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# Mean invariance identity

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*Dedicated to Professor Zoltán Daróczy on the occasion of his 80th birthday*

**Abstract.** For a continuous and increasing function $f$ in a real interval $I$, and a bivariable mean $P$ defined in $I^2$, we prescribe a pair of bivariable means $M$ and $N$ such that the quasiarithmetic mean $A_f$ generated by $f$ is invariant with respect to the mean-type mapping $(M, N)$. This allows to find effectively the limit of the iterates of the mean-type mapping $(M, N)$. The means $M$ and $N$ are equal iff $P$ is the arithmetic mean $A$; they are symmetric iff so so is $P$. Treating $f$ and $P$ as the parameters, we obtain the family of all pairs of means $(M, N)$ such that the quasiarithmetic mean $A_f$ is invariant with respect to $(M, N)$. In particular, we indicate the function $f$ and the mean $P$ such that the invariance identity $A_f \circ (M, N) = A_f$ coincides with the equality $G \circ (H, A)$, where $G$ and $H$ are the geometric and harmonic means, equivalent to the classical Pythagorean harmony proportion.

Some examples and an application are also presented.

## 1. Introduction

A function $M : I^2 \to I$ is a *mean in an interval* $I \subset \mathbb{R}$ if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

The mean $M$ is called *strict* if these inequalities are sharp for all $x, y \in I$; $x \neq y$.

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Let $K, M, N : I^2 \to I$ be means in $\dot{I}$. The mean $K$ is called *invariant* with respect to the mean-type mapping $(M, N)$ (briefly, $(M, N)$-*invariant* mean) if

$$K \circ (M, N) = K.$$

The invariance of a mean with respect to a mean-type mapping is essential in applications. For instance, the invariance of the geometric bivariable mean $G$ with respect to the mean-type mapping $(A, H)$, where $A$ and $H$ are the arithmetic and harmonic bivariable means, i.e. the identity

$$G \circ (A, H) = G,$$

equivalent to the classical Pythagorean harmony proportion, allows to deduce the convergence of the sequence of iterates of the mapping $(A, H)$ to $(G, G)$ in $(0, \infty)^2$. Of course, each of the means occurring here is quasiarithmetic. In the present paper, for every bivariable mean and an increasing generator of a quasiarithmetic mean, we establish a bivariable mean-type mapping with respect to which the quasi-arithmetic mean is invariant. More precisely, given a continuous strictly increasing function $f$ defined in an interval $I \subset \mathbb{R}$ and a bivariable function $P : I^2 \to I$, in Section 2. we construct two functions $M, N : I^2 \to \mathbb{R}$ such that $P$ is a mean in $I$ iff so are the functions $M, N : I^2 \to \mathbb{R}$. Moreover, $M = N$ iff $P$ is the arithmetic mean $A$; the functions $M$ and $N$ are symmetric iff so is $P$; the quasiarithmetic mean $A_f$ generated by $f$ is invariant with respect to the mapping $(M, N)$, i.e. $A_f \circ (M, N) = A_f$, which allows to determine effectively the limit of the sequence of its iterates (Theorem 1).

Treating $f$ and $P$ as the parameters, we obtain the family of all pairs of bivariable means $(M, N)$ such that the quasiarithmetic mean $A_f$ is invariant with respect to $(M, N)$. In particular, taking $I = (0, \infty)$, $f = \log$, and $P$ defined by

$$P(x, y) = \frac{\log \left(\frac{2x}{x+y}\right)^y \left(\frac{x+y}{2y}\right)^x}{\log \frac{x}{y}}, \quad x, y > 0,$$

we get $A_f = G$, $M = H$, $N = A$, where $G$ and $H$ denote the geometric and harmonic means and, moreover, the invariance identity $A_f \circ (M, N) = A_f$ coincides with the equality $G \circ (H, A) = G$.

In Section 3, we prove Theorem 2, which is in a sense "dual" to Theorem 1.

Let us note that, in the case of symmetric means, a general invariance identity for $k$-variables generalized quasiarithmetic means introduced in [6], [9] trivializes. The invariance formulas proposed here are nontrivial and may be interesting also for symmetric means.

Some examples and an application for effectively solving a functional equation are presented.

## 2. Family of mean-type mappings and invariant quasiarithmetic means

We shall need the following ([8]; see also [5]).

**Lemma 1.** *Let $I \subset \mathbb{R}$ be an interval. Suppose that $M, N : I^2 \to I$ are continuous means such that, for all $x, y \in I$, $x \neq y$,*

$$|M(x,y) - N(x,y)| < |x - y| . \tag{1}$$

*Then there exists a unique $(M,N)$-invariant mean $K$. Moreover, $K$ is continuous and the sequence of iterates $(M_n, N_n) := (M,N)^n$, $n \in \mathbb{N}$, of the mean-type mapping $(M,N)$ converges to the mean-type mapping $(K,K)$ in $I^2$; in particular,*

$$\lim_{n\to\infty} M_n(x,y) = \lim_{n\to\infty} N_n(x,y) = K(x,y), \quad x, y \in I.$$

**Theorem 1.** *Let $f : I \to \mathbb{R}$ be a continuous strictly increasing function in an interval $I$, and let $P : I^2 \to I$ be such that for all $x, y \in I$, $x \neq y$, the numbers $\frac{x-P(x,y)}{x-y} f(x) + \frac{P(x,y)-y}{x-y} f(y)$ and $\frac{x-P(x,y)}{x-y} f(y) + \frac{P(x,y)-y}{x-y} f(x)$ belong to $f(I)$. Then it holds:*

(i) *The function $M : I^2 \to I$ defined by*

$$M(x,y) := \begin{cases} f^{-1}\left(\frac{x-P(x,y)}{x-y} f(x) + \frac{P(x,y)-y}{x-y} f(y)\right), & \text{for } x \neq y, \\ x, & \text{for } x = y, \end{cases} \tag{2}$$

*is a (strict) mean in $I$ if, and only if, $P$ is a (strict) mean.*

(ii) *The function $N : I^2 \to I$ defined by*

$$N(x,y) := \begin{cases} f^{-1}\left(\frac{x-P(x,y)}{x-y} f(y) + \frac{P(x,y)-y}{x-y} f(x)\right), & \text{for } x \neq y, \\ x, & \text{for } x = y, \end{cases} \tag{3}$$

*is a (strict) mean in $I$ if, and only if, $P$ is a (strict) mean.*

(iii) *$M = N$ iff $P = A$, where $A$ is the arithmetic mean.*

(iv) *$M$ $(N)$ is symmetric if so is $P$.*

(v) *The quasiarithmetic mean $A_f : I^2 \to I$,*

$$A_f(x,y) := f^{-1}\left(\frac{f(x) + f(y)}{2}\right)$$

*is invariant with respect to the mapping $(M,N) : I^2 \to I^2$, i.e.*

$$A_f \circ (M,N) = A_f.$$

(vi) *If $P$ is a continuous and strict mean, then the sequence $((M,N)^n : n \in \mathbb{N}_0)$ of iterates of the mean-type mapping $(M,N)$ converges pointwise in $I^2$ and*

$$\lim_{n\to\infty} (M,N)^n = (A_f, A_f)\,.$$

Proof. (i) Assume that $P$ is a strict mean in $I$, and take arbitrary $x, y \in I$, $x \neq y$. Without any loss of generality, we can assume that $x < y$. Since $x < P(x,y) < y$, we have

$$\frac{x - P(x,y)}{x-y}, \frac{P(x,y)-y}{x-y} \in (0,1) \quad \text{and} \quad \frac{x - P(x,y)}{x-y} + \frac{P(x,y)-y}{x-y} = 1.$$

Hence, as $f$ is increasing,

$$f(x) < \frac{x - P(x,y)}{x-y} f(x) + \frac{P(x,y)-y}{x-y} f(y) < f(y)\,,$$

whence

$$x < f^{-1}\left(\frac{x - P(x,y)}{x-y} f(x) + \frac{P(x,y)-y}{x-y} f(x)\right) < y,$$

that is $\min(x,y) = x < M(x,y) < y = \max(x,y)$, which shows that $M$ is a mean in $I$.

Assume that $M$ is a mean and take $x, y \in I$, $x < y$. The definition of $M$ and the monotonicity of $f$ imply that

$$f(x) < \frac{x - P(x,y)}{x-y} f(x) + \frac{P(x,y)-y}{x-y} f(y) < f(y)\,,$$

or equivalently, that

$$0 < \frac{P(x,y)-y}{x-y}\left[f(y) - f(x)\right] \quad \text{and} \quad 0 < \frac{x-P(x,y)}{x-y}\left[f(y) - f(x)\right].$$

Since $f(y) - f(x) > 0$, we hence get

$$\frac{P(x,y)-y}{x-y} > 0 \quad \text{and} \quad \frac{x-P(x,y)}{x-y} > 0,$$

which implies that $x = \min(x,y) < P(x,y) < \max(x,y) = y$, showing that $P$ is a strict mean in $I$.

(ii) Since the proof is similar as in the case (i), we omit it.

(iii) It is easy to verify that, for $x \neq y$, the equality $M(x,y) = N(x,y)$ holds if

$$\frac{x-P(x,y)}{x-y}f(x)+\frac{P(x,y)-y}{x-y}f(y)=\frac{x-P(x,y)}{x-y}f(y)+\frac{P(x,y)-y}{x-y}f(x),$$

that is, iff

$$[f(x)-f(y)][2P(x,y)-x-y]=0,$$

whence, taking into account that $f$ is one-to-one, iff

$$P(x,y)=\frac{x+y}{2}.$$

(iv) If $M(x,y) = M(y,x)$ for $x \neq y$, then, by the definition of $M$,

$$\frac{x-P(x,y)}{x-y}f(x)+\frac{P(x,y)-y}{x-y}f(y)=\frac{y-P(y,x)}{y-x}f(y)+\frac{P(y,x)-y}{y-x}f(x),$$

whence, after simple calculations,

$$[f(x)-f(y)][P(x,y)-P(y,x))]=0,$$

which holds true only if $P(x,y) = P(y,x))$ for all $x \neq y$.

(v) From the definitions of the means $M, N$ and $A_f$, we have for all $x, y \in I$,

$$\begin{aligned} A_f\circ(M,N)(x,y) &= f^{-1}\left(\frac{f\left(f^{-1}\left(\frac{x-P(x,y)}{x-y}f(x)+\frac{P(x,y)-y}{x-y}f(y)\right)\right)}{2}\right.\\ &\qquad\left.+\frac{f\left(f^{-1}\left(\frac{x-P(x,y)}{x-y}f(y)+\frac{P(x,y)-y}{x-y}f(x)\right)\right)}{2}\right)\\ &= f^{-1}\left(\frac{f(x)+f(y)}{2}\right)=A_f(x,y), \end{aligned}$$

so $A_f$ is $(M,N)$-invariant.

(vi) Since every strict mean satisfies condition (1), this result follows from Lemma 1. □

*Remark 1.* If $P$ is a mean in $I$, then it is continuous on the diagonal $\{(x,x) : x \in I\}$ of the square $I \times I$, ([8]).

*Remark 2.* If $P$ is a mean in $I$, then the numbers $\frac{x-P(x,y)}{x-y}f(x)+\frac{P(x,y)-y}{x-y}f(y)$ and $\frac{x-P(x,y)}{x-y}f(y)+\frac{P(x,y)-y}{x-y}f(x)$ belong to $f(I)$. To see that the converse is not true, take, for instance, $I=\mathbb{R}$, $f(x)=x^3$ and $P(x,y)=2x-y$.

The following remark shows that the above theorem contains, as a very special case, the Pythagorean harmony proportion invariance identity $G\circ(H,A)=G$.

*Remark 3.* Let $P:(0,\infty)^2\to(0,\infty)$ be defined by

$$P(x,y):=\begin{cases}\frac{\log\left(\frac{2x}{x+y}\right)^y\left(\frac{x+y}{2y}\right)^x}{\log\frac{x}{y}}, & \text{if } x\neq y,\\ x, & \text{if } x=y,\end{cases}\tag{4}$$

and let $f=\log$. From (2) an (3) we get

$$M(x,y)=H(x,y)=\frac{2xy}{x+y},\quad N(x,y)=A(x,y)=\frac{x+y}{2},\qquad x,y>0.$$

Applying Theorem 1 (i) and (ii), we conclude that $P$ is a strict mean and, of course,

$$A^{[f]}(x,y)=G(x,y)=\sqrt{xy},\quad x,y>0,$$

Moreover, $P$ is continuous. By part (iv) of Theorem 1, we obtain the classical invariance identity $G\circ(H,A)=G$, equivalent to the Pythagorean harmony proportion, which, in view of part (v), implies

$$\lim_{n\to\infty}(H,A)^n=(G,G)\quad\text{pointwise in }(0,\infty)^2.$$

Proof. By (2) with $P$ given by (4) and $f=\log$, by simple calculations, for $x\neq y$, we get

$$M(x,y)=\exp\left(\frac{x-\frac{\log\left(\frac{2x}{x+y}\right)^y\left(\frac{x+y}{2y}\right)^x}{\log\frac{x}{y}}}{x-y}\log x+\frac{\frac{\log\left(\frac{2x}{x+y}\right)^y\left(\frac{x+y}{2y}\right)^x}{\log\frac{x}{y}}-y}{x-y}\log y\right)$$

$$=\frac{2xy}{x+y}=H(x,y),$$

and, by (3),

$$N(x,y)=\exp\left(\frac{x-\frac{\log\left(\frac{2x}{x+y}\right)^y\left(\frac{x+y}{2y}\right)^x}{\log\frac{x}{y}}}{x-y}\log y+\frac{\frac{\log\left(\frac{2x}{x+y}\right)^y\left(\frac{x+y}{2y}\right)^x}{\log\frac{x}{y}}-y}{x-y}\log x\right)$$

$$=\frac{x+y}{2}=A(x,y).$$

Since $M$ is a strict mean, in view of Theorem 1, so is $P$. The remaining facts follow from Theorem 1 and Remark 1. □

Simple calculations allow to verify the following:

*Remark 4.* The mean $M$ given by (2) in Theorem 1 is quasiarithmetic iff there is a continuous and strictly increasing $g : I \to \mathbb{R}$ such that

$$P(x,y) = \frac{yf(y) - xf(x) + (x-y) f(A_g(x,y))}{f(y) - f(x)}, \quad x,y \in I,\ x \neq y.$$

The mean $N$ given by (3) in Theorem 1 is quasiarithmetic iff there is a continuous and strictly increasing $h : I \to \mathbb{R}$ such that

$$P(x,y) = \frac{yf(x) - xf(y) + (x-y) f(A_h(x,y))}{f(x) - f(y)}, \quad x,y \in I,\ x \neq y.$$

Thus both $M$ and $N$ are quasiarithmetic iff there are continuous and strictly increasing functions $g, h : I \to \mathbb{R}$ such that the right-hand sides of the above expressions are equal. Comparing them, we get the equality

$$f(A_g(x,y)) + f(A_h(x,y)) = f(x) + f(y), \quad x,y \in I,$$

first considered by O. Sutô [10] for analytic functions, then by the present author [4] in the class of twice differentiable functions, and in the general case by Z. Daróczy and Zs. Páles [1].

This remark remains true if we replace $A_g$ and $A_h$ by the weighted quasiarithmetic means $A_{g,t}$ and $A_{h,1-t}$, where

$$A_{g,t}(x,y) := g^{-1}(tg(x) + (1-t)g(y)), \quad x,y \in I.$$

(see also [2] and J. Jarczyk [3]). Hence, applying the main result of [1], we get

*Remark 5.* Let $f, g, h : I \to \mathbb{R}$ be a continuous strictly increasing function in an interval $I$, and let $t \in (0,1)$. Then the following conditions are equivalent:

(i) $A_f$ is invariant with respect to the mean type mapping $(A_{g,t}, A_{h,1-t})$, i.e.

$$A_f \circ (A_{g,t}, A_{h,1-t}) = A_f.$$

(ii) The function $P : I^2 \to \mathbb{R}$,

$$P(x,y) := \begin{cases} \frac{yf(y) - xf(x) + (x-y)f(A_{g,t}(x,y))}{f(y) - f(x)}, & \text{if } x \neq y, \\ x, & \text{if } x = y, \end{cases}$$

is a mean.

(iii) There is a real number $a > 0$, $a \neq 1$, such that for all $x, y \in I$,

$$A_{g,t}(x,y) = f^{-1}\left(\log_a\left(ta^{f(x)} + (1-t)\,a^{f(y)}\right)\right);$$
$$A_{h,1-t}(x,y) = f^{-1}\left(\log_a\left((1-t)\,a^{f(x)} + ta^{f(y)}\right)\right).$$

In particular, we hence get the following:

*Invariance Identity.* If $f : I \to \mathbb{R}$ *is continuous and strictly monotonic in an interval* $I$, $t \in (0,1)$, and $a > 0$, $a \neq 1$, then

$$A_f\left(f^{-1}\left(\log_a\left(ta^{f(x)}+(1-t)\,a^{f(y)}\right)\right), f^{-1}\left(\log_a\left((1-t)\,a^{f(x)}+ta^{f(y)}\right)\right)\right) = A_f(x,y),$$

for all $x, y \in I$.

Taking here $f = \log$, $a = e$ and $t = 1/2$, we get $G \circ (A, H) = G$.

## 3. A dual result

**Theorem 2.** *Let* $f : I \to \mathbb{R}$ *be a continuous strictly increasing function in an interval* $I$*, and let* $M, N : I^2 \to I$*. Then it holds:*

(i) $M$ *is a (strict) mean in* $I$ *iff the function* $P : I^2 \to \mathbb{R}$ *defined by*

$$P(x,y) := \begin{cases} \frac{xf(x)-yf(y)+(y-x)f(M(x,y))}{f(x)-f(y)}, & \text{if } x \neq y, \\ x, & \text{if } x = y, \end{cases}$$

*is a (strict) mean in* $I$.

(ii) $N$ *is a (strict) mean in* $I$ *iff the function* $Q : I^2 \to \mathbb{R}$ *defined by*

$$Q(x,y) := \begin{cases} \frac{yf(x)-xf(y)+(x-y)f(N(x,y))}{f(x)-f(y)}, & \text{if } x \neq y, \\ x, & \text{if } x = y, \end{cases}$$

*is a (strict) mean in* $I$.

(iii) $M$ $(N)$ *is symmetric iff so is* $P$ $(Q)$.

(iv) *The quasiarithmetic mean* $A_f$ *is invariant with respect to the mapping* $(M, N) : I^2 \to I^2$ *iff*

$$P = Q.$$

Proof. (i) Assume that $M$ is a mean in $I$, and take arbitrary $x, y \in I$, $x \neq y$. Assuming (without loss of generality) that $x < y$, i.e. that $x = \min(x, y)$ and $y = \max(x, y)$, we hence get

$$y \leq M(x, y) \leq y,$$

whence, as $f$ is increasing,

$$f(x) \leq f(M(x, y)) \leq f(y).$$

Consequently, we have

$$(y - x) f(x) \leq (y - x) f(M(x, y)), \quad (y - x) f(M(x, y)) \leq (y - x) f(y),$$

or, equivalently,

$$yf(x) \leq xf(x) + (y - x) f(M(x, y)), \quad -yf(y) + (y - x) f(M(x, y)) \leq -xf(y).$$

Adding $(-yf(y))$ to both of the first of these inequalities, and $xf(x)$ to both sides of the second one, gives

$$yf(x) - yf(y) \leq xf(x) - yf(y) + (y - x) f(M(x, y))$$

and

$$xf(x) - yf(y) + (y - x) f(M(x, y)) \leq xf(x) - xf(y),$$

whence

$$y[f(x) - f(y)] \leq xf(x) - yf(y) + (y - x) f(M(x, y)) \leq x[f(x) - f(y)].$$

Since $f$ is strictly increasing and $x < y$, we hence get

$$\min(x, y) = x \leq \frac{xf(x) - yf(y) + (y - x) f(M(x, y))}{f(x) - f(y)} \leq y = \max(x, y),$$

which proves that $P$ is a mean.

Clearly, in the case when $M$ is a strict mean, all these inequalities are sharp, implying that $P$ is also strict.

As all the above inequalities are equivalent, the converse implication holds true.

(ii) We omit similar argument as in case (i), as well as easy calculations needed to verify (iii).

To prove (iv) note that equality $P = Q$ is holds iff, for all $x, y \in I$, $x \neq y$,

$$\frac{xf(x) - yf(y) + (y-x) f(M(x,y))}{f(x) - f(y)} = \frac{yf(x) - xf(y) + (x-y) f(N(x,y))}{f(x) - f(y)},$$

which simplifies to the equality

$$f(M(x,y)) + f(M(x,y)) = f(x) + f(y), \quad x, y \in I, x \neq y,$$

which is equivalent to the invariance relation $A_f \circ (M, N) = A_f$. □

*Remark 6.* The functions $M$ and $N$ in Theorem 2 (as well as the function $P$ in Theorem 1) need not be means.

To see it, consider two examples.

*Example 1.* Take $I = \mathbb{R}$ or $I = (0, \infty)$; $f := \mathrm{id}$; $a, b \in (0,1)$, $b \neq 1 - a$, and define $M, N : I^2 \to I$ by

$$M(x,y) = ax + by, \quad N(x,y) = (1-a)x + (1-b)y, \quad x, y \in I.$$

Then neither $M$ nor $N$ is a mean. However, we have $A_f \circ (M, N) = A_f$.

*Example 2.* Take $I = (0, \infty)$, $f = \log$. Let $M : I^2 \to I$ be an arbitrary function, and let

$$N(x,y) = \frac{xy}{M(x,y)}, \quad x, y > 0.$$

Since $A_f = G$ and, for all $x, y > 0$,

$$G \circ (M, N)(x,y) = \sqrt{\frac{2x^2 + y^2}{x + 2y} \frac{x + 2y}{2x^2 + y^2} xy} = \sqrt{xy} = G(x,y),$$

the invariance equality holds.

## 4. An application

**Theorem 3.** *Let $f : I \to \mathbb{R}$ be a continuous strictly increasing function in an interval $I$, let $P : I^2 \to I$ a continuous a strict mean, and let $M, N : I^2 \to I$ be defined by* (2) *and* (3)*. Assume that $\Phi : I^2 \to \mathbb{R}$ is continuous on the diagonal $\{(x,x) : x \in I\}$.*

*Then the function $\Phi$ satisfies the functional equation*

$$\Phi(M(x,y), N(x,y)) = \Phi(x,y), \quad x, y \in I, \tag{5}$$

*if, and only if, there exists a continuous single variable function $\varphi : I \to \mathbb{R}$ such that $\Phi = \varphi \circ A_f$.*

Proof. If $\Phi : I^2 \to \mathbb{R}$ satisfies (5), then, by induction, we get

$$\Phi(x,y) = \Phi \circ ((M,N)^n)(x,y), \quad n \in \mathbb{N},\ x,y \in I, \tag{6}$$

where $(M,N)^n$ is the $n$-th iterate of the mean-type mapping. In view of Theorem 1 (v)–(vi), the means $M$ and $N$ are strict means

$$\lim_{n\to\infty} (M,N)^n(x,y) = (A_f(x,y), A_f(x,y)), \quad x,y \in I.$$

Therefore, letting $n \to \infty$ in (6) and making use of the continuity of $\Phi$ on the diagonal, we obtain

$$\Phi(x,y) = \Phi \circ (A_f(x,y), A_f(x,y)), \quad x,y \in I.$$

Setting

$$\varphi(t) := \Phi(t,t), \quad t \in I,$$

we obtain

$$\Phi(x,y) = \varphi \circ (A_f)(x,y), \quad x,y \in I.$$

The converse implication is easy to verify. □

For other applications of invariance, see [7].

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