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On additive decompositions with uniqueness properties of rational integers

By J. FEHÉR (Pécs), B. KOVÁCS and CS. SÁRVÁRI (Pécs)

In memoriam Béla Kovács

1. Introduction

1.1. Definition. The system $S = \{S_0, S_1, \dots\}$ of sets is called an *S-system* if $0 \in S_i \subset \mathbb{Z},\ 1 < \operatorname{card} S_i < \infty$ $(i = 0, 1, \dots)$ and every integer $n \in \mathbb{Z}$ admits a unique decomposition of the form

$$n = \sum_{i=0}^{L} s_i \qquad (s_i \in S_i,\ L \geq 1)\,. \tag{1.2}$$

The simplest examples of S-systems are the so-called *canonical number systems* defined as follows: given a fixed integer $q > 1,\ S_i := (-q)^i\ \{0, 1, \dots, q-1\}$.

It is well-known that such decompositions with card $S \geq 2$ and $\mathbb{N}_0$ instead of $\mathbb{Z}$ were characterized completely by N. G. de Bruijn [1]. His characterization has useful applications e.g. in the investigations of certain arithmetic functions [2]. However, the decompositions defined in 1.1 cannot be described in such a simple manner. The following theorem seems to be a good characterization from several viewpoints.

1.3. Theorem. *Let $A, B \subset \mathbb{Z},\ 0 \in A,\ 0 \in B$ and $1 < \operatorname{card} B < \infty$. Furthermore assume that each integer n admits a unique decomposition $n = a + b$ $(a \in A,\ b \in B)$. Then there exists $M \in \mathbb{N}$ such that*

$$\bar{A} = \{\bar{0}, \overline{a_2}, \dots, \overline{a_H}\}, \quad \bar{B} = \{\bar{0}, \overline{b_2}, \dots, \overline{b_R}\} \subset \mathbb{Z}_M$$

where $\mathbb{Z}_M$ is the additive group of all $\bmod M$*-cosets* $k + M\cdot\mathbb{Z}$ *and*

$\mathbb{Z}_M = \bar{A} + \bar{B}$ $(H\cdot R = M)$ *and* $B = \{0, b_2, \ldots, b_R\}$, $A = \bar{0}\cup\overline{a_2}\cup\cdots\cup\overline{a_H}$.

After the proof of Theorem 1.3 we are going to show an application.

1.4. Definition. The function $f:\mathbb{Z}\to\mathbb{C}$ is said to be S-additive (with respect to a given system S) if $f(n) = \sum_{i=0}^{L} f(s_i)$ with the decomposition (1.2) of the integer n.

Given an S-system, the set of all S-additive functions is a linear space over $\mathbb{C}$, the linear functions $f(n) = c\cdot n$ are S-additive with respect to any S-system. It is also clear that $f(0) = 0$ for every S-additive function f.

For a polynomial $P(z) = a_k z^k + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$ and function $f:\mathbb{Z}\to\mathbb{C}$ we set

$$P(E)f(n) := a_k f(n+k) + \cdots + a_1 f(n+1) + a_0 f(n)\,.$$

1.5. Theorem. *Given an S-system and a polynomial $P(z)\in\mathbb{C}[z]$ of at least first degree, the S-additive solutions of the equation*

$$P(E)f(n) = 0 \quad (\forall n\in\mathbb{Z}) \tag{1.6}$$

are of the form $f(n) = cn + g(n)$ where $g(n)$ is a periodic S-additive function.

An analogous result with $\mathbb{N}_0$ instead of $\mathbb{Z}$ and under a stronger hypothesis on the decompositions is given in [2].

2. Proof of Theorem 1.3[1]

Suppose that $A+B=\mathbb{Z}$ is a normal direct decomposition (i.e. $0\in A\cap B$ and the sums of the form $a+b$ $(a\in A,\ b\in B)$ are unique). Let b^* be the module of the least element of B and let $B' = B + b^* (:= \{b+b^* \mid b\in B\})$. Then $A + B' = \mathbb{Z}$ is also a normal direct decomposition. Thus we may assume that B consists of non-negative elements and b_r is the greatest one among them.

Let us color the point in the real line corresponding to the integer x by red if $x\subset A$ and by blue if $x\neq A$. It is clear that x is blue if and only if $x = a+b$ $(a\in A,\ b\in B)$ and $b\neq 0$.

[1] This proof has been communicated to us by Béla Kovács in a private letter (1987). Apparently, he was not aware of the fact that the problem apppearing in the theorem had already been proposed by N. G. de Bruijn in [4] (Problem 12) and V. T. Sós [5] had given a solution similar to the one to be described here.

Let us consider the intervals (open from the left and closed from the right)

$$I_T := (Tb_r, (T+1)b_r] \quad (T \in \mathbb{Z})\,.$$

(a) If the colorings of I_T and I_L are the same then the colorings of I_{T+1} and I_{L+1} are also the same. Suppose — contrarily to the statement — that $c_{T+1} = (T+1)b_r + h_0$ is red, $c_{L+1} = (L+1)b_r + h_0$ is blue for some $0 < h_0 \leq b_r$ and $h_0 = 1$ or $(T+1)b_r + h$ and $(L+1)b_r + h$ have the colors for $0 < h < h_0$. Then the colorings of the intervals $[c_{T+1} - b_r, c_{T+1})$ and $[c_{L+1} - b_r, c_{L+1})$ coincide. Since the point c_{L+1} is blue, there exists $0 < k < b_r$ such that $c_{L+1} - b_r$ is red and $b_r - k \in B$. But then $c_{T+1} - b_r + k$ is red and hence $c_{T+1} - b_r + k$ is blue, a contradiction.

(b) One can show in a similar manner that if I_T and I_L have the same colorings then the colorings of I_{T-1} and I_{L-1} are the same, too.

(c) Since the intervals I_T have only finitely many colorings, there exist T_0 and a minimal integer M such that the colorings of I_{T_0} and I_{T_0+M} are the same. By observations (a) and (b), the coloring of the whole line is $\pmod{Mb_r}$-periodic.

(d) *Construction.* Consider the red points in the interval $(0, Mb_r]$ and let us denote them by $a_1 < a_2 < \cdots < a_H$. Since the point 0 is colored red, $a_H = Mb_r$. Therefore $A = \overline{a_1} \cup \overline{a_2} \cup \cdots \cup \bar{0}$ where each $\overline{a_i}$ (resp. $\bar{0}$) is a $\pmod{Mb_r}$-coset and $\bar{A} = \{\overline{a_1}, \ldots, \overline{a_{H-1}}, \bar{0}\}$. Since $\bar{B} = \{\bar{0}, \overline{b_2}, \ldots, \overline{b_r}\}$, it is clear that the decomposition $\mathbb{Z}_{Mb_r} = \bar{A} + \bar{B}$ is normal direct.

3. Proof of Theorem 1.5

We need the following two lemmas.

3.1. Lemma. *Let* $P(z) = a_k z^k + \cdots + a_1 z + a_0 = a_k z^{s_0} \prod_{j=1}^{h} (z - \varrho_j)^{s_j}$ $(0 \neq \varrho_j \in \mathbb{C})$ *be a given polynomial with complex coefficients and let* $f : \mathbb{Z} \to \mathbb{C}$*. Then the solutions of the equation*

$$P(E)f(n) = 0 \quad (\forall n \in \mathbb{Z})$$

are exactly the functions

$$f(n) = \sum_{j=1}^{n} q_j(n) \varrho_j^n$$

where each q_j $(j = 1, \ldots, n)$ *is an arbitrary polynomial of degree at most* $(s_j - 1)$ *with complex coefficients.*

Proof. See e.g. in [3].

3.2. Lemma. *Let $\varrho_1, \varrho_2, \dots, \varrho_r$ be distinct complex numbers ($r \geq 1$) and $q_1, \dots, q_r$ polynomials with complex coefficients, respectively. If*

$$q_1(n)\varrho_1^n + \dots + q_r(n)\varrho_r^n = 0 \quad (\forall n \in \mathbb{Z})$$

then

$$q_1 = q_2 = \dots = q_r = 0\,.$$

Proof. See e.g. in [3].

Let us now consider those blocks S_i from the given S-system $\{S_0, S_1, \dots\}$ which are indispensable to the decomposition (1.2) of the numbers $1, 2, \dots, k$. Let B be the direct sum of these blocks and let A denote the direct sum of the remaining ones. Then the decomposition $\mathbb{Z} = A + B$ is normal direct and $0, 1, \dots, k \in B$. By Theorem 1.3 there exists an integer $D > k$ such that $D \cdot \mathbb{Z} \subset A$. On the other hand, from the additivity with respect to the given S-system of the function f it follows

$$f(a+b) = f(a) + f(b) \quad (\forall a \in A \text{ and } \forall b \in B)\,. \tag{3.3}$$

The solutions of (1.6) are of the form

$$f(n) = q_0(n)1^n + q_1(n)\varrho_1^n + \dots + q_T(n)\varrho_T^n \tag{3.4}$$

where $P(z) = a_k z^\alpha (z-1)^{s_0} \prod_{j=1}^T (z - \varrho_j)^{s_j}$, $(\varrho_j \notin \{0,1\})$. Then

$$f(ND) = q_0(ND) + q_1(ND)(\varrho_1^D)^N + \dots + q_T(ND)(\varrho_T^D)^N\,. \tag{3.5}$$

Let $\varrho_j^D = \delta_j$ $(j = 1, \dots, T)$. Then

$$f(ND + r) = q_0(ND + r) + \sum_{j=1}^T q_j(ND + r)\delta_j^N \varrho_j\,.$$

By the additivity

$$f(ND + r) - f(ND) - f(r) = 0 \quad (\forall N \in \mathbb{Z};\ r = 1, \dots, k)\,. \tag{3.7}$$

From (3.5), (3.6) and (3.7) we get the condition

$$Q_0^{(r)}(N) + \sum_{i=1}^h Q_i^{(r)}(N)\tau_i^N = 0 \quad (\forall N \in \mathbb{Z};\ r = 1, \dots, k)$$

where each $Q_i^{(r)}$ is a polynomial of the variable N and $\tau_1, \dots, \tau_h$ are distinct complex numbers not taking the values 0 and 1. By Lemma 3.2,

$$Q_j^{(r)} = 0 \quad (j = 0, \dots, h;\ r = 1, \dots, k)\,.$$

This means that

$$Q_j^{(r)}(z) = q_\ell(z+r)\varrho_\ell^r - q_\ell(z) + \cdots + q_\nu(z+r)\varrho_\nu^r - q_\nu(z) = 0\,. \tag{3.9}$$

We show that $q_i = 0$ $(i = \ell, \dots, \nu)$. Suppose the contrary and let max deg $q_i = m$. Let $a_{m,j}$ denote the coefficient of the term of degree m in q_j. From (3.9) we infer

$$\begin{gathered} a_{m,\ell}(\varrho_\ell^r - 1) + \cdots + a_{m,\nu}(\varrho_\nu^r - 1) = 0 \\ r = 1, \dots, \nu - \ell + 1 \le k\,. \end{gathered} \tag{3.10}$$

Since the determinant of (3.10) cannot be 0, we have $a_{m,j} = 0$ $(j = \ell, \dots, \nu)$, a contradiction. Finally, considering the detailed form of the condition $Q_0^{(r)} = 0$, we get

$$\begin{aligned} q_0(z+r) - q_0(z) - f(r) + q_1(z+r)\varrho_1^r - q_1(z) + \dots \\ \cdots + q_u(z+r)\varrho_u^r - q_u(z) = 0\,. \end{aligned} \tag{3.11}$$

We show that $q_0 =$ constant or deg $q_0 = 1$ and $q_i =$ constant $(i = 1, \dots, u)$. Indeed, if

(a) deg $q_0 \ge 1$ and $\max\deg q_i \ge \deg q_0$ then we get a contradiction in a similar way as above from the fact that $q_0(z+r) - q_0(z) - f(r) = 0$ or $\deg(q_0(z+r) - q_0(z)f(r)) < \deg q_0$.

(b) If $\deg q_0 \ge 1$ and $\max\deg q_i < \deg q_0$ or $q_i = 0$ $(i = 1, \dots, u)$ then there exists $1 \le r \le k$ such that $Q_0^{(r)} \ne 0$ which is impossible.

(c) The assumptions $q_0 =$ constant and $\max\deg q_i > 0$ lead to a contradiction similarly as in (b).

Remark that we have $\varrho_i^D = 1$ for the roots ϱ_i of the polynomial $Q_0^{(r)}$. Therefore

$$f(n) = c \cdot n + \sum_{j=0}^{D-1} b_j \varrho^{jn} \quad \text{where} \quad \varrho = \exp(\pi i/D)$$

and the coefficients c, b_j $(j = 0, 1, \dots, D-1)$ are complex numbers with the following properties:

(i) $$c = 0 \ \text{ if } \ P(1) \ne 0 \ \text{ or } \ P'(1) \ne 0\,,$$

(ii) $$b_j = 0 \ \text{ if } \ P(\varrho^j) \ne 0\,,$$

(iii) $$\sum_{j=0}^{D-1} b_j = 0 \ \text{ since } \ f(0) = 0\,.$$

It is clear that the function $g(n) = \sum_{j=0}^{D-1} b_j \varrho^{jn}$ is periodic.

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JÁNOS FEHÉR
DEPT. OF MATHEMATICS
JPTE UNIVERSITY, PÉCS
IFJÚSÁG ÚT 6
H–7624 PÉCS
HUNGARY

BÉLA KOVÁCS

CSABA SÁRVÁRI
DEPT. OF MATH. AND INFORMATICS
PMMF, PÉCS
RÓKUS U. 2
H–7624 PÉCS
HUNGARY

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