An annihilator condition on Leavitt path algebras

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Abstract. Let E be a row-finite graph, and let K be a field. In the present paper, necessary and sufficient conditions are established for E to get the Leavitt path algebra $L_K(E)$ which satisfies Property (A).

1. Introduction and preliminaries

The Leavitt path algebra $L_K(E)$ has its origins in works of ABRAMS and ARANDA PINO [1], [2], and ARA, MORENO and PARDO [8]. In the first two mentioned works, the authors provide characterizations of the simplicity and purely infinite simplicity, respectively, of the Leavitt path algebra $L_K(E)$ in terms of properties of the graph E only. That result in some way determines one of the directions of research in the area, which aims to express the properties of rings in graph language. In the third paper, Ara et al. explicitly described the natural isomorphism between the lattice of graded ideals of the Leavitt path algebra $L_K(E)$ and the lattice of order ideals of the monoid $V(L_K(E))$. The object discussed is of great interest to researchers, as evidenced by the large number of published articles and the multitude of results obtained. It is worth mentioning that the Leavitt path algebra $L_K(E)$ is the algebraic analogue of the Cuntz–Krieger algebra $C^*(E)$ considered in [24].

A commutative ring R has *Property* (A) if every finitely generated ideal of R consisting entirely of zero-divisors has a non-zero annihilator. This property was

introduced by HUCKABA and KELLER [19], and has been called Condition (C) by Quentel. The class of commutative rings with Property (A) is quite large and has been studied by many authors (see [7], [12], [15], [19]–[20], [22]–[23]). Polynomial rings, rings whose classical ring of quotients is von Neumann regular, Noetherian rings [21, p. 56] and rings whose prime ideals are maximal [15] are well-known examples of rings in this class. On the other hand, KAPLANSKY [21, p. 63] showed that there are non-Noetherian rings which do not have Property (A). As an application of the considered property we want to mention that HINKLE and HUCKABA [16] using it widened the concept of Kronecker function rings from integral domains to rings with zero-divisors.

In [17], HONG *et al.* extended Property (A) to the non-commutative setting as follows: a ring R has *right (left) Property* (A) if every finitely generated two-sided ideal of R consisting entirely of left (right) zero-divisors has a right (left) non-zero annihilator. A ring R is said to have Property (A) if R has right and left Property (A). By [17, Example 1.2] Property (A) is not left-right symmetric.

Another condition often considered in tandem with Property (A) (in the commutative case see [19], [22], and in the non-commutative case see [18], [25]), is the following: a ring R has *right annihilator condition* (for short, R has right (a.c.)), if, for every 2-generated right ideal $J = aR + bR$ of R, there is $c \in R$ such that $\operatorname{ann}_r^R(J) = \operatorname{ann}_r^R(cR)$. Left (a.c.) is defined similarly.

By [5] for an arbitrary graph E and a field K, in the Leavitt path algebra $L_K(E)$ all finitely generated one-sided ideals are principal (such rings in the literature are called Bézout). Thus $L_K(E)$ satisfies always right and left (a.c.).

By the above it is natural, and it is our main motivation, to ask about Property (A) in the context of Leavitt path algebras. This motivation is reinforced by the fact that every commutative Bézout ring R satisfies Property (A), which is easy to see.

It occurs that we are able to give the full answer to the described problem. More precisely, we give necessary and sufficient conditions on a row-finite graph E to get $L_K(E)$ with Property (A) (see Theorems 2.11 and 2.12). This allows us to construct an example of a non-commutative algebra which is Bézout and does not satisfy left and right Property (A) (see Example 2.13). But we want to stress that the mentioned example is only the consequence of our main findings and it is not the main purpose of the work.

Now we want to recall some basic definitions.

A directed graph $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. For each edge e, $r(e)$ is the range of e and $s(e)$ is the source of e.

A vertex which emits no edges is called a *sink*. A graph is called *row-finite* if $s^{-1}(v)$ is a finite set for each vertex v.

A *path* π in a graph E is a sequence of edges $\pi = e_1 e_2 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. We define $s(\pi) = s(e_1)$ and $r(\pi) = r(e_n)$. If $r(\pi) = s(\pi)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then π is called *a cycle*. An edge e is an *exit* of a cycle $\pi = e_1 e_2 \ldots e_n$ if $s(e) = s(e_i)$ for some i, and $e \neq e_i$. If π is a cycle in E, then by $E^0(\pi)$ we denote the set of all vertices which are sources of edges appearing in π. The length of a path π is denoted by $|\pi|$.

As in many places in the literature, we will consider the relation $\geq$ on E^0 in the following way: $v \geq w$ for $v, w \in E^0$ if there is a path $\sigma \in E$ (possibly empty) such that $s(\sigma) = v$ and $r(\sigma) = w$. If S is a subset of E^0, then for a vertex v, $v \geq S$ (resp., $S \geq v$) means that there is $w \in S$ such that $v \geq w$ (resp., $w \geq v$). For sets $S, S' \subseteq E^0$, $S \geq S'$ means that there is $v \in S$ such that $v \geq S'$. By $H(S)$ we denote the set of all vertices $u \in E^0$ such that $S \geq u$.

For a ring R, the right (left) annihilator of a set $X \subseteq R$ is denoted by $\operatorname{ann}_r^R(X)$ $(\operatorname{ann}_\ell^R(X))$.

Let $E = (E^0, E^1, r, s)$ be a directed graph, and let K be a field. We define *the Leavitt path K-algebra $L_K(E)$ ($L(E)$ for short)* as the K-algebra generated by the set E^0 together with $\{e, e^* : e \in E^1\}$ which satisfy the following relations:

(1) $vv' = \delta_{v,v'} v$ for all $v, v' \in E^0$,

(2) $s(e)e = er(e) = e$ for all $e \in E^1$,

(3) $e^* s(e) = r(e)e^* = e^*$ for all $e \in E^1$,

(4) $e^* f = \delta_{e,f} r(f)$ for all $e, f \in E^1$,

(5) $v = \sum_{\{e \in E^1 : s(e) = v\}} ee^*$ for every vertex v which is not a sink and emits finite number of edges.

For an edge $e \in E^1$, the element e^* is called a *ghost-edge*, and for a path $\sigma = e_1 e_2 \ldots e_n$ we denote the so called *ghost-path* $e_n^* e_{n-1}^* \ldots e_1^*$ by σ^*.

For the general notation, terminology and results in Leavitt path algebras, we refer to [1], [4], [6] and [10].

We will need the following lemma which can be found in [1].

Lemma 1.1. *Let E be a graph, and let K be a field. Then every monomial in $L_K(E)$ is of the following form:*

(a) *kv with $k \in K$ and $v \in E^0$, or*

(b) *$k e_{i_1} \cdot \ldots \cdot e_{i_n} f_{j_1}^* \cdot \ldots \cdot f_{j_m}^*$ with $k \in K$, $n, m \geq 0$, $n + m > 0$, $e_{i_k}, f_{j_\ell} \in E^1$ for $1 \leq k \leq n, 1 \leq \ell \leq m$.*

We will say that for the paths $\sigma = e_1 e_2 \ldots e_n, \tau = f_1 f_2 \ldots f_m$, an edge e is *a factor* of the monomial $\sigma \cdot \tau^*$ if either $e = e_i$ for some i, or $e = f_j$ for some j.

We leave the proof of the following lemma to the reader.

Lemma 1.2. *Let E be a graph. For the vertices $v, w \in E^0$, there exist paths σ and τ such that $v \sigma \tau^* w \neq 0$ if and only if there is $u \in E^0$ such that $v \geq u$ and $w \geq u$ (in other words $H(v) \cap H(w) \neq \emptyset$).*

By [13, Theorem 8] every ideal I of $L(E)$ is generated by elements of the form $v + \sum_{i=1}^{q} k_i \pi^i$, for some $q \geq 1$, where $v \in E^0$, π is a cycle based at v and $k_i \in K$ (hoping that this will not lead to a misunderstanding, the above generators will be denoted by $v + \sum_i k_i \pi^i$). For a set T of generators of I which are of the above form, we consider the set

$$E_T^0(I) = \left\{ v \in E^0 : v + \sum_i k_i \pi^i \in T \text{ for some cycle } \pi \text{ based at } v \text{ and } k_i \in K \right\}.$$

Theorem 1.3. *Let E be a row-finite graph, K a field, and let I be a non-zero ideal of $L_K(E)$. Then the following are equivalent:*

(1) *The right annihilator of I is equal to 0; $\mathrm{ann}_r^{L_K(E)}(I) = 0$.*

(2) *The left annihilator of I is equal to 0; $\mathrm{ann}_\ell^{L_K(E)}(I) = 0$.*

(3) *For any set T of generators of I which consists of elements of the form $v + \sum_i k_i \pi^i$, where $v \in E^0$, π is a cycle based at v and $k_i \in K$, and for any vertex $u \in E^0$, $H(E_T^0(I)) \cap H(u) \neq \emptyset$.*

PROOF. (1) $\Rightarrow$ (3). Let I be an ideal of $L(E)$ with $\mathrm{ann}_r^{L(E)}(I) = 0$. For a contradiction, suppose that for a set T consisting of elements of the form $v + \sum_i k_i \pi^i$ and generating I, $H(E_T^0(I)) \cap H(u) = \emptyset$ for some $u \in E^0$. Then taking any element $v + \sum_i k_i \pi^i \in T$ and a monomial $\sigma \tau^*$ of the form presented in Lemma 1.1, we can see that $(v + \sum_i k_i g^i) \sigma \tau^* u = 0$. Indeed, as $L(E)$ is $\mathbb{Z}$-graded (see [1, Lemma 1.7]), it is enough to show that $v \sigma \tau^* u = 0$ and for any i, $g^i \sigma \tau^* u = 0$. But this follows, as for any i, $r(g^i) = v$ and we assumed $H(E_T^0(I)) \cap H(u) = \emptyset$, which in particular gives $H(v) \cap H(u) = \emptyset$. Thus $Iu = 0$, a contradiction.

(3) $\Rightarrow$ (1). Assume that (3) holds, and to get a contradiction, suppose that $Q = \mathrm{ann}_r^{L(E)}(I)$ is not equal to 0. It is easy to see that Q is an ideal of $L(E)$. Let $u + \sum_j r_j \xi^j$ with a vertex u and a cycle ξ based at u, be one of non-zero elements generating Q. By assumption, there is $v + \sum_i k_i \pi^i \in I$ such that for some $w \in E^0$, $v \geq w$ and $u \geq w$. Let α, β be paths such that $s(\alpha) = u$, $r(\alpha) = w$

and $s(\beta) = v$, $r(\beta) = w$. As $L(E)$ is $\mathbb{Z}$-graded and

$$\left(v + \sum_i k_i \pi^i\right) \beta \alpha^* \left(u + \sum_j r_j \xi^j\right) = 0,$$

we have $\beta \alpha^* = v \beta \alpha^* u = 0$, a contradiction. Thus $Q = 0$.

Since $(2) \Leftrightarrow (3)$ can be proved in a similar way as the previous equivalence, the proof is finished. □

We want to mention that another way of proving the above theorem is to use the fact that for an arbitrary graph E and a field K, $L_K(E)$ is nonsingular (see [6, Proposition 2.3.7]).

2. Leavitt path algebras satisfying Property (A)

It is well known (see [9, Corollary 3.5]) that if E is a finite acyclic graph, then $L_K(E)$ is a K-matricial algebra, which means that $L(E) \cong \bigoplus_{i=1}^{t} M_{n_i}(K)$ for some positive integers $t, n_1, \ldots, n_t$. Thus in this case $L_K(E)$ satisfies Property (A) by [17, Proposition 1.3]. Therefore, in this section, considering graphs with a finite number of vertices, we focus on graphs with at least one non-trivial cycle.

Lemma 2.1. *Let E be a row-finite graph with finite E^0, and let K be a field. If E has a cycle π and a vertex $v \in E^0$ such that $E^0(\pi) \geq v$ and $v \not\geq E^0(\pi)$, then $L_K(E)$ has neither right nor left Property (A).*

PROOF. Let v be a vertex, and π be a cycle based at a vertex v', which satisfy conditions presented above. Assume that ϵ is a path such that $s(\epsilon) = v'$ and $r(\epsilon) = v$. Let

$$P(v) = \{w \in E^0 : w \geq H(v)\},$$

and for $T = E^0 \setminus P(v)$ consider the ideal I of $L(E)$ generated by the finite set $\{v\} \cup T$. Notice that we have $E^0(\pi) \subseteq P(v)$. Let x be a non-zero element of I. Then

$$x = k_v v + \sum_{q \in T} k_q q + \sum_{i=1}^{n} k_i \cdot \alpha_i \overline{\alpha}_i^* \cdot v \cdot \beta_i \overline{\beta}_i^* + \sum_{p \in T} \sum_{j=1}^{m_p} k_{pj} \cdot \sigma_{pj} \overline{\sigma}_{pj}^* \cdot p \cdot \delta_{pj} \overline{\delta}_{pj}^*,$$

where $\alpha_i, \overline{\alpha}_i, \beta_i, \overline{\beta}_i, \sigma_{pj}, \overline{\sigma}_{pj}, \delta_{pj}, \overline{\delta}_{pj}$ are paths (possibly empty), $k_\mu, k_{\zeta\eta}$ are elements of K, and n, m_p are positive integers. Let

$$\overline{s} = \max(\{|\overline{\beta}_i| : i = 1, \ldots, n\} \cup \{|\overline{\delta}_{pj}| : p \in T, j = 1, \ldots, m_p\}),$$

and let

$$s = \max(\{|\beta_i| : i = 1, \ldots, n\} \cup \{|\delta_{pj}| : p \in T, j = 1, \ldots, m_p\}).$$

Take $p \in T$ and $j \in \{1, \ldots, m_p\}$, and consider the product

$$\sigma_{pj}\overline{\sigma}_{pj}^* \cdot p \cdot \delta_{pj}\overline{\delta}_{pj}^* \cdot \pi^{2\overline{s}}.$$

As $|\overline{\delta}_{pj}| \leq \overline{s}$ and $|\pi^{2\overline{s}}| \geq 2\overline{s}$, we can see that either this product is zero or there is a path γ such that

$$\sigma_{pj}\overline{\sigma}_{pj}^* \cdot p \cdot \delta_{pj}\overline{\delta}_{pj}^* \cdot \pi^{2\overline{s}} = \sigma_{pj}\overline{\sigma}_{pj}^* \cdot p \cdot \gamma \cdot \pi \neq 0.$$

If the latter holds, then $p \geq v' \geq v$, which means that $p \in P(v)$, a contradiction. Thus

$$\left(\sum_{p \in T}\sum_{j=1}^{m_p} k_{pj} \cdot \sigma_{pj}\overline{\sigma}_{pj}^* \cdot p \cdot \delta_{pj}\overline{\delta}_{pj}^*\right) \cdot \pi^{2\overline{s}} = 0.$$

By similar arguments, we get

$$k_v v \cdot \pi^{2\overline{s}} = \left(\sum_{q \in T} k_q q\right) \cdot \pi^{2\overline{s}} = \left(\sum_{i=1}^{n} k_i \cdot \alpha_i\overline{\alpha}_i^* \cdot v \cdot \beta_i\overline{\beta}_i^*\right) \cdot \pi^{2\overline{s}} = 0,$$

and finally $x \cdot \pi^{2\overline{s}} = 0$. Also, using arguments as above, one can see that $(\pi^{2s})^* \cdot x = 0$. Thus we showed that any element of I is a left and right zero-divisor.

As it is easy to see, using Theorem 1.3, that $\text{ann}_r^{L(E)}(I) = 0 = \text{ann}_\ell^{L(E)}(I)$, $L(E)$ has neither right nor left Property (A). □

Inspired by [11, Definition 2.1], we formulate the following.

Definition 2.2. Let E be a row-finite graph. Suppose that there are subsets E_1^0, E_2^0, E_3^0 of E^0 satisfying the following conditions:
   (i) $E_1^0 \cup E_2^0 \cup E_3^0 = E^0$,
  (ii) $E_i^0 \cap E_j^0 = \emptyset$ for any $i \neq j$,
 (iii) E_1^0 is not empty set, and E_3^0 is finite,
  (iv) $E_1^0 \not\geq E_2^0 \cup E_3^0$ and $E_2^0 \not\geq E_1^0 \cup E_3^0$,
   (v) for each $v \in E_3^0$, $v \geq E_1^0$ and $v \geq E_2^0$,
  (vi) for each cycle π in E, $E^0(\pi) \subseteq E_1^0 \cup E_2^0$.
Then the triple (E_1^0, E_2^0, E_3^0) will be called a partition of E^0.

For a row-finite graph E, the partition $(E^0, \emptyset, \emptyset)$ will be called a *trivial partition*.

Remark 2.3. (a) Notice that by points (iv) and (vi), if π is a cycle in E and (E_1^0, E_2^0, E_3^0) is a partition of E^0, then either $E^0(\pi) \subseteq E_1^0$ or $E^0(\pi) \subseteq E_2^0$.

(b) As it will be presented in Example 2.7, a non-trivial partition of E^0 is not uniquely determined.

(c) Notice that if $E_2^0 = \emptyset$, then also $E_3^0 = \emptyset$ (see Definition 2.2(v)). On the other hand, it is possible that $E_3^0 = \emptyset$ and $E_2^0 \neq \emptyset$. In this case, E is a disconnected graph and it should be clear that by properties presented in the above definition and well-known facts on decompositions of Leavitt path algebras of disconnected graphs, $L(E) \cong L(F_1) \times L(F_2)$, where for $i = 1, 2$, F_i is the subgraph of E such that $F_i^0 = E_i^0$ and F_i^1 is the set of all edges e appearing in E with $s(e), r(e) \in E_i^0$.

Proposition 2.4. *If E is a row-finite graph and (E_1^0, E_2^0, E_3^0) is not the trivial partition of E^0, then there are proper subgraphs $\mathcal{E}_1$ and $\mathcal{E}_2$ of E such that $L(E) \cong L(\mathcal{E}_1) \times L(\mathcal{E}_2)$.*

PROOF. By Remark 2.3(c), the thesis follows if E_3^0 is an empty set. Therefore, in the rest of the proof, we assume that E_3^0 is not an empty set, which implies that also E_2^0 is not empty.

Let for $i = 1, 2, 3$,
$$E_i^1 = \{e \in E^1 : r(e) \in E_i^0\}.$$

Then $E_i^1 \cap E_j^1 = \emptyset$ for any $i \neq j$ and $E_1^1 \cup E_2^1 \cup E_3^1 = E^1$.

Notice that if an edge e belongs to E_1^1, then $r(e) \in E_1^0$, and it follows that $s(e) \in E_1^0 \cup E_3^0$. If $e \in E_3^1$, then $s(e), r(e) \in E_3^0$. Thus $\mathcal{E}_1 = (E_1^0 \cup E_3^0, E_1^1 \cup E_3^1)$ is a proper subgraph of E. By similar reasons, $\mathcal{E}_2 = (E_2^0 \cup E_3^0, E_2^1 \cup E_3^1)$ is a proper subgraph of E. We will show that $L(E) \cong L(\mathcal{E}_1) \times L(\mathcal{E}_2)$.

We define a map $\phi : L(E) \to L(\mathcal{E}_1) \times L(\mathcal{E}_2)$ on generators of $L(E)$ as follows, and then we extend it linearly and multiplicatively (see [11, Proposition 2.4]). For a vertex $v \in E^0$ and an edge $e \in E^1$,

$$\phi(v) = \begin{cases} (v, 0), & \text{if } v \in E_1^0 \\ (0, v), & \text{if } v \in E_2^0 \\ (v, v), & \text{if } v \in E_3^0 \end{cases}, \quad \phi(e) = \begin{cases} (e, 0), & \text{if } e \in E_1^1 \\ (0, e), & \text{if } e \in E_2^1 \\ (e, e), & \text{if } e \in E_3^1 \end{cases},$$

$$\phi(e^*) = \begin{cases} (e^*, 0), & \text{if } e \in E_1^1 \\ (0, e^*), & \text{if } e \in E_2^1 \\ (e^*, e^*), & \text{if } e \in E_3^1 \end{cases}. \tag{2.1}$$

Now, we will show that ϕ preserves relations in $L(E)$, which together with the above will guarantee that ϕ is an algebra homomorphism.

Let $e \in E_3^1$. Then $r(e) \in E_3^0$, so it must be also $s(e) \in E_3^0$. Thus direct calculation shows that $\phi(s(e)e) = \phi(e) = \phi(er(e))$. Also direct calculations show that the appropriate relations are preserved by ϕ for $e \in E_1^1 \cup E_2^1$ and for ghost-edges e^*.

Clearly, if $e \neq e'$ for edges e, e', then $\phi(e^* e') = 0$. Whereas for an edge e, $\phi(e^* e) = \phi(r(e))$. Indeed, if $e \in E_1^1$, then $r(e) \in E_1^0 \subseteq \mathcal{E}_1^0$ and $\phi(e^* e) = \phi(e^*)\phi(e) = (e^*, 0)(e, 0) = (e^* e, 0) = (r(e), 0) = \phi(r(e))$. The same holds for $e \in E_2^1$. For $e \in E_3^1$, we have $r(e) \in E_3^0$. As $E_3^0 \subseteq \mathcal{E}_1^0 \cap \mathcal{E}_2^0$ and $E_3^1 \subseteq \mathcal{E}_1^1 \cap \mathcal{E}_2^1$, we get $\phi(e^* e) = \phi(e^*)\phi(e) = (e^*, e^*)(e, e) = (e^* e, e^* e) = (r(e), r(e)) = \phi(r(e))$.

Suppose now that v is a vertex of E which is not a sink. Then in $L(E)$ we have

$$v = \sum_{\{e \in E^1 : s(e) = v\}} ee^*.$$

We will only consider the case $v \in E_3^0$, because the other cases result from similar considerations. Let $e_1, \ldots, e_k$ be all edges such that $s(e_i) = v$ and $r(e_i) \in E_1^0$, let $f_1, \ldots f_t$ be all edges such that $s(f_j) = v$ and $r(f_j) \in E_2^0$, and finally let $h_1, \ldots h_s$ be all edges such that $s(h_\ell) = v$ and $r(h_\ell) \in E_3^0$. Then

$$v = \sum_{i=1}^{k} e_i \cdot e_i^* + \sum_{j=1}^{t} f_j \cdot f_j^* + \sum_{\ell=1}^{s} h_\ell \cdot h_\ell^*,$$

and by definition,

$$\phi\left(\sum_{i=1}^{k} e_i \cdot e_i^* + \sum_{j=1}^{t} f_j \cdot f_j^* + \sum_{\ell=1}^{s} h_\ell \cdot h_\ell^*\right)$$
$$= \left(\sum_{i=1}^{k} e_i \cdot e_i^* + \sum_{\ell=1}^{s} h_\ell \cdot h_\ell^*, \sum_{j=1}^{t} f_j \cdot f_j^* + \sum_{\ell=1}^{s} h_\ell \cdot h_\ell^*\right) = (v, v) = \phi(v).$$

Thus it follows that ϕ is a well-defined K-algebra homomorphism.

Although injectivity of ϕ can be justified using the Graded Uniqueness Theorem (see [6, Theorem 2.2.15]), for the sake of completeness, we provide a direct proof.

We will need the following pretty obvious fact: for paths σ, δ in $L(E)$, if for $p \in \{1, 2\}$ an edge $e \in E_p^1$ is a factor of $\sigma \cdot \delta^*$, then all factors of $\sigma \cdot \delta^*$ are from $E_p^1 \cup E_3^1$.

To get a contradiction, suppose that there is a non-zero $x \in L(E)$ such that $\phi(x) = 0$. We can see that

$$x = \sum_i k_{1i} u_i + \sum_j k_{2j} v_j + \sum_t k_{3t} w_t + \sum k_1 \alpha_1 \beta_1^* + \sum k_2 \alpha_2 \beta_2^* + \sum k_3 \alpha_3 \beta_3^*,$$

where all k_{pq}, k_g are from K, and for any i, j, t, $u_i \in E_1^0$, $v_j \in E_2^0$ and $w_t \in E_3^0$, respectively. Moreover, in any $\alpha_1 \beta_1^*$, at least one of the factors is from E_1^1, in any $\alpha_2 \beta_2^*$, at least one of the factors is from E_2^1, and in any $\alpha_3 \beta_3^*$, all factors are from E_3^1. By definition of ϕ, we get $\phi(x) = (a, b)$, where

$$a = \sum_i k_{1i} u_i + \sum_t k_{3t} w_t + \sum k_1 \alpha_1 \beta_1^* + \sum k_3 \alpha_3 \beta_3^*,$$

$$b = \sum_j k_{2j} v_j + \sum_t k_{3t} w_t + \sum k_2 \alpha_2 \beta_2^* + \sum k_3 \alpha_3 \beta_3^*,$$

and the fact that $\phi(x) = 0$ gives $a = 0 = b$. As $a = 0$, we have $x = \sum_j k_{2j} v_j + \sum k_2 \alpha_2 \beta_2^*$. Since in any $\alpha_2 \beta_2^*$ there is an edge from E_2^1, we get by definition of ϕ, $\phi(x) = \phi(\sum_j k_{2j} v_j + \sum k_2 \alpha_2 \beta_2^*) = (0, \sum_j k_{2j} v_j + \sum k_2 \alpha_2 \beta_2^*)$, which implies with what we already said that $x = 0$, a contradiction. Thus ϕ is injective.

We will prove that ϕ is surjective. As the main step to achieve the intended goal, we will show that for any $v \in E_3^0$, there are a and b such that $\phi(a) = (v, 0)$, and $\phi(b) = (0, v)$. Let

$$D_0 = \{ v \in E_3^0 : \text{ for each edge } e \text{ such that } s(e) = v, \text{ we have } r(e) \in E_1^0 \cup E_2^0 \}.$$

It is not hard to see that if E_3^0 is not empty, then D_0 is also not empty. Now let k be a positive integer, and suppose that we already defined $D_0, \ldots, D_{k-1}$. Then we define D_k as follows:

$$D_k = \left\{ v \in E_3^0 \setminus \bigcup_{i=0}^{k-1} D_i : \text{ for each edge } e \text{ such that } s(e) = v, \right.$$

$$\left. \text{we have } r(e) \in E_1^0 \cup E_2^0 \cup D_0 \cup \ldots \cup D_{k-1} \right\}.$$

As by Definition 2.2(iii) the set E_3^0 is finite, there is a non-negative integer ℓ such that the sets $D_0, \ldots, D_\ell$ are not empty, and starting from this point, we have $D_{\ell+1} = D_{\ell+2} = \ldots = \emptyset$. Notice that $D_0 \cup D_1 \cup \ldots \cup D_\ell = E_3^0$.

Now, let $v \in D_0$, and let $e_1, \ldots, e_n$ be all edges such that $s(e_i) = v$ and $r(e_i) \in E_1^0$, and let $f_1, \ldots, f_m$ be all such edges that $s(f_j) = v$ and $r(f_j) \in E_2^0$.

Then in $L(E)$, $v = \sum_{i=1}^{n} e_i e_i^* + \sum_{j=1}^{m} f_j f_j^*$, and $\phi(\sum_{i=1}^{n} e_i e_i^*) = (\sum_{i=1}^{n} e_i e_i^*, 0) = (v, 0)$. Also $\phi(\sum_{j=1}^{m} f_j f_j^*) = (0, v)$.

Suppose now that for some positive integer k, if $v' \in D_0 \cup \ldots \cup D_{k-1}$, then there are $a, b \in L(E)$ such that $\phi(a) = (v', 0)$ and $\phi(b) = (0, v')$, and suppose that $D_k \neq \emptyset$. Consider a vertex $v \in D_k$. Then there are vertices $v'_1, \ldots, v'_t \in D_0 \cup \ldots \cup D_{k-1}$ such that for any v'_j there is an edge $\overline{h}_j$, $j = 1, \ldots, t$, such that $s(\overline{h}_j) = v$ and $r(\overline{h}_j) = v'_j$, and any other edge with v as a source has the range in $E_1^0 \cup E_2^0$. Furthermore, for any $j = 1, \ldots, t$, there are $a_j, b_j \in L(E)$ with $\phi(a_j) = (v'_j, 0), \phi(b_j) = (0, v'_j)$. Notice that $\phi(a_j v'_j + b_j v'_j) = \phi(a_j)\phi(v'_j) + \phi(b_j)\phi(v'_j) = (v'_j, 0)(v'_j, v'_j) + (0, v'_j)(v'_j, v'_j) = (v'_j, v'_j) = \phi(v'_j)$, which means that $a_j v'_j + b_j v'_j = v'_j = v'_j a_j v'_j + v'_j b_j v'_j$ for any j.

Let $\overline{e_1}, \ldots, \overline{e_n}$ be all edges such that $s(\overline{e}_i) = v$ and $r(\overline{e}_i) \in E_1^0$, and let $\overline{f}_1, \ldots, \overline{f}_m$ be all such that $s(\overline{f}_j) = v$ and $r(\overline{f}_j) \in E_2^0$. Then

$$v = \sum_i \overline{e_i} \cdot \overline{e_i}^* + \sum_j \overline{f_j} \cdot \overline{f_j}^* + \sum_\ell \overline{h_\ell} \cdot \overline{h_\ell}^*$$

$$= \sum_i \overline{e_i} \cdot \overline{e_i}^* + \sum_j \overline{f_j} \cdot \overline{f_j}^* + \sum_\ell \overline{h_\ell}(v'_\ell a_\ell v'_\ell + v'_\ell b_\ell v'_\ell)\overline{h_\ell}^*$$

$$= \sum_i \overline{e_i} \cdot \overline{e_i}^* + \sum_j \overline{f_j} \cdot \overline{f_j}^* + \sum_\ell \overline{h_\ell}(v'_\ell a_\ell v'_\ell)\overline{h_\ell}^* + \sum_\ell \overline{h_\ell}(v'_\ell b_\ell v'_\ell)\overline{h_\ell}^*,$$

and it follows that

$$\phi(v) = \left(\sum_i \overline{e_i} \cdot \overline{e_i}^* + \sum_\ell \overline{h_\ell}(v'_\ell a_\ell v'_\ell)\overline{h_\ell}^*, \sum_j \overline{f_j} \cdot \overline{f_j}^* + \sum_\ell \overline{h_\ell}(v'_\ell b_\ell v'_\ell)\overline{h_\ell}^* \right).$$

As $\phi(v) = (v, v)$, we get that in $L(\mathcal{E}_1)$, $v = \sum_i \overline{e_i} \cdot \overline{e_i}^* + \sum_\ell \overline{h_\ell}(v'_\ell a_\ell v'_\ell)\overline{h_\ell}^*$, and in $L(\mathcal{E}_2)$, $v = \sum_i \overline{e_i} \cdot \overline{e_i}^* + \sum_\ell \overline{h_\ell}(v'_\ell b_\ell v'_\ell)\overline{h_\ell}^*$, which means that

$$\phi\left(\sum_i \overline{e_i} \cdot \overline{e_i}^* + \sum_\ell \overline{h_\ell}(v'_\ell a_\ell v'_\ell)\overline{h_\ell}^* \right) = (v, 0)$$

and

$$\phi\left(\sum_i \overline{e_i} \cdot \overline{e_i}^* + \sum_\ell \overline{h_\ell}(v'_\ell b_\ell v'_\ell)\overline{h_\ell}^* \right) = (0, v).$$

We have proved above that for any $v \in E_3^0$, there are a and b such that $\phi(a) = (v, 0)$, and $\phi(b) = (0, v)$.

For any edge $e \in E_3^1$, we have $e = er(e)$ and $e^* = r(e)e^*$ with $r(e) \in E_3^0$. Let $\phi(a) = (r(e), 0)$ and $\phi(b) = (0, r(e))$ for $a, b \in L(E)$. Then $\phi(ea) = (e, e)(r(e), 0) = (e, 0)$, and similarly, $\phi(eb) = (0, e)$. Moreover, $\phi(ae^*) = (e^*, 0)$ and $\phi(be^*) = (0, e^*)$. Using (2), (2.1), we just showed that indeed ϕ is surjective. $\qquad\square$

The pair $(\mathcal{E}_1, \mathcal{E}_2)$ of proper subgraphs of E related to a non-trivial partition (E_1^0, E_2^0, E_3^0), which is constructed at the beginning of the above proof, will be called an *iso-partition* of E.

Recall (see [14]) that a cycle π in a graph E is an *extreme cycle* if π has an exit and for any $v \in E^0 \setminus E^0(\pi)$, if $E^0(\pi) \geq v$, then $v \geq E^0(\pi)$. Notice that if π is an extreme cycle, then for a vertex $v \in E^0$, $v \geq E^0(\pi)$ if and only if $v \geq H(E^0(\pi))$.

Lemma 2.5. *Let E be a row-finite graph with finite E^0. Assume that in E there are a cycle π and a vertex v such that $v \not\geq E^0(\pi)$. Moreover, assume that in E every cycle with an exit is an extreme cycle. Then there exists a non-trivial partition (E_1^0, E_2^0, E_3^0) of E^0 and for related iso-partition $(\mathcal{E}_1, \mathcal{E}_2)$ of E, in $\mathcal{E}_1$ and in $\mathcal{E}_2$ any cycle with an exit is an extreme cycle. Moreover, π is a cycle in $\mathcal{E}_1$ and for any vertex $w \in \mathcal{E}_1^0$, $w \geq E^0(\pi)$.*

PROOF. For the cycle π, consider the following sets:

$$B = \{w \in E^0 : w \geq H(E^0(\pi)) \text{ and } H(E^0(\pi)) \not\geq w\}.$$
$$E_2^0 = \{w \in E^0 : \text{ for every } u \in H(E^0(\pi)), w \not\geq u\},$$
$$E_3^0 = \{w \in B : w \geq E_2^0\}, \qquad E_1^0 = H(E^0(\pi)) \cup (B \setminus E_3^0).$$

Notice that $E_1^0 \neq \emptyset$, and it is not difficult to deduce that E_2^0 is also not empty. If $E_3^0 = \emptyset$, then $(E_1^0, E_2^0, \emptyset)$ is a non-trivial partition such that the related iso-partition satisfies required properties (see Remark 2.3(c)). Therefore, in the rest of the proof, we assume that E_3^0 is not empty.
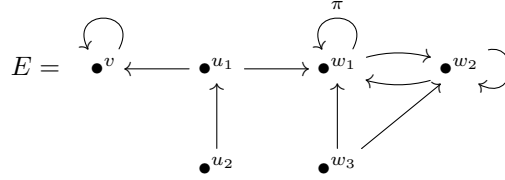
The proofs of the conditions (i)–(v) of Definition 2.2 are straightforward. To see (vi), notice that as in E any cycle with an exit is an extreme cycle, every cycle λ in E must contain an edge e such that $r(e) \in E_1^0 \cup E_2^0$. Indeed, otherwise $E^0(\lambda) \subseteq E_3^0$. By definition of the sets B and E_3^0, there is $c \in H(E^0(\pi))$ such that $E^0(\lambda) \geq c$. Since $E_3^0 \subseteq B$, we have $c \not\geq E^0(\lambda)$, a contradiction.

On the other hand, by definition of E_1^0, if there is an edge e appearing in a cycle λ such that $r(e) \in E_1^0$, then $E^0(\lambda) \subseteq E_1^0$. The same holds for cycles which have an edge f such that $r(f) \in E_2^0$. Thus for each cycle λ in E, either $E^0(\lambda) \subseteq E_1^0$ or $E^0(\lambda) \subseteq E_2^0$.
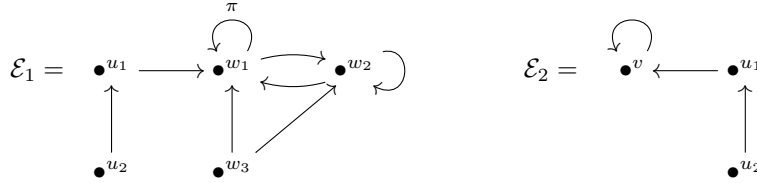
Now, we consider related iso-partition $(\mathcal{E}_1, \mathcal{E}_2)$ of E with $\mathcal{E}_1 = (E_1^0 \cup E_3^0, E_1^1 \cup E_3^1)$, $\mathcal{E}_2 = (E_2^0 \cup E_3^0, E_2^1 \cup E_3^1)$, where for $i = 1, 2, 3$, $E_i^1 = \{e \in E^1 : r(e) \in E_i^0\}$. As $E_i^1 \subseteq \mathcal{E}_i^0$ for $i = 1, 2$, the fact that in $\mathcal{E}_1$ and $\mathcal{E}_2$ any cycle with an exit is an extreme cycle follows from what we said and definitions of E_1^1 and E_2^1. Since the rest follows from the construction of $\mathcal{E}_1$, the proof is complete. $\square$

We would like to emphasize here that the above proof starts with a fixed cycle π, and all constructions are related to π. Therefore (E_1^0, E_2^0, E_3^0) will be called a π-*partition* of E^0 and eventually we will consider π-iso-partition $(\mathcal{E}_1, \mathcal{E}_2)$ of E.

*Example 2.6.* To illustrate the above construction, consider the following graph with the highlighted cycle π
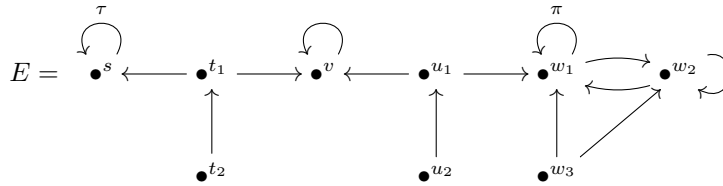
Then we get π-partition (E_1^0, E_2^0, E_3^0) of E^0 with $E_1^0 = \{w_1, w_2, w_3\}, E_2^0 = \{v\}, E_3^0 = \{u_1, u_2\}$. Moreover, we have π-iso-partition $(\mathcal{E}_1, \mathcal{E}_2)$ of E, where

and finally $L(E) \cong L(\mathcal{E}_1) \times L(\mathcal{E}_2)$.

Example 2.7. As mentioned before, for a graph E, a partition of E^0, as defined above, is not unique. Consider the following graph:

Then we get π-partition (E_1^0, E_2^0, E_3^0) with $E_1^0 = \{w_1, w_2, w_3\}$, $E_2^0 = \{v, t_1, t_2, s\}$ and $E_3^0 = \{u_1, u_2\}$.

On the other hand, we consider τ-partition $(\overline{E}_1^0, \overline{E}_2^0, \overline{E}_3^0)$ with

$$\overline{E}_1^0 = \{s\}, \quad \overline{E}_2^0 = \{v, u_1, u_2, w_1, w_2, w_3\} \quad \text{and} \quad \overline{E}_3^0 = \{t_1, t_2\}.$$

It is clear that in any case we get a related iso-partition of E.

We want to recall some well-known definitions.

Definition 2.8. A graph E is said to be *downward directed* if for any vertices $v, w \in E^0$, $H(v) \cap H(w) \neq \emptyset$.

A subset X of E^0 is *hereditary* if $w \in X$ and $w \geq v$ imply $v \in X$.

We say that $X \subseteq E^0$ is *saturated* if whenever $s^{-1}(v) \neq \emptyset$ and $\{r(e) : s(e) = v\} \subseteq X$, then $v \in X$.

Lemma 2.9. *Let E be a row-finite graph with finite E^0, and let K be a field. Then the following conditions are equivalent:*

(1) *Graph E has the following properties:*

   (a) *graph E is downward directed;*

   (b) *every cycle with an exit is an extreme cycle;*

   (c) *if graph E contains cycles, then there is at least one cycle with an exit.*

(2) *Every cycle in E has an exit and E^0 is the only non-empty hereditary and saturated set in E^0.*

(3) *$L_K(E)$ is a simple algebra with an identity.*

PROOF. $(2) \Leftrightarrow (3)$. As E^0 is finite, $L(E)$ has an identity, so the equivalence holds by [1, Theorem 3.11].

$(1) \Rightarrow (2)$. Suppose, for a contradiction, that there exists a cycle κ without an exit. By (c), there is another cycle π, which has an exit f. Notice that $E^0(\pi) \cap E^0(\kappa) = \emptyset$. Indeed, if there is a vertex $v \in E^0(\pi) \cap E^0(\kappa)$, then $s(f) \geq v$. By assumption (b), $v \geq s(f)$ also. Hence, κ has an exit, a contradiction.

Let us consider some vertices $v \in E^0(\kappa)$ and $w \in E^0(\pi)$. By (a) there exists a vertex u, such that $v \geq u$ and $w \geq u$. Since κ does not have an exit, $u \in E^0(\kappa)$, but in E any cycle with an exit is an extreme cycle which yields $v \geq u \geq E^0(\pi)$, a contradiction. We conclude that in E every cycle has an exit.

Now, suppose for a contradiction that there exists a non-empty hereditary and saturated set F that is not equal to E^0. Let $v \in F$. We have that for all cycles π in the considered graph E, $E^0(\pi) \subseteq H(v) \subseteq F$. Indeed, let $u \in E^0(\pi)$.

Then by (a) there is a vertex s such that $s \in H(v) \cap H(u)$. By (b) we have $s \geq E^0(\pi)$. Thus $H(v) \geq E^0(\pi)$, so $E^0(\pi) \subseteq H(v)$.

Note that (a) assures that graph E is not disjoint, and that it cannot contain any isolated vertices. Moreover, for any sink $w \in E^0$, $H(w) \cap H(v) \neq \emptyset$ implies $w \in H(v) \subseteq F$. So F contains all the sinks and all the cycles in E.

As E is finite, by using what we have proved, it is not hard to see that there exists a vertex $w \in E^0 \setminus F$ which is not a sink and any edge that it emits is ranged at a vertex in F. Then as F is saturated, we get $w \in F$, a contradiction.

$(3) \Rightarrow (1)$. Since we have already $(2) \Leftrightarrow (3)$, E satisfies property (c), and as $L(E)$ has an identity, (a) follows from Theorem 1.3.

If (b) does not hold, then there is a cycle π and a vertex v such that $E^0(\pi) \geq v$ and $v \not\geq E^0(\pi)$. Let

$$Q(\pi) = \{u \in E^0 : u \geq E^0(\pi)\}.$$

It is easy to check, that $E^0 \setminus Q(\pi)$ is hereditary. It is also saturated, because for any vertex in $Q(\pi)$, there exists a path to a vertex of $E^0(\pi)$. Notice that $v \in E^0 \setminus Q(\pi)$, so the set $E^0 \setminus Q(\pi)$ is non-empty. By (2), which is equivalent to (3), we have $Q(\pi) = \emptyset$, a contradiction, as $E^0(\pi) \subseteq Q(\pi)$. $\qquad \square$

The following lemma will be useful in our further considerations.

Lemma 2.10. *Let E be a row-finite graph with E^0 finite, and let K be a field. Suppose that there is a cycle π in E such that for any $v \in E^0$, $v \geq E^0(\pi)$. Moreover, suppose that in E any cycle with an exit is an extreme cycle. Then $L(E)$ has Property (A).*

Proof. If π has an exit, then using Lemma 2.9 we can see that $L(E)$ is a simple algebra with 1. Thus $L(E)$ has Property (A). If π does not have an exit, then π is the only cycle in E (justification for this fact is presented below). Thus if $v \in E^0 \setminus E^0(\pi)$, then we have $v \geq E^0(\pi)$ and $E^0(\pi) \not\geq v$. Hence by [3, Theorem 3.3], $L(E) \cong M_d(K[x, x^{-1}])$ for some d. Then using [17, Proposition 1.3 and Theorem 2.1], we can see that $L(E)$ has Property (A).

To see that π is the only cycle in E if π does not have an exit, suppose that there is a cycle π' in E such that $\pi' \neq \pi$. It is not hard to see that there is an edge e which appears in π' and does not appear in π (remember that π has no exits). If $s(e) \in E^0(\pi)$, then π has an exit, a contradiction. Thus $s(e) \in E^0 \setminus E^0(\pi)$. By assumption, $s(e) \geq E^0(\pi)$. It implies that for any vertex $w \in E^0(\pi)$, $E^0(\pi') \geq w$, and π' has an exit. As in E any cycle with an exit is an extreme cycle, we also have $w \geq E^0(\pi')$, and $w \geq s(e)$. In particular, it follows that π has an exit, a contradiction. $\qquad \square$

Collecting all information received so far, we are ready to prove the main results of this paper.

Theorem 2.11. *Let K be a field. If E is a row-finite graph and E^0 is finite, then the following are equivalent:*

(1) *$L_K(E)$ has Property (A).*

(2) *$L_K(E)$ has left Property (A).*

(3) *$L_K(E)$ has right Property (A).*

(4) *In graph E, any cycle with an exit is an extreme cycle.*

PROOF. Implications $(1) \Rightarrow (4)$, $(2) \Rightarrow (4)$, $(3) \Rightarrow (4)$ follow from Lemma 2.1.

We will show now that (4) implies (1), which is obviously enough to finish our proof. By the paragraph immediately preceding Lemma 2.1, it is enough to work with the assumption that there is at least one cycle π in E.

If for any vertex $v \in E^0$, $v \geq E^0(\pi)$, then the result follows from Lemma 2.10.

There remains to be considered the case where $v \not\geq E^0(\pi)$ for some vertex $v \in E^0$. By Proposition 2.4 and Lemma 2.5, there is a non-trivial π-partition (E_1^0, E_2^0, E_3^0) of E^0 and related π-iso-partition $(\mathcal{E}_1, \mathcal{E}_2)$ such that $L(E) \cong L(\mathcal{E}_1) \times L(\mathcal{E}_2)$. In this case, by Lemma 2.10 the algebra $L(\mathcal{E}_1)$ has Property (A). As graph $\mathcal{E}_2$ is row-finite and in $\mathcal{E}_2$ any cycle with an exit is an extreme cycle, we can replace E by $\mathcal{E}_2$ and work now with $L(\mathcal{E}_2)$ in the same way as we did with $L(E)$. It is easy to see that after finite number of steps we will get graphs $F_1 = \mathcal{E}_1, F_2 \ldots, F_n$ for some $n \geq 1$ such that $L(E) \cong L(F_1) \times \ldots \times L(F_n)$ and for any i, $L(F_i)$ has Property (A). Thus by [17, Proposition 1.3] $L(E)$ has Property (A). $\square$

Theorem 2.12. *Let K be a field. If E is a row-finite graph and E^0 is infinite, then the following are equivalent:*

(1) *$L_K(E)$ has Property (A).*

(2) *$L_K(E)$ has left Property (A).*

(3) *$L_K(E)$ has right Property (A).*

(4) *For any finite set $F \subseteq E^0$, there is a vertex v_F such that $H(F) \cap H(v_F) = \emptyset$.*

PROOF. We will show $(3) \Leftrightarrow (4)$. If E^0 is infinite, then any element of $L(E)$ is a zero-divisor. Thus $L(E)$ has right Property (A) if and only if any finitely generated ideal I of $L(E)$ has a non-zero right annihilator.
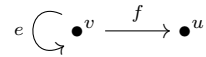
Assume that (4) holds and let $a_1, \ldots, a_n$ be elements of $L(E)$, where n is a positive integer. Assume that all a_j's are monomials. It is easy to see that the set of all vertices u such that $a_i u \neq 0$ for some $i = 1, \ldots, n$ is finite. Denote this set by F. Take $j \in \{1, \ldots, n\}$, and consider any paths α, β. We claim $(a_j \alpha \beta^*) v_F = 0$.

Indeed, if we assume the contrary, then $a_j\alpha \neq 0$ and, in particular, $a_jw \neq 0$ where $w = s(\alpha)$. So $w \in F$. Since $v_F = s(\beta)$ and $r(\alpha) = r(\beta)$, we conclude that $r(\alpha) \in H(w) \cap H(v_F)$ and this contradicts (4). Thus v_F annihilates the ideal generated by $a_1, \ldots, a_n$. As any finitely generated ideal of $L(E)$ is contained in an ideal generated by finitely many monomials, by the above, $L(E)$ has right Property (A), so (3) holds.

To show opposite implication, suppose that there is a finite set F such that for any vertex v, $H(F) \cap H(v) \neq \emptyset$. Then, by Theorem 1.3, the right annihilator of the ideal I of $L(E)$ which is generated by F is equal zero. Thus by what we said at the beginning, $L(E)$ does not have Property (A). It follows that we showed that (3) $\Leftrightarrow$ (4).

As (2) $\Leftrightarrow$ (4) can be proved in a similar way, the proof is finished. $\square$

Example 2.13. In this example we want to consider graph E of the following form:

$$e \,\circlearrowright\, \bullet^v \xrightarrow{\;f\;} \bullet^u$$

and the Toeplitz algebra $L_K(E)$. As in E, the cycle e based at v with exit f is not extreme, $L_K(E)$ is an example of algebra which is Bézout (by [5]) and has neither right nor left Property (A).

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