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Commutativity of torsion and normal Jacobi operators on real hypersurfaces in the complex quadric

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Abstract. On a real hypersurface in the complex quadric we can consider the Levi-Civita connection and, for any non-zero real constant k, the k-th generalized Tanaka–Webster connection. Associated to this connection we can define a differential operator whose difference with the Lie derivative is the torsion operator of the k-th generalized Tanaka–Webster connection. We prove the non-existence of real hypersurfaces in the complex quadric for which the torsion operators commute with the normal Jacobi operator of the real hypersurface.

1. Introduction

The complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is a compact Hermitian symmetric space of rank 2. It is also a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ (see [5], [6], [8]). The space Q^m is equipped with two geometric structures: a Kaehler structure J and a parallel circle subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$, which consists of all the real structures on the tangent space of Q^m. For any $A \in \mathfrak{A}$ the following relations hold: $A^2 = I$ and $AJ = -JA$. A nonzero tangent vector W at a point of Q^m is called *singular* if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for Q^m: $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic vectors.

Real hypersurfaces M are immersed submanifolds of real co-dimension 1 in a Hermitian manifold. Since Q^m is a compact Hermitian symmetric space with

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rank 2, it is interesting to study real hypersurfaces M in Q^m. The Kaehler structure J of Q^m induces on M an almost contact metric structure (ϕ, ξ, η, g), where ϕ is the structure tensor field, ξ is the Reeb vector field, η is a 1-form and g is the induced Riemannian metric of Q^m.

The study of real hypersurfaces M in Q^m is initiated by Berndt and Suh in [1]. In this paper the geometric properties of real hypersurfaces M in complex quadric Q^m, which are tubes of radius r, $0 < r < \pi/2$, around the totally geodesic $\mathbb{C}P^k$ in Q^m, when $m = 2k$ or tubes of radius r, $0 < r < \pi/2\sqrt{2}$, around the totally geodesic Q^{m-1} in Q^m, are presented. The condition of isometric Reeb flow is equivalent to the commuting condition of the shape operator S with the structure tensor ϕ of M. The classification of such real hypersurfaces in Q^m is obtained in [2].

Given a Riemannian manifold $(\tilde{M}, \tilde{g})$, Jacobi fields along geodesics satisfy a differential equation which results in the notion of Jacobi operator. That is, if $\tilde{R}$ is the Riemannian curvature tensor of $\tilde{M}$, and X is a tangent vector field on $\tilde{M}$, then the Jacobi operator with respect to X at a point $p \in \tilde{M}$ is given by

$$(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p),$$

and becomes a self adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$, i.e., $\tilde{R}_X \in \mathrm{End}(T_p\tilde{M})$. In the case of real hypersurfaces M in Q^m, we can consider the normal Jacobi operator $\bar{R}_N$, where $\bar{R}$ is the Riemannian curvature tensor of Q^m and N is the unit normal vector field on the real hypersurface M.

As M has an almost contact metric structure, for any non-zero real constant k, we can define the so called k-th generalized Tanaka–Webster connection $\hat{\nabla}^{(k)}$ on M by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi SX, Y)\xi - \eta(Y)\phi SX - k\eta(X)\phi Y$$

for any X, Y tangent to M, where ∇ is the Levi-Civita connection on M, and S denotes the shape operator on M associated to N (see [3]). Let us call $F^{(k)}_X Y = g(\phi SX, Y)\xi - \eta(Y)\phi SX - k\eta(X)\phi Y$, for any X, Y tangent to M. $F^{(k)}_X$ is called the k-th Cho operator on M associated to X. Notice that if $X \in \mathcal{C}$, the maximal holomorphic distribution on M, given by all the vector fields orthogonal to ξ, the associated Cho operator does not depend on k and we will denote it simply by F_X. Then, given a symmetric tensor field L of type (1,1) on M, $\nabla_X L = \hat{\nabla}^{(k)}_X L$ for a tangent vector field X on M if and only if $F^{(k)}_X L = LF^{(k)}_X$, that is, the eigenspaces of L are preserved by $F^{(k)}_X$. If $L = \bar{R}_N$, in [4] we proved

Theorem 1.1. *There do not exist real hypersurfaces M in Q^m, $m \geq 3$, such that $\nabla \bar{R}_N = \hat{\nabla}^{(k)} \bar{R}_N$, for any non-zero real constant k.*

The torsion of the k-th generalized Tanaka–Webster connection is given by $T^{(k)}(X,Y) = F_X^{(k)}Y - F_Y^{(k)}X$ for any X, Y tangent to M. For any X tangent to M, we define the torsion operator associated to X by $T_X^{(k)}Y = T^{(k)}(X,Y)$ for any Y tangent to M.

Let $\mathcal{L}$ denote the Lie derivative on M. Associated to the k-th generalized Tanaka–Webster connection, we can define the differential operator of first order $\mathcal{L}^{(k)}$ by $\mathcal{L}_X^{(k)}Y = \hat{\nabla}_X^{(k)}Y - \hat{\nabla}_Y^{(k)}X = \mathcal{L}_XY + T_X^{(k)}Y$, for any X, Y tangent to M. Then for a symmetric tensor of type (1,1) on M, $\mathcal{L}_XL = \mathcal{L}_X^{(k)}L$ for a tangent vector field X on M if and only if $T_X^{(k)}L = LT_X^{(k)}$.

In this paper we study real hypersurfaces M in Q^m such that the Lie derivative and the differential operator $\mathcal{L}^{(k)}$ associated to the k-th generalized Tanaka–Webster connection coincide when we apply them to the normal Jacobi operator $\bar{R}_N$, that is

$$\mathcal{L}\bar{R}_N = \mathcal{L}^{(k)}\bar{R}_N \tag{1.1}$$

for some non-zero real constant k. We will prove the following

Theorem 1.2. *There do not exist real hypersurfaces M in Q^m, $m \geq 3$, such that $\mathcal{L}\bar{R}_N = \mathcal{L}^{(k)}\bar{R}_N$, for any non-zero real constant k.*

2. The space Q^m

The complex projective space $\mathbb{C}P^{m+1}$ is considered as the Hermitian symmetric space of the special unitary group SU_{m+2}, namely

$$\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1).$$

The symbol $o = [0, ..., 0, 1]$ in $\mathbb{C}P^{m+1}$ is the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The action of the special orthogonal group $SO_{m+2} \subset SU_{m+2}$ on $CP^{m+1}$is of cohomogeneity one. A totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$ is an orbit containing point o. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. It is a homogeneous model, which interprets geometrically the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. Thus, the complex quadric Q^m is considered as a Hermitian space of rank 2. For $m = 1$, the complex quadric Q^1 is isometric to a sphere S^2 of constant curvature. For $m = 2$, the complex quadric Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. Therefore, we assume the dimension of complex quadric Q^m to be greater than or equal to 3.

Moreover, the complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ defined by the homogeneous quadric equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where z_i, $i = 1, ..., m+2$, are homogeneous coordinates on $\mathbb{C}P^{m+1}$. The Kaehler structure of complex projective space $\mathbb{C}P^{m+1}$ induces canonically a Kaehler structure (J, g) on Q^m, where g is a Riemannian metric with maximal holomorphic sectional curvature 4 induced by the Fubini Study metric of $\mathbb{C}P^{m+1}$.

Consider the Riemannian fibration $\pi : S^{2m+3} \subset \mathbb{C}^{m+2} \longrightarrow \mathbb{C}P^{m+1}$, $z \mapsto [z]$. Then $\mathbb{C}^{m+2} \ominus [z]$ is the horizontal space of π at $z \in S^{2m+3}$. Then at each $[z]$ in Q^m the tangent space $T_{[z]}Q^m$ can be identified canonically with the orthogonal complement of $\mathbb{C}^{m+2} \ominus ([z] \oplus [\bar{z}])$ of $[z] \oplus [\bar{z}]$ in $\mathbb{C}^{m+2}$. Thus $\pi_*|_z \bar{z}$ is a unit normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

The shape operator $A_{\bar{z}}$ of Q^m with respect to the unit normal vector $\bar{z}$ is given by

$$A_{\bar{z}} \pi_*|_z w = \pi_*|_z \bar{w},$$

for all $w \in T_{[z]}Q^m$. The shape operator $A_{\bar{z}}$ is a complex conjugation restricted to $T_{[z]}Q^m$. The complex vector space $T_{[z]}Q^m$ is decomposed into

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (+1)-eigenspace of $A_{\bar{z}}$, i.e., $A_{\bar{z}}X = X$, and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$, i.e., $A_{\bar{z}}JX = -JX$ for any $X \in V(A_{\bar{z}})$. Geometrically, it means that $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, which is an antilinear involution. The set of all such shape operators $A_{\bar{z}}$ defines a parallel circle subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$, which consists of all the real structures on the tangent space of Q^m. For any $A \in \mathfrak{A}$ the following relations hold:

$$A^2 = I \quad \text{and} \quad AJ = -JA.$$

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ yields that the Riemannian curvature tensor R of Q^m is given by

$$\begin{aligned} \bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY, \end{aligned}$$

where J is the complex structure, g is the Riemannian metric and A is a real structure in $\mathfrak{A}$.

A nonzero tangent vector $W \in T_{[z]}Q^m$ is called *singular* if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for Q^m:

(1) $\mathfrak{A}$-*principal*. In this case, there exists a real structure $A \in \mathfrak{A}$ such that $W \in V(A)$.

(2) $\mathfrak{A}$-*isotropic*. In this case, there exists a real structure $A \in \mathfrak{A}$ and orthonormal vectors X, $Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$.

For every unit vector field W tangent to Q^m, there is a complex conjugation $A \in \mathfrak{A}$ and orthonormal vectors X, $Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY, \tag{2.1}$$

for some $t \in [0, \pi/4]$. The singular vectors correspond to the values $t = 0$ and $t = \pi/4$.

3. Real hypersurfaces in Q^m

Let M be a real hypersurface in Q^m and N a unit normal vector field of M. Any vector field X tangent to M satisfies the relation

$$JX = \phi X + \eta(X)N. \tag{3.1}$$

The tangential component of the above relation defines on M a skew-symmetric tensor field of type (1,1) ϕ, named the structure tensor. The structure vector field ξ is defined by $\xi = -JN$ and is called the Reeb vector field. The 1-form η is given by $\eta(X) = g(X, \xi)$ for any vector field X tangent to M. So, on M an almost contact metric structure (ϕ, ξ, η, g) is defined. The elements of the almost contact structure satisfy the following relations:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3.2}$$

for all tangent vectors X, Y to M. Relation (3.2) implies

$$\phi\xi = 0.$$

The tangent bundle TM of M splits orthogonally into

$$TM = \mathcal{C} \oplus \mathcal{F},$$

where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J.

The shape operator of a real hypersurface M in Q^m is denoted by S. The real hypersurface is called *Hopf hypersurface* if the Reeb vector field is an eigenvector of the shape operator, i.e.,

$$S\xi = \alpha\xi, \tag{3.3}$$

where $\alpha = g(S\xi, \xi)$ is the Reeb function.

At each point $[z] \in M$, we choose a real structure $A \in \mathfrak{A}_{[z]}$ such that

$$N_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2, \qquad AN_{[z]} = \cos(t)Z_1 - \sin(t)JZ_2, \tag{3.4}$$

where Z_1, Z_2 are orthonormal vectors in $V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Moreover, the above relations due to $\xi = -JN$ imply

$$\xi_{[z]} = -\cos(t)JZ_1 + \sin(t)Z_2, \qquad A\xi_{[z]} = \cos(t)JZ_1 + \sin(t)Z_2. \tag{3.5}$$

So, we have $g(AN_{[z]}, \xi_{[z]}) = 0$.

The Codazzi equation of M is given by

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ &\quad + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ &\quad + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z) \end{aligned} \tag{3.6}$$

for any X, Y, Z tangent to M.

The normal Jacobi operator of a real hypersurface in Q^m is calculated by the Gauss equation for $Y = Z = N$ and, because of (3.4), is given by

$$\bar{R}_N(X) = X + 3\eta(X)\xi + \cos(2t)AX - g(AX, N)AN - g(AX, \xi)A\xi, \tag{3.7}$$

for any $X \in TM$, where $g(AN, N) = \cos(2t) = -g(A\xi, \xi)$. Let us suppose that $(\mathcal{L}_X \bar{R}_N)Y = (\mathcal{L}_X^{(k)} \bar{R}_N)Y$ for any X, Y tangent to M. This yields $F_X^{(k)} \bar{R}_N Y - F_{\bar{R}_N Y}^{(k)} X - \bar{R}_N F_X^{(k)} Y + F_Y^{(k)} X = 0$, for any X, Y tangent to M. That is

$$\begin{aligned} & g(\phi SX, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi SX - k\eta(X)\phi \bar{R}_N Y - g(\phi S\bar{R}_N Y, X)\xi \\ &\quad + \eta(X)\phi S\bar{R}_N Y + k\eta(\bar{R}_N Y)\phi X - g(\phi SX, Y)\bar{R}_N\xi + \eta(Y)\bar{R}_N\phi SX \\ &\quad + k\eta(X)\bar{R}_N\phi Y + g(\phi SY, X)\bar{R}_N\xi - \eta(X)\bar{R}_N\phi SY - k\eta(Y)\bar{R}_N\phi Y = 0. \end{aligned} \tag{3.8}$$

If we take $X = \xi$ in (3.8), we obtain

$$g(\phi S\xi, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi S\xi - k\phi\bar{R}_N Y + \phi S\bar{R}_N Y \\ - g(\phi S\xi, Y)\bar{R}_N\xi + \eta(Y)\bar{R}_N\phi S\xi + k\bar{R}_N\phi Y - \bar{R}_N\phi SY = 0 \tag{3.9}$$

for any Y tangent to M.

Taking $X \in \mathcal{C}$ in (3.8), we get

$$g((\phi S + S\phi)X, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi SX + k\eta(\bar{R}_N Y)\phi X \\ - g((\phi S + S\phi)X, Y)\bar{R}_N\xi + \eta(Y)\bar{R}_N\phi SX - k\eta(Y)\bar{R}_N\phi X = 0 \tag{3.10}$$

for any $X \in \mathcal{C}$, Y tangent to M.

We finish this section with the following Proposition, which concerns Hopf hypersurfaces in Q^m whose shape operator commutes with the structure tensor, see [2].

Proposition 3.1. *The following statements hold for a tube M of radius r, $0 < r < \pi/2$ around the totally geodesic $\mathbb{C}P^k$ in Q^m, $m = 2k$:*

(1) *M is a Hopf hypersurface.*

(2) *The normal bundle of M consists of $\mathfrak{A}$-isotropic singular tangent vectors of Q^m.*

(3) *M has four distinct principal curvatures, unless $m = 2$, in which case M has two distinct principal curvatures.*

(4) *The shape operator commutes with the structure tensor field ϕ, i.e., $S\phi = \phi S$.*

(5) *M is a homogeneous hypersurface.*

And see also [7]:

Proposition 3.2. *Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then $\alpha = g(S\xi, \xi)$ is constant, and if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$, and ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

4. Proof of Theorem 1.2. The case of Hopf real hypersurfaces

All the following calculations take place at an arbitrary point $[z] \in M$, but we can omit the subscript $_{[z]}$ from the vector fields and other objects for the sake of brevity.

Let us suppose that M is Hopf at $[z]$, i.e., that $S\xi = \alpha\xi$ holds. We will first prove the following:

Lemma 4.1. *Let M be a Hopf real hypersurface in Q^m, $m \geq 3$. If $\mathcal{L}\bar{R}_N = \mathcal{L}^{(k)}\bar{R}_N$ for some non-zero real constant k, then N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. As M is Hopf, (3.9) becomes

$$-k\phi\bar{R}_N Y + \phi S\bar{R}_N Y + k\bar{R}_N\phi Y - \bar{R}_N\phi SY = 0 \tag{4.1}$$

for any Y tangent to M. If, in particular, $Y = \xi$, we get

$$-k\phi\bar{R}_N\xi + \phi S\bar{R}_N\xi = 0. \tag{4.2}$$

If in (3.10) we take $Y = \xi$, we have

$$g((\phi S+S\phi)X, \bar{R}_N\xi)\xi - \eta(\bar{R}_N\xi)\phi SX + k\eta(\bar{R}_N\xi)\phi X + \bar{R}_N\phi SX - k\bar{R}_N\phi X = 0 \tag{4.3}$$

for any $X \in \mathcal{C}$.

Taking the scalar product of both sides of (4.3) by ξ gives $2g(\phi SX, \bar{R}_N\xi) + g(S\phi X, \bar{R}_N\xi) - kg(\phi X, \bar{R}_N\xi) = 0$ for any $X \in \mathcal{C}$. From (4.2) we obtain $g(\bar{R}_N\xi, \phi SX) = 0$, for any $X \in \mathcal{C}$. As $\bar{R}_N\xi = 4\xi + 2\cos(2t)A\xi$, it follows that

$$2\cos(2t)g(A\xi, \phi SX) = 0 \tag{4.4}$$

for any $X \in \mathcal{C}$. From (4.4), if $\cos(2t) = 0$, N is $\mathfrak{A}$-isotropic. If $\cos(2t) \neq 0$, $g(A\xi, \phi SX) = 0$ for any $X \in \mathcal{C}$. In this case, from (3.10), if $X \in \mathcal{C}$ satisfies $SX = \lambda X$, where $\lambda \neq k$, then $g(A\xi, X) = 0$.

Therefore, if in $\mathcal{C}$ k does not appear as an eigenvalue of S or k is the unique eigenvalue of S, $g(A\xi, X) = 0$ for any $X \in \mathcal{C}$ and N is $\mathfrak{A}$-principal. If the unique eigenvalue of S in $\mathcal{C}$ is k, $\phi S = S\phi$ and N should be $\mathfrak{A}$-isotropic, which is a contradiction. Therefore, if in $\mathcal{C}$ k does not appear as an eigenvalue of S, N must be $\mathfrak{A}$-principal.

Thus we must suppose there exists $X \in \mathcal{C}$ such that $SX = kX$, and therefore $g(AN, X) = 0$, and there exists $Z \in \mathcal{C}$ such that $SZ = \lambda Z$, $\lambda \neq k$, and then $g(A\xi, Z) = 0$. Moreover, we must suppose there exists $W \in \mathcal{C}$ such that $\eta(\bar{R}_N W) = g(A\xi, W) \neq 0$. If not, N should be $\mathfrak{A}$-principal.

Let $X \in \mathcal{C}$ such that $SX = kX$. From (3.10) we have $g((\phi S+S\phi)X, \bar{R}_N Y)\xi - g((\phi S + S\phi)X, Y)\bar{R}_N\xi = 0$ for any Y tangent to M. Its scalar product with W yields $g((\phi S + S\phi)X, Y) = 0$ for any Y tangent to M, that is, $\phi SX = -S\phi X = k\phi X$. Therefore $S\phi X = -k\phi X$. Again from (3.10) we have $g((\phi S + S\phi)\phi X, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi S\phi X - k\eta(\bar{R}_N Y)X - g((\phi S+S\phi)\phi X, Y)\bar{R}_N\xi + \eta(Y)\bar{R}_N\phi S\phi X + k\eta(Y)\bar{R}_N X = 0$. But $\phi S\phi X + S\phi^2 X = 0$. Therefore, it follows that $-2k\eta(\bar{R}_N Y)X + 2k\eta(Y)\bar{R}_N X = 0$. If $Y = \xi$, we have $-\eta(\bar{R}_N\xi)X + \bar{R}_N X = 0$. Its scalar product with ξ implies $\eta(\bar{R}_N X) = 0$. As for any $Z \in \mathcal{C}$ such that $SZ = \lambda Z$, $\lambda \neq k$, we have $g(A\xi, Z) = 0$, we arrive to a contradiction and we have finished the proof. □

Lemma 4.2. *There do not exist Hopf real hypersurfaces in Q^m, $m \geq 3$, such that $\mathcal{L}\bar{R}_N = \mathcal{L}^{(k)}\bar{R}_N$ for a non-zero real constant k if N is $\mathfrak{A}$-isotropic.*

Proof. If N is $\mathfrak{A}$-isotropic, $\bar{R}_N\xi = 4\xi$. Let $X \in \mathcal{C}$ be a unit vector field such that $SX = \lambda X$. Introducing it in (4.3), we have $-4\lambda\phi X + 4k\phi X + \lambda\bar{R}_N\phi X - k\bar{R}_N\phi X = 0$. That is, $(k-\lambda)\bar{R}_N\phi X = 4(k-\lambda)\phi X$. There are two possibilities, either $\lambda = k$ or if $\lambda \neq k$, $\bar{R}_N\phi X = 4\phi X$.

In the second case, $4\phi X = \phi X - g(\phi X, AN)AN - g(\phi X, A\xi)A\xi$. Its scalar product with ϕX gives $3 = -g(\phi X, AN)^2 - g(\phi X, A\xi)^2$, which is impossible.

Therefore $SX = kX$ for any $X \in \mathcal{C}$. Take $X, Y \in \mathcal{C}$ in (3.10). This yields $g((\phi S + S\phi)X, \bar{R}_N Y)\xi - 4g((\phi S + S\phi)X, Y)\xi = 0$. That is, $2kg(\phi X, \bar{R}_N Y)\xi - 8kg(\phi X, Y)\xi = 0$ for any $X, Y \in \mathcal{C}$. Therefore $g(\phi X, \bar{R}_N Y) = 4g(\phi X, Y)$. Taking $Y = \phi X$, we arrive to the same contradiction, finishing the proof. □

Lemma 4.3. *There do not exist Hopf real hypersurfaces in Q^m, $m \geq 3$, such that $\mathcal{L}\bar{R}_N = \mathcal{L}^{(k)}\bar{R}_N$ for some non-zero real constant k if N is $\mathfrak{A}$-principal.*

Proof. As we suppose N is $\mathfrak{A}$-principal, we can write $AN = N$, $A\xi = -\xi$ and $\bar{R}_N\xi = 2\xi$. We also know that α is constant, and that if $X \in \mathcal{C}$ satisfies $SX = \lambda X$, then $S\phi X = \mu\phi X$, with $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$.

Let $\{E_1, ..., E_{2m-2}\}$ be an orthonormal basis of eigenvectors of S in $\mathcal{C}$ such that $SE_i = \lambda_i E_i$, $i = 1, ..., 2m-2$. For any $X \in \mathcal{C}$, $\bar{R}_N X = X + AX$. As there exists $Y \in \mathcal{C}$ such that $AY = -Y$, for such a vector field, $\bar{R}_N Y = 0$. For such a Y and $X \in \mathcal{C}$, (3.10) yields $g((\phi S + S\phi)X, Y) = 0$. Therefore $(\lambda_i + \mu_i)g(\phi E_i, Y) = 0$, for any $i = 1, ..., 2m-2$. As $\{\phi E_1, ..., \phi E_{2m-2}\}$ is also an orthonormal basis of $\mathcal{C}$, there exists $j \in \{1, ..., 2m-2\}$ such that $g(\phi E_j, Y) \neq 0$. Therefore $\lambda_j + \mu_j = 0$.

From the Codazzi equation,

$$\begin{aligned}
&g((\nabla_{E_j}S)\phi E_j - (\nabla_{\phi E_j}S)E_j, \xi) = -2g(\phi E_j, \phi E_j) = -2\\
&\quad = g(\nabla_{E_j}(-\lambda_j\phi E_j) - S\nabla_{E_j}\phi E_j - \nabla_{\phi E_j}(\lambda_j E_j) + S\nabla_{\phi E_j}E_j, \xi)\\
&\quad = \lambda_j g(\phi E_j, \phi S E_j) + \alpha g(E_j, \phi S E_j) + \lambda_j g(E_j, \phi S\phi E_j) - \alpha g(E_j, \phi S\phi E_j)\\
&\quad = \lambda_j^2 + \alpha\lambda_j + \lambda_j^2 - \alpha\lambda_j = 2\lambda_j^2,
\end{aligned}$$

which is impossible and finishes the proof. □

The proof of Theorem 1.2 for Hopf real hypersurfaces follows from the Lemmas above.

5. Proof of Theorem 1.2. The case of non-Hopf real hypersurfaces

If M is not Hopf at $[z]$, we write $S\xi = \alpha\xi + \beta U$, where U is a unit vector in $\mathcal{C}$, and β is a nonzero number. Let us call $\mathcal{C}_U = \{X \in \mathcal{C} | g(X,U) = g(X,\phi U) = 0\}$. We will prove the following:

Lemma 5.1. *Let M be a non-Hopf real hypersurface in Q^m, $m \geq 3$, such that $\mathcal{L}\bar{R}_N = \mathcal{L}^{(k)}\bar{R}_N$, for some non-zero real constant k. Then N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. As M is non-Hopf, (3.9) becomes

$$\begin{aligned}&\beta g(\phi U, \bar{R}_N Y)\xi - \beta\eta(\bar{R}_N Y)\phi U - k\phi\bar{R}_N Y + \phi S\bar{R}_N Y \\ &\quad - \beta g(\phi U, Y)\bar{R}_N\xi + \beta\eta(Y)\bar{R}_N\phi U + k\bar{R}_N\phi Y - \bar{R}_N\phi SY = 0 \qquad (5.1)\end{aligned}$$

for any Y tangent to M. Taking $Y = \xi$ in (5.1), we get $\beta g(\phi U, \bar{R}_N\xi)\xi - \beta\eta(\bar{R}_N\xi)\phi U - k\phi\bar{R}_N\xi + \phi S\bar{R}_N\xi = 0$. Its scalar product with ξ gives

$$g(\phi U, \bar{R}_N\xi) = 0, \qquad (5.2)$$

that is, $2\cos(2t)g(A\phi U, \xi) = 0$. Therefore, if $\cos(2t) = 0$, N is $\mathfrak{A}$-isotropic. Thus we suppose $\cos(2t) \neq 0$, and then

$$g(A\phi U, \xi) = 0. \qquad (5.3)$$

From (5.2) the above expression becomes

$$-\beta\eta(\bar{R}_N\xi)\phi U - k\phi\bar{R}_N\xi + \phi S\bar{R}_N\xi = 0. \qquad (5.4)$$

Its scalar product with ξ, bearing in mind (5.2), yields

$$g(\bar{R}_N\xi, S\phi U) = 0, \qquad (5.5)$$

and its scalar product with $X \in \mathcal{C}_U$ implies

$$kg(\bar{R}_N\xi, \phi X) - g(\bar{R}_N\xi, S\phi X) = 0 \qquad (5.6)$$

for any $X \in \mathcal{C}_U$.

The scalar product of (3.10) and ϕU yields

$$\begin{aligned}&- \eta(\bar{R}_N Y)g(SX, U) + k\eta(\bar{R}_N Y)g(X, U) \\ &\quad + \eta(Y)g(\bar{R}_N\phi SX, \phi U) - k\eta(Y)g(\bar{R}_N\phi X, \phi U) = 0\end{aligned}$$

for any $X \in \mathcal{C}$, Y tangent to M. If also $Y \in \mathcal{C}$, we get $-\eta(\bar{R}_N Y)(g(SX,U) - kg(SX,U)) = 0$ for any $X, Y \in \mathcal{C}$. Therefore, if for any $Y \in \mathcal{C}$, $g(Y, \bar{R}_N\xi) = 0 = 2\cos(2t)g(Y, A\xi)$, as $\cos(2t) \neq 0$, we obtain $g(Y, A\xi) = 0$ for any $Y \in \mathcal{C}$, and therefore N is $\mathfrak{A}$-principal. Let us suppose now that there exists $Z \in \mathcal{C}$ such that $\eta(\bar{R}_N Z) \neq 0$, and that for any $X \in \mathcal{C}$, $g(SU, X) = kg(U, X)$. That is,

$$SU = g(SU,\xi)\xi + g(SU,U)U = \beta\xi + kU. \tag{5.7}$$

Taking $X = U$ in (3.10), we get $g((\phi S + S\phi)U, \bar{R}_N Y)\xi - g((\phi S + S\phi)U, Y)$ $\bar{R}_N\xi = 0$, for any Y tangent to M. Its scalar product with Z yields $g((\phi S + S\phi)U, Y) = 0$ for any Y tangent to M. Therefore $S\phi U = -\phi SU$, that is,

$$S\phi U = -k\phi U. \tag{5.8}$$

Take $X = \phi U$ in (3.10). Then $-2k\eta(\bar{R}_N Y) - g((\phi S + S\phi)\phi U, Y)\eta(\bar{R}_N U) = 0$, that is,

$$-2k\eta(\bar{R}_N Y) - g(\phi S\phi U, Y)\eta(\bar{R}_N U) + g(SU, Y)\eta(\bar{R}_N U) = 0$$

for any $Y \in \mathcal{C}$. This implies $-2k\eta(\bar{R}_N Y) = 0$ for any $Y \in \mathcal{C}$, which contradicts the existence of Z and finishes the proof. □

From Lemma 5.1, N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. Suppose first that N is $\mathfrak{A}$-isotropic. Then $\bar{R}_N\xi = 4\xi$. Moreover, $\bar{R}_N\phi U = \phi U - g(A\phi U, N)AN - g(A\phi U, \xi)A\xi$. If (5.1) is satisfied, we have $\beta g(\phi U, \bar{R}_N\phi U)\xi - k\phi\bar{R}_N\phi U + \phi S\bar{R}_N\phi U - \beta\bar{R}_N\xi - k\bar{R}_N U - \bar{R}_N\phi S\phi U = 0$. Its scalar product with ξ yields $\beta g(\phi U, \bar{R}_N\phi U) - 4\beta = 0$. That is, $4 = g(\bar{R}_N\phi U, \phi U) = 1 - g(A\phi U, N)^2 - g(A\phi U, \xi)^2$, which is impossible.

Let us suppose now N is $\mathfrak{A}$-principal. In this case, $AN = N$, $A\xi = -\xi$, $\bar{R}_N\xi = 2\xi$, and for any $X \in \mathcal{C}$, $\bar{R}_N X = X + AX$.

Take $Y = \phi U$ in (5.1). We obtain

$$\beta g(\phi U, \bar{R}_N\phi U)\xi - k\phi\bar{R}_N\phi U + \phi S\bar{R}_N\phi U - \beta\bar{R}_N\xi - k\bar{R}_N U - \bar{R}_N\phi S\phi U = 0. \tag{5.9}$$

Its scalar product with ξ gives $g(\phi U, \bar{R}_N\phi U) = 2 = 1 + g(A\phi U, \phi U)$. This yields $g(A\phi U, \phi U) = 1$, which implies $A\phi U = \phi U$. Therefore $\bar{R}_N\phi U = 2\phi U$. As $A\phi U = \phi U$, we have $AJU = JU = -JAU$. This gives $AU = -U$ and $\bar{R}_N U = 0$. Thus (5.9) becomes $2kU + 2\phi S\phi U - \bar{R}_N\phi S\phi U = 0$. Its scalar product with U implies $2k - 2g(S\phi U, \phi U) = 0$. That is,

$$g(S\phi U, \phi U) = k. \tag{5.10}$$

Taking $Y = U$ in (5.1), we have $k\bar{R}_N\phi U - \bar{R}_N\phi SU = 0$. Its scalar product with ϕU gives $2k - 2g(\phi SU, \phi U) = 0 = 2k - 2g(SU, U)$. Then

$$g(SU, U) = k. \tag{5.11}$$

Taking $Y = U$ in (3.10), we have $-g((\phi S + S\phi)X, U)\bar{R}_N\xi = 0$, that is, $-2g((\phi S + S\phi)X, U)\xi = 0$ for any $X \in \mathcal{C}$. If $X = \phi U$, we obtain $g(\phi S\phi U, U) + g(S\phi^2 U, U) = 0 = -g(S\phi U, \phi U) - g(SU, U) = -2k$, which is impossible, finishing the proof of Theorem 1.2.

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