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Remarks on nonlinear Schrödinger equations with the harmonic potential

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Abstract. We show that the small solution for a type of nonlinear Schrödinger equation with the harmonic potential exists over a longer time interval than the one given by local existence theory. We also get a control of the Sobolev norm of the solution on that time interval. We exploit the structure of nonlinearity to estimate the small divisor and perform a normal form process.

1. Introduction and the main result

We are interested in lower bounds for the lifespan of the solution to the nonlinear Schrödinger equation with the harmonic potential

$$i\partial_t u = (-\Delta + |x|^2)u - u^p, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d \tag{1.1}$$

with small initial data, where integer d is the space dimension and $p > 0$ is an integer. If we neglect the harmonic potential, then from the point view of scaling, the critical regularity is $s_c = \frac{d}{2} - \frac{2}{p-1}$. Thus $s_c > 1$ if $d \geq 3$ and p is large. This falls into the interesting energy-supercritical case, about which there are few results. Almost global existence for solutions of equation

$$i\partial_t u = (-\Delta + |x|^2 + M)u - u^p, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d$$

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with small initial data has been obtained in [2], where M is a Hermite multiplier operator. The operator M plays an important role in their proof, actually, it was used to avoid the resonance. We are curious about what happens if M is set to be zero, i.e., what could we say about (1.1)?

There are some results for the similar equation:

$$i\partial_t u = (-\Delta + |x|^2)u + \lambda |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.2}$$

where $p \in (1, +\infty)$ when $d = 1, 2$, and $p \in (1, 1 + \frac{4}{d-2}]$ when $d \geq 3$. If $\lambda \geq 0$ or if $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$, then there exists a global in time solution to (1.2) for any initial datum $u_0 \in \Sigma^1$ defined in (1.4), while if $\lambda < 0$ and if $1 + \frac{4}{d} \leq p < 1 + \frac{4}{d-2}$ when $d \geq 3$, the solution to (1.2) still exists globally for any initial datum $u_0 \in \Sigma^1$ with small norm. However, it may happen that the solution to (1.2) blows up in finite time if $\lambda < 0$ and if $1 + \frac{4}{d} \leq p < 1 + \frac{4}{d-2}$ when $d \geq 3$. Lifespan estimates were obtained when a blow-up happens. We refer to [1] and references therein for details in this energy-subcritical case. The energy-critical case, i.e., (1.2) with $p = 1 + \frac{4}{d-2}$ ($d \geq 3$), is more subtle and has been studied in [3].

We also mention the results about long-time existence for nonlinear Klein–Gordon and wave equations with the harmonic potential [5]–[7].

We are in the position to state our main result. Consider (1.1) with initial data

$$u(t,x)\,|_{t=0} = \varepsilon u_0\ , \tag{1.3}$$

where $\varepsilon > 0$ is a parameter. Because of the harmonic potential, it is natural to consider the solution in the space

$$\Sigma^k := \{u \in L^2(\mathbb{R}^d) : x^\alpha \partial_x^\beta u \in L^2(\mathbb{R}^d), \forall |\alpha| + |\beta| \leq k\} \tag{1.4}$$

for some nonnegative integer k. By local existence theory, problems (1.1) and (1.3) admit a unique solution defined on the time interval $|t| \leq c\varepsilon^{-(p-1)}$ for any u_0 in the unit ball of Σ^k, provided k is large enough and $\varepsilon > 0$ is small enough. The main result of this paper is the following:

Theorem 1.1. *Let d be odd and p a positive even integer. Then there exists s_0, ε_0 and $c, C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any integer $s \geq s_0$, any $u_0 \in \Sigma^s$ with $||u_0||_{\Sigma^s} \leq 1$, there exists a unique solution*

$$u \in C^1\left(\left(-T_\varepsilon, T_\varepsilon\right), \Sigma^s\right)$$

to (1.1), (1.3) with $T_\varepsilon \geq c\varepsilon^{-(2p-2)}$. Moreover, $\forall\ t \in (-T_\varepsilon, T_\varepsilon)$, one has

$$\|u(t,\cdot)\|_{\Sigma^s} \leq C\varepsilon. \tag{1.5}$$

Remark 1.1. The result still holds if u^p in (1.1) is replaced by

$$\sum_{p_1,p_2\in\mathbb{N},p_1+p_2=p} c(p_1,p_2)u^{p_1}\overline{u}^{p_2}$$

for any real constant $c(p_1,p_2)$.

Remark 1.2. It seems that the other cases, for instance, when d is even and p is odd, are difficult to deal with, because of the following reason: in those cases (3.4) does not hold true, so there are resonant terms which are difficult to treat using this method. However, by the same analysis, the result of the above theorem holds for the following equation:

$$i\partial_t u = (-\Delta + |x|^2 + 1)u - u^p, \quad x \in \mathbb{R}^d,$$

for even d and positive even p.

In the next section, we provide some preliminaries, and the last section is devoted to the proof of the theorem. Let us explain the main idea. We want to control the Sobolev norm of the solution, whose time derivative is a multilinear expression in u homogenous of order $p+1$ (see (3.1)). We then perturb it so that (i) the time derivative of the perturbation cancels out the right-hand side of (3.1), up to a high-order term $O\left(\|u\|^{2p}_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}\right)$; (ii) the perturbation is controlled by the power of $\|u\|_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}$ (see Lemma 3.1). In the end, the standard continuous argument allows one to show that the Sobolev norm of the solution is bounded in the time interval.

2. Preliminaries

Let $-\Delta+|x|^2$ be the harmonic oscillator on $L^2(\mathbb{R}^d)$. Its eigenvalues are given by λ_n^2 with

$$\lambda_n = \sqrt{2n+d}, \quad n = 0, 1, 2, \ldots.$$

We denote by Π_n the orthogonal projection onto the eigenspace associated to λ_n^2, and by $\mathcal{E}$, the algebraic direct sum of the ranges of the $\Pi_n's, n \in \mathbb{N}$. Denote

$$\Lambda = \sqrt{-\Delta + |x|^2}.$$

We shall work in the space

$$\widetilde{\mathcal{H}}^s(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) : \sum_{n\in\mathbb{N}} \lambda_n^{2s} \|\Pi_n u\|^2_{L^2(\mathbb{R}^d)} < +\infty \right\},$$

which is equivalent to Σ^s defined in (1.4) when s is a natural number (see [5]). Thus we shall not distinguish Σ^s with $\widetilde{\mathcal{H}}^s(\mathbb{R}^d)$ when s is a nonnegative integer. The constant C in the paper could be different from line to line. From now on, we shall also denote by $\max_1, \max_2, \max_3$, respectively, the largest, the second largest and the third largest element among $\lambda_{n_0}, \lambda_{n_1}, \dots, \lambda_{n_p}$ for natural numbers $n_0, n_1, \dots, n_p$. One should easily distinguish these notations from the functions "max" in the following context.

We shall need the following proposition.

Proposition 2.1. *Let $p > 1$ be a natural number. There is a positive constant ν such that for any natural number N, there is $C_N > 0$ satisfying that for any $n_0, n_1, \dots, n_p \in \mathbb{N}$, any $u_0, u_1, \dots, u_p \in L^2(\mathbb{R}^d)$, one has for any $\delta \in \left(0, \frac{1}{d+3}\right)$*

$$\left|\int_{\mathbb{R}^d} (\Pi_{n_0} u_0)(\Pi_{n_1} u_1) \cdots (\Pi_{n_p} u_p)\, dx\right| \le C_N \left(\max_1 \cdot \max_2\right)^{-\left(\frac{1}{d+3} - \delta\right)} \times \frac{\max_3^{\nu} \cdot (\max_2 \cdot \max_3)^N}{(|\max_1^2 - \max_2^2| + \max_2 \cdot \max_3)^N} \prod_{j=0}^{p} \|u_j\|_{L^2(\mathbb{R}^d)}.$$

Proof. The proof is similar to that of [7, Proposition 2.1]. We give it here for convenience of the reader. By symmetry we may assume

$$\lambda_{n_0} \ge \lambda_{n_1} \ge \cdots \ge \lambda_{n_p},$$

i.e.,

$$\max_1 = \lambda_{n_0}, \qquad \max_2 = \lambda_{n_1}, \qquad \max_3 = \lambda_{n_2}.$$

Thus we are reduced to showing

$$\left|\int_{\mathbb{R}^d} (\Pi_{n_0} u_0)(\Pi_{n_1} u_1) \cdots (\Pi_{n_p} u_p)\, dx\right| \le C_N \left(\lambda_{n_0}\lambda_{n_1}\right)^{-\left(\frac{1}{d+3} - \delta\right)} \frac{\lambda_{n_2}^{\nu} \cdot (\lambda_{n_1} \cdot \lambda_{n_2})^N}{\left(\left|\lambda_{n_0}^2 - \lambda_{n_1}^2\right| + \lambda_{n_1}\lambda_{n_2}\right)^N} \prod_{j=0}^{p} \|u_j\|_{L^2(\mathbb{R}^d)}. \tag{2.1}$$

On the one hand, by Hölder's inequalities,

$$\left|\int_{\mathbb{R}^d} (\Pi_{n_0} u_0)(\Pi_{n_1} u_1) \cdots (\Pi_{n_p} u_p)\, dx\right| \le \prod_{j=0}^{p} \left\|\Pi_{n_j} u_j\right\|_{L^{q_j}(\mathbb{R}^d)} \tag{2.2}$$

with $q_0, q_1, \ldots, q_p \in [2, +\infty)$ satisfying

$$\frac{1}{q_0} + \frac{1}{q_1} + \cdots + \frac{1}{q_p} = 1.$$

From [4, Corollary 3.2], we know

$$\|\Pi_n u\|_{L^q(\mathbb{R}^d)} \leq C\lambda_n^{\rho(q)} \|u\|_{L^2(\mathbb{R}^d)}, \tag{2.3}$$

where

$$\rho(q) = \begin{cases} -\left(\frac{1}{2} - \frac{1}{q}\right), & \text{if } \frac{d+1}{2(d+3)} < \frac{1}{q} \leq \frac{1}{2}, \\ -\frac{1}{3} + \frac{d}{3}\left(\frac{1}{2} - \frac{1}{q}\right), & \text{if} \max\left\{\frac{d-2}{2d}, 0\right\} \leq \frac{1}{q} < \frac{d+1}{2(d+3)}, \\ -1 + d\left(\frac{1}{2} - \frac{1}{q}\right), & \text{if } 0 < \frac{1}{q} \leq \max\left\{\frac{d-2}{2d}, 0\right\}. \end{cases} \tag{2.4}$$

We set for any $\delta_1 \in (0, \delta)$ with $\delta \in \left(0, \frac{1}{3+d}\right)$,

$$\frac{1}{q_0} = \frac{1}{q_1} = \frac{d+1}{2(d+3)} + \delta_1, \qquad \frac{1}{q_2} = 2\left(\frac{1}{d+3} - \delta_1\right), \qquad q_j = \infty, \qquad j = 3, \cdots, p.$$

Then from (2.2)–(2.4) and Sobolev's inequalities, we deduce

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} (\Pi_{n_0} u_0)(\Pi_{n_1} u_1) \cdots (\Pi_{n_p} u_p)\, dx\right| \\ &\leq C(\lambda_{n_0}\lambda_{n_1})^{-\left(\frac{1}{2} - \frac{d+1}{2(d+3)} - \delta_1\right)} \lambda_{n_2}^{\nu_1} \prod_{j=0}^{p} \|u_j\|_{L^2(\mathbb{R}^d)} \\ &= C(\lambda_{n_0}\lambda_{n_1})^{-\left(\frac{1}{d+3} - \delta_1\right)} \lambda_{n_2}^{\nu_1} \prod_{j=0}^{p} \|u_j\|_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{2.5}$$

for some $\nu_1 > 0$, where we also used the symmetric assumption $\lambda_{n_2} \geq \cdots \geq \lambda_{n_p}$.

On the other hand, it follows from (1.3.6) in [5] that there exists ν_2, and for any $N_1 \in \mathbb{N}$, there exists a constant $C_{N_1} > 0$, such that

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} (\Pi_{n_0} u_0)(\Pi_{n_1} u_1) \cdots (\Pi_{n_p} u_p)\, dx\right| \\ &\leq C_{N_1} \frac{\lambda_{n_2}^{\nu_2} \cdot (\lambda_{n_1} \cdot \lambda_{n_2})^{N_1}}{\left(\left|\lambda_{n_0}^2 - \lambda_{n_1}^2\right| + \lambda_{n_1}\lambda_{n_2}\right)^{N_1}} \prod_{j=0}^{p} \|u_j\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.6}$$

Then (2.1) follows from (2.5) and (2.6). This concludes the proof. □

3. Proof of the main theorem

By the local theory, it suffices to show

$$\|u(t,\cdot)\|^2_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)} \le C^2\varepsilon^2, \quad \forall\ t\in[-c\varepsilon^{-(2p-2)}, c\varepsilon^{-(2p-2)}],$$

for some $C, c>0$. We only need to prove

$$E_s(u)(t) := \langle \Lambda^s u, \Lambda^s u\rangle \le C^2\varepsilon^2, \quad \forall\ t\in[-c\varepsilon^{-(2p-2)}, c\varepsilon^{-(2p-2)}],$$

since

$$E_s(u)(t) = \|u(t,\cdot)\|^2_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}.$$

Thus we have by (1.1)

$$\begin{aligned}\frac{d}{dt}E_s(u)(t) &= 2\operatorname{Re} i\langle \Lambda^s u, \Lambda^s i\partial_t u\rangle\\ &= 2\operatorname{Re} i\langle \Lambda^s u, \Lambda^{s+2}u\rangle - 2\operatorname{Re} i\langle \Lambda^s u, \Lambda^s u^p\rangle = 2\operatorname{Im}\langle \Lambda^s u, \Lambda^s u^p\rangle.\end{aligned}$$

Decomposing

$$u = \sum_{n\in\mathbb{N}} \Pi_n u,$$

we obtain, denoting $\vec{n} = (n_0, n_1, \dots, n_p)\in\mathbb{N}^{p+1}$,

$$\frac{d}{dt}E_s(u)(t) = 2\operatorname{Im}\sum_{\vec{n}\in\mathbb{N}^{p+1}} \langle \Lambda^s \Pi_{n_0}u, \Lambda^s\left((\Pi_{n_1}u)\cdots(\Pi_{n_p}u)\right)\rangle. \tag{3.1}$$

Lemma 3.1. *Assume that d is odd and that $p>1$ is even. Then there exists a quantity $Q=Q(t)$ such that*

$$\frac{d}{dt}\left(E_s(u)(t) - Q(t)\right) = O\left(\|u\|^{2p}_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}\right), \qquad Q(t) = O\left(\|u\|^{p+1}_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}\right)$$

if s is large enough.

Proof of Lemma 3.1. Set

$$L_{\vec{n}}(u)(t) = \langle \Lambda^s \Pi_{n_0}u, \Lambda^s\left((\Pi_{n_1}u)\cdots(\Pi_{n_p}u)\right)\rangle. \tag{3.2}$$

Then we get by (1.1), denoting $D_t = i\partial_t$,

$$\begin{aligned}
\frac{d}{dt}L_{\vec{n}}(u)(t) &= (-i)\langle \Lambda^s \Pi_{n_0} D_t u, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_p} u)\right)\rangle \\
&\quad + i\sum_{j=1}^{p}\langle \Lambda^s \Pi_{n_0} u, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_j} D_t u)\cdots(\Pi_{n_p} u)\right)\rangle \\
&= (-i)\langle \Lambda^s \Pi_{n_0} \Lambda^2 u, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_p} u)\right)\rangle \\
&\quad + i\sum_{j=1}^{p}\langle \Lambda^s \Pi_{n_0} u, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_j} \Lambda^2 u)\cdots(\Pi_{n_p} u)\right)\rangle \\
&\quad + i\langle \Lambda^s \Pi_{n_0} u^p, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_p} u)\right)\rangle \\
&\quad - i\sum_{j=1}^{p}\langle \Lambda^s \Pi_{n_0} u, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_j} u^p)\cdots(\Pi_{n_p} u)\right)\rangle \\
&= (-i)\left(\lambda_{n_0}^2 - \sum_{j=1}^{p}\lambda_{n_j}^2\right)\langle \Lambda^s \Pi_{n_0} u, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_p} u)\right)\rangle \\
&\quad + i\langle \Lambda^s \Pi_{n_0} u^p, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_p} u)\right)\rangle \\
&\quad - i\sum_{j=1}^{p}\langle \Lambda^s \Pi_{n_0} u, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_j} u^p)\cdots(\Pi_{n_p} u)\right)\rangle. \qquad (3.3)
\end{aligned}$$

Since we assume that d is odd and that $p > 1$ is even, we have for any $\vec{n} \in \mathbb{N}^{p+1}$,

$$\left|\lambda_{n_0}^2 - \sum_{j=1}^{p}\lambda_{n_j}^2\right| = \left|2n_0 - 2(n_1 + n_2 + \cdots + n_p) - (p-1)d\right| \geq 1. \qquad (3.4)$$

Therefore it is meaningful to define

$$Q(t) = 2\,\mathrm{Im}\sum_{\vec{n}\in\mathbb{N}^{p+1}}\left((-i)\left(\lambda_{n_0}^2 - \sum_{j=1}^{p}\lambda_{n_j}^2\right)\right)^{-1} L_{\vec{n}}(u)(t). \qquad (3.5)$$

By similar computation as in (3.3), we obtain

$$\frac{d}{dt}Q(t) = 2\,\mathrm{Im}\sum_{\vec{n}\in\mathbb{N}^{p+1}}\left(L_{\vec{n}}(u)(t) - R_{\vec{n}}(u)(t)\right), \qquad (3.6)$$

where

$$\begin{aligned}
R_{\vec{n}}(u)(t) = \left(\lambda_{n_0}^2 - \sum_{j=1}^{p}\lambda_{n_j}^2\right)^{-1} \Big[&\left\langle \Lambda^s \Pi_{n_0} u^p, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_p} u)\right)\right\rangle \\
&- \sum_{j=1}^{p}\left\langle \Lambda^s \Pi_{n_0} u, \Lambda^s \left((\Pi_{n_1} u)\cdots(\Pi_{n_j} u^p)\cdots(\Pi_{n_p} u)\right)\right\rangle \Big].
\end{aligned}$$

Subtracting (3.6) from (3.1), we obtain

$$\frac{d}{dt}\big(E_s(u)(t) - Q(t)\big) = R(t),$$

where

$$R(t) = 2\,\mathrm{Im} \sum_{\vec{n}\in\mathbb{N}^{p+1}} R_{\vec{n}}(u)(t).$$

Thus to prove Lemma 3.1, we are left to showing the following claim.

Claim.

$$Q(t) = O\left(\|u\|^{p+1}_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}\right) \quad \text{and} \quad R(t) = O\left(\|u\|^{2p}_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}\right) \text{ for large } s. \tag{3.7}$$

Since Λ is self-adjoint, in view of (3.2) and (3.5), one has

$$\begin{aligned} Q(t) &= 2\,\mathrm{Im} \sum_{\vec{n}\in\mathbb{N}^{p+1}} \left((-i)\left(\lambda_{n_0}^2 - \sum_{j=1}^{p}\lambda_{n_j}^2\right)\right)^{-1} \left\langle \Lambda^{2s}\Pi_{n_0}u, (\Pi_{n_1}u)\cdots(\Pi_{n_p}u)\right\rangle \\ &= 2\,\mathrm{Im} \sum_{\vec{n}\in\mathbb{N}^{p+1}} \left((-i)\left(\lambda_{n_0}^2 - \sum_{j=1}^{p}\lambda_{n_j}^2\right)\right)^{-1} \int_{\mathbb{R}^d} \lambda_{n_0}^{2s}(\Pi_{n_0}u)\prod_{j=1}^{p}\left(\Pi_{n_j}\overline{u}\right)dx. \end{aligned}$$

It follows from Proposition 2.1 with $\delta = \frac{1}{2(d+3)}$ and Hölder's inequalities that

$$\begin{aligned} |Q(t)| &\le C_N \sum_{\vec{n}\in\mathbb{N}^{p+1}} \frac{1}{(\lambda_{n_0}^2 - \sum_{j=1}^{p}\lambda_{n_j}^2)} \cdot \frac{\lambda_{n_0}^{2s}}{(\max_1\cdot\max_2)^{\frac{1}{2(d+3)}}} \\ &\quad \times \frac{\max_3^{\nu}\cdot(\max_2\cdot\max_3)^N}{(|\max_1^2 - \max_2^2| + \max_2\cdot\max_3)^N} \|\Pi_{n_0}u\|_{L^2(\mathbb{R}^d)} \prod_{j=1}^{p}\left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)} \\ &\le C_N\,\|u\|_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)} \left(\sum_{n_0\in\mathbb{N}} T(n_0)^2\right)^{1/2}, \end{aligned}$$

where

$$T(n_0) = \sum_{\vec{n}'\in\mathbb{N}^p} F(n_0, \vec{n}', \overline{u})$$

with $\vec{n}' = (n_1, \ldots, n_p)$ and

$$F(n_0,\vec{n}',\overline{u}) = \frac{1}{(\lambda_{n_0}^2 - \sum_{j=1}^p \lambda_{n_j}^2)} \cdot \frac{\lambda_{n_0}^s}{(\max_1 \cdot \max_2)^{\frac{1}{2(d+3)}}} \times \frac{\max_3^\nu \cdot (\max_2 \cdot \max_3)^N}{(|\max_1^2 - \max_2^2| + \max_2 \cdot \max_3)^N} \prod_{j=1}^p \left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)}. \tag{3.8}$$

Now to prove the first estimate in (3.7), it suffices to show that there exists a constant $C_s > 0$ depending on s only such that

$$\sum_{n_0\in\mathbb{N}} T(n_0)^2 \leq C_s \left\|\overline{u}\right\|_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}^{2p}.$$

To show this inequality, without loss of generality, we assume

$$\lambda_{n_1} \geq \lambda_{n_2} \geq \cdots \geq \lambda_{n_p}.$$

Note that $\lambda_{n_i} \geq \lambda_{n_j}$ is equivalent to $n_i \geq n_j$. So setting

$$S = \{\vec{n} \in \mathbb{N}^{p+1} : n_1 \geq n_2 \geq \cdots \geq n_p\},$$
$$A = \{\vec{n} \in \mathbb{N}^{p+1} : n_0 \leq n_2\}, \qquad A^c = \{\vec{n} \in \mathbb{N}^{p+1} : n_0 > n_2\},$$

we only need to show

$$\sum_{n_0\in\mathbb{N}} T_A(n_0)^2 \leq C_s \left\|\overline{u}\right\|_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}^{2p}, \qquad \sum_{n_0\in\mathbb{N}} T_{A^c}(n_0)^2 \leq C_s \left\|\overline{u}\right\|_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}^{2p}, \tag{3.9}$$

where

$$T_A(n_0) = \sum_{\vec{n}'\in\mathbb{N}^p} \mathbf{1}_{S\cap A} F(n_0,\vec{n}',\overline{u}), \qquad T_{A^c}(n_0) = \sum_{\vec{n}'\in\mathbb{N}^p} \mathbf{1}_{S\cap A^c} F(n_0,\vec{n}',\overline{u}).$$

To estimate $\sum_{n_0\in\mathbb{N}} T_A(n_0)^2$, we note that on the support of $\mathbf{1}_{S\cap A} F(n_0,\vec{n}',\overline{u})$,

$$\max_1 = \lambda_{n_1}, \qquad \max_2 = \lambda_{n_2}, \qquad \max_3 = \max(\lambda_{n_3},\lambda_{n_0}).$$

By the definition of $T_A(n_0)$, (3.8), (3.4), Hölder's inequalities and the fact that

$$\frac{(\max_2 \cdot \max_3)^N}{(\max_1 \cdot \max_2)^{\frac{1}{2(d+3)}} (|\max_1^2 - \max_2^2| + \max_2 \cdot \max_3)^N} \leq 1,$$

one has

$$\begin{aligned}\sum_{n_0\in\mathbb{N}} T_A(n_0)^2 &\le C\sum_{n_0\in\mathbb{N}}\left(\sum_{\vec{n}'\in A}\mathbf{1}_{S\cap A}\lambda_{n_0}^s\max(\lambda_{n_3},\lambda_{n_0})^\nu\prod_{j=1}^p\left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)}\right)^2\\ &\le C\sum_{n_0\in\mathbb{N}}\left(\sum_{\vec{n}'\in A}\mathbf{1}_{S\cap A}\lambda_{n_0}^{-2}\lambda_{n_1}^{\frac{s+\nu+2}{2}}\lambda_{n_2}^{\frac{s+\nu+2}{2}}\prod_{j=1}^p\left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)}\right)^2\\ &\le C\sum_{n_0\in\mathbb{N}}\lambda_{n_0}^{-4}\left\|\overline{u}\right\|^{2p}_{\widetilde{\mathcal{H}}^{\frac{s+\nu+5}{2}}(\mathbb{R}^d)}\le C\left\|\overline{u}\right\|^{2p}_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}\end{aligned}$$

if $s\ge\nu+5$. This gives the first inequality in (3.9).

To obtain the second inequality in (3.9), we note that on the support of $\mathbf{1}_{S\cap A^c}F(n_0,\vec{n}',\overline{u})$, one has

$$\max{}_1=\max(\lambda_{n_0},\lambda_{n_1}),\qquad \max{}_2=\min(\lambda_{n_0},\lambda_{n_1}),\qquad \max{}_3=\lambda_{n_2}.$$

So we get by (3.8),

$$\begin{aligned}\left|\mathbf{1}_{S\cap A^c}F(n_0,\vec{n}',\overline{u})\right| &= \mathbf{1}_{S\cap A^c}\left|\frac{1}{(\lambda_{n_0}^2-\sum_{j=1}^p\lambda_{n_j}^2)}\right|\frac{\lambda_{n_0}^s}{(\lambda_{n_0}\lambda_{n_1})^{\frac{1}{2(d+3)}}}\\ &\times\frac{\lambda_{n_2}^\nu\cdot(\min(\lambda_{n_0},\lambda_{n_1})\cdot\lambda_{n_2})^N}{\left(\left|\lambda_{n_0}^2-\lambda_{n_1}^2\right|+\min(\lambda_{n_0},\lambda_{n_1})\cdot\lambda_{n_2}\right)^N}\prod_{j=1}^p\left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)}. \end{aligned}\tag{3.10}$$

Remark that

$$\mathbf{1}_{S\cap A^c}\frac{\lambda_{n_0}\cdot\min(\lambda_{n_0},\lambda_{n_1})\cdot\lambda_{n_2}}{\left|\lambda_{n_0}^2-\lambda_{n_1}^2\right|+\min(\lambda_{n_0},\lambda_{n_1})\cdot\lambda_{n_2}}\le 2\lambda_{n_1},$$

since when $|\lambda_{n_0}^2-\lambda_{n_1}^2|\le\frac{1}{2}\lambda_{n_0}^2$, one has $\lambda_{n_0}<2\lambda_{n_1}$, so this holds trivially, while when $|\lambda_{n_0}^2-\lambda_{n_1}^2|>\frac{1}{2}\lambda_{n_0}^2$, the left-hand side is less than

$$\mathbf{1}_{S\cap A^c}\frac{\lambda_{n_0}\cdot\min(\lambda_{n_0},\lambda_{n_1})\cdot\lambda_{n_2}}{\frac{1}{2}\lambda_{n_0}^2}\le 2\min(\lambda_{n_0},\lambda_{n_1})\le 2\lambda_{n_1}.$$

Consequently, taking $N=s$, we get by (3.10),

$$\begin{aligned}&\left|\mathbf{1}_{S\cap A^c}F(n_0,\vec{n}',\overline{u})\right|\\ &\quad\le C\mathbf{1}_{S\cap A^c}\left|\frac{1}{(\lambda_{n_0}^2-\sum_{j=1}^p\lambda_{n_j}^2)}\right|\frac{\lambda_{n_1}^s\lambda_{n_2}^\nu}{(\lambda_{n_0}\lambda_{n_1})^{\frac{1}{2(d+3)}}}\prod_{j=1}^p\left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

As a result, when $|\lambda_{n_0}^2 - \lambda_{n_1}^2| > C\lambda_{n_2}^2$ for some large C, one has

$$\left|\frac{1}{(\lambda_{n_0}^2 - \sum_{j=1}^{p} \lambda_{n_j}^2)}\right| \leq C\frac{1}{|\lambda_{n_0}^2 - \lambda_{n_1}^2| + 1},$$

so we obtain in this case,

$$|\mathbf{1}_{S\cap A^c} F(n_0, \vec{n}', \overline{u})| \leq C\frac{1}{|\lambda_{n_0}^2 - \lambda_{n_1}^2| + 1} \cdot \frac{\lambda_{n_1}^s \lambda_{n_2}^\nu}{(\lambda_{n_0}\lambda_{n_1})^{\frac{1}{2(d+3)}}} \prod_{j=1}^{p} \left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)}; \tag{3.11}$$

while when $|\lambda_{n_0}^2 - \lambda_{n_1}^2| \leq C\lambda_{n_2}^2$, by (3.4) we have

$$|\mathbf{1}_{S\cap A^c} F(n_0, \vec{n}', \overline{u})| \leq C\frac{\lambda_{n_2}^2}{|\lambda_{n_0}^2 - \lambda_{n_1}^2| + 1} \frac{\lambda_{n_1}^s \lambda_{n_2}^\nu}{(\lambda_{n_0}\lambda_{n_1})^{\frac{1}{2(d+3)}}} \prod_{j=1}^{p} \left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)}. \tag{3.12}$$

Since

$$(\lambda_{n_0}\lambda_{n_1})^{\frac{1}{2(d+3)}} \geq (|\lambda_{n_0}^2 - \lambda_{n_1}^2| + 1)^\tau$$

for some $\tau \in \left(0, \frac{1}{4(d+3)}\right)$, we deduce from (3.11), (3.12), Hölder's and Young's inequalities:

$$\begin{aligned}
\sum_{n_0\in\mathbb{N}} T_{A^c}(n_0)^2 &= \sum_{n_0\in\mathbb{N}} \left(\sum_{\vec{n}'\in\mathbb{N}^p} \mathbf{1}_{S\cap A^c} F(n_0, \vec{n}', \overline{u})\right)^2 \\
&\leq C\sum_{n_0\in\mathbb{N}} \left(\sum_{\vec{n}'\in\mathbb{N}^p} \frac{\lambda_{n_1}^s \lambda_{n_2}^{\nu+2}}{(|\lambda_{n_0}^2 - \lambda_{n_1}^2| + 1)^{1+\tau}} \prod_{j=1}^{p} \left\|\Pi_{n_j}\overline{u}\right\|_{L^2(\mathbb{R}^d)}\right)^2 \\
&\leq C\sum_{n_0\in\mathbb{N}} \left(\sum_{n_1\in\mathbb{N}} \frac{\lambda_{n_1}^s}{(|n_0 - n_1| + 1)^{1+\tau}} \|\Pi_{n_1}\overline{u}\|_{L^2(\mathbb{R}^d)} \|\overline{u}\|_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}^{p-1}\right)^2 \\
&\leq C\left(\sum_{n_0\in\mathbb{N}} \frac{1}{(n_0+1)^{1+\tau}}\right)^2 \left(\sum_{n_1\in\mathbb{N}} \lambda_{n_1}^{2s} \|\Pi_{n_1}\overline{u}\|_{L^2(\mathbb{R}^d)}^2\right) \|\overline{u}\|_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}^{2p-2} \leq C\, \|\overline{u}\|_{\widetilde{\mathcal{H}}^s(\mathbb{R}^d)}^{2p},
\end{aligned}$$

provided $s > \nu + 4$. This finishes the proof of the first estimate in the claim. The other estimate in (3.7) can be shown in the same way by noting that $\widetilde{\mathcal{H}}^s\left(\mathbb{R}^d\right)$ is an algebra when s is large. This concludes the proof of the claim and thus that of Lemma 3.1. □

Now we go back to prove the main theorem. Note that there is a constant C_1 independent of t such that

$$C_1^{-2}\left\|u(t)\right\|_{\Sigma^s}^2 \le E_s(u)(t) \le C_1\left\|u(t)\right\|_{\Sigma^s}^2,$$

and that there is another constant $M>0$ such that

$$E_s(u)(0)\le M^2\varepsilon^2, \qquad Q(0)\le M^{p+1}\varepsilon^{p+1}\le M^2\varepsilon^2, \quad \text{if } M\varepsilon<1,$$

since u_0 belongs to the unit ball of Σ^s. Let K be a constant such that $K>2M$. We claim that if ε is small enough, we have

$$E_s(u)(t)\le K^2\varepsilon^2\le 1, \qquad \forall\, t\in[-c\varepsilon^{-(2p-2)}, c\varepsilon^{-(2p-2)}], \tag{3.13}$$

for some $K>0$. We use the standard continuity argument to prove this claim. By the choice of K, we know (3.13) holds when $t=0$. Thus it suffices to show that the bound of form (3.13) actually implies the stronger estimate

$$E_s(u)(t)\le \frac{3}{4}K^2\varepsilon^2, \qquad \forall\, t\in[-c\varepsilon^{-(2p-2)}, c\varepsilon^{-(2p-2)}].$$

We deduce from Lemma 3.1

$$E_s(u)(t)\le Q(t)+E_s(u)(0)-Q(0)+c\int_0^t \left\|u(\tau)\right\|_{\Sigma^s}^{2p} d\tau.$$

Since the bounds of (3.13) imply

$$||u(\tau)||_{\Sigma^s(\mathbb{R}^d)}\le c_1\sqrt{E_s(u)(\tau)}\le c_1K\varepsilon,$$

we obtain form Lemma 3.1

$$E_s(u)(t)\le cc_1^{p+1}K^{p+1}\varepsilon^{p+1}+2M^2\varepsilon^2+cc_1^{2p}K^{2p}\varepsilon^{2p}|t|\le \frac{3}{4}K^2\varepsilon^2,$$

for any $|t|\le \frac{1}{8}(cc_1^{2p}K^{2p-2})^{-1}\varepsilon^{-(2p-2)}$, provided ε is so small that

$$cc_1^{p+1}K^{p-1}\varepsilon^{p-1}<\frac{1}{8}.$$

This concludes the proof of the claim. So the solution exists at least over a time interval of length of order $c\varepsilon^{-(2p-2)}$ with uniformly bounded Sobolev norm estimate (1.5) holding on that interval.

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