsup_{n\to\infty} \frac{B(A,\underline{\lambda},n)}{\sqrt{n}} = v(N+1).$$

This completes the proof of Lemma 2. □

Proof of Theorem 1. By Lemmas 1 and 2, we have

$$\limsup_{n\to\infty} \frac{B(A,\underline{\lambda},n)}{\sqrt{n}} = v(N+1)$$

and

$$\begin{aligned} \left|\sum_{i=0}^{d} \lambda_i R_A(n-i)\right| &\le \sum_{i=0}^{d} |\lambda_i| \max_n R_A(n) \le 2\sum_{i=0}^{d} |\lambda_i|((N+1)^2 + (N+1)) \\ &= \left(1 + \frac{1}{N+1}\right) \frac{2\sum_{i=0}^{d}|\lambda_i|}{v^2} \limsup_{n\to\infty} \left(\frac{B(A,\underline{\lambda},n)}{\sqrt{n}}\right)^2. \end{aligned}$$

This completes the proof of Theorem 1. □

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