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On properties derived from different types of asymptotic distribution functions of ratio sequences

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Dedicated to Professor Oto Strauch on the occasion of his 75th birthday

Abstract. Let $X = \{x_1 < x_2 < \cdots\}$ be an infinite subset of positive integers and $X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$, $n = 1, 2, \dots$. In this paper we give new necessary and sufficient conditions for X for that the sequence of blocks X_n has an asymptotic distribution function.

1. Introduction

Denote by $\mathbb{N}$ the set of all positive integers. Let $X = \{x_1 < x_2 < x_3 < \cdots\}$ be an infinite subset of $\mathbb{N}$.

The following sequence of finite sequences derived from X

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

is called *the ratio block sequence* of the sequence X.

It is formed by the blocks $X_1, X_2, \dots, X_n, \dots$, where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \quad n = 1, 2, \dots$$

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is called the *n-th block*. This kind of block sequences was introduced by O. Strauch and J. T. Tóth [11] and they studied the set $G(X_n)$ of its distribution functions.

In this paper we prove that a block sequence X_n has an asymptotic distribution function if and only if the arithmetic mean of the elements of X_n tends to a non-negative real number not greater than $\frac{1}{2}$ for $n \to \infty$. Further, we show that if the asymptotic distribution function is of the form x^λ, then $\ln x_n$ is asymptotically equal to $\frac{1}{\lambda} \ln n$.

The rest of our paper is organized as follows. In Section 2 and Section 3, we recall some known definitions, notations and theorems, which will be used and extended. In Section 4, our new results are presented.

2. Definitions

The following basic definitions are from O. Strauch's paper [9].

- $1 \le x_1 < x_2 < \cdots$ denotes a sequence of positive integers, and x denotes an element of $(0,1]$.
- For each $n \in \mathbb{N}$, consider the *step distribution function*

$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$

  for $x \in [0,1)$, and for $x = 1$, we define $F(X_n, 1) = 1$.
- A non-decreasing function $g : [0,1] \to [0,1]$, $g(0) = 0$, $g(1) = 1$ is called a *distribution function* (abbreviated d.f.). We shall identify any two d.f.s coinciding at common points of continuity.
- A d.f. $g(x)$ is a d.f. of the sequence of blocks X_n, $n = 1, 2, \ldots$, if there exists an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

  a.e. on $[0,1]$. This is equivalent to the weak convergence, i.e., the preceding limit holds for every point $x \in [0,1]$ of continuity of $g(x)$.
- Denote by $G(X_n)$ the set of all d.f.s of X_n, $n = 1, 2, \ldots$. The set of distribution functions of ratio block sequences was studied in [1]–[6], [9].

  If $G(X_n) = \{g(x)\}$ is a singleton, the d.f. $g(x)$ is also called the *asymptotic distribution function* (abbreviated a.d.f.) of X_n.

Especially, if $G(X_n) = \{x\}$, then we say that the sequence of blocks X_n is uniformly distributed (abbreviated as u.d.) in $[0, 1]$.

We will use some auxiliary results based on the following two theorems of Helly (see the First and Second Helly theorem [10, Th. 4.1.0.10 and Th. 4.1.0.11, p. 45]).

- *Helly's selection principle*: For any sequence $g_n(x)$, $n = 1, 2, \dots$, of d.f.s in $[0, 1]$, there exists a subsequence $g_{n_k}(x)$, $k = 1, 2, \dots$, and a d.f. $g(x)$ such that $\lim_{k\to\infty} g_{n_k}(x) = g(x)$ a.e.
- *Second Helly theorem*: If we have $\lim_{n\to\infty} g_{n_k}(x) = g(x)$ a.e. in $[0, 1]$, then for every continuous function $f : [0, 1] \to \mathbb{R}$, we have $\lim_{n\to\infty} \int_0^1 f(x)\mathrm{d}g_n(x) = \int_0^1 f(x)\mathrm{d}g(x)$.
- Note that applying Helly's selection principle, from the sequence $F(X_n, x)$, $n = 1, 2, \dots$, one can select a subsequence $F(X_{n_k}, x)$, $k = 1, 2, \dots$, such that $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$ holds not only for the continuity points x of $g(x)$, but also for all $x \in [0, 1]$.
- We will use the one-step d.f. $c_\alpha(x)$ with step 1 at α defined on $[0, 1]$ via

$$c_\alpha(x) = \begin{cases} 0, & \text{if } x \le \alpha, \\ 1, & \text{if } x > \alpha. \end{cases}$$

In particular, we always have $c_\alpha(0) = 0$ and $c_\alpha(1) = 1$.

3. Overwiew of known results

In this section we mention known results related to the topic of this paper and some other ones we use in the proofs of our theorems.

(A1) If $g(x) \in G(X_n)$ increases and is continuous at $x = \beta$ and $g(\beta) > 0$, then there exists $1 \le \alpha < \infty$ such that $\alpha g(x\beta) \in G(X_n)$. If every d.f. of $G(X_n)$ is continuous at 1, then $\alpha = 1/g(\beta)$, [11, Prop. 3.1 and Th. 3.2].

(A2) Assume that $G(X_n)$ is singleton, i.e., $G(X_n) = \{g(x)\}$. Then either $g(x) = c_0(x)$ for $x \in [0, 1]$; or $g(x) = x^\lambda$ for some $0 < \lambda \le 1$ and $x \in [0, 1]$, [11, Th. 8.2].

(A3) Let $0 < \lambda \le 1$ be a real number. Then $G(X_n) = \{x^\lambda\}$ if and only if for every $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\frac{1}{\lambda}}, \tag{2}$$

[5, Th. 1].

(A4) Let $0 < \lambda \le 1$ be a real number. If $G(X_n) = \{x^\lambda\}$, then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1,$$

[3, Remark 3].

(A5) Assume that all d.f.s in $G(X_n)$ are continuous at 1. Then all d.f.s in $G(X_n)$ are continuous on $(0, 1]$, i.e., the only discontinuity point can be 0 [11, Th. 4.1].

(A6) Assume that all d.f. in $G(X_n)$ are continuous at 1 and that, for every $g \in G(X_n)$, we have $\int_0^1 g(x)\mathrm{d}x = c$, where $c > 0$ is a constant. Then:

(i) If $g \in G(X_n)$ increases at every point $\beta \in (0, 1)$, then $g(x) = x^{\frac{1-c}{c}}$ for every $x \in [0, 1]$.

(ii) If $g \in G(X_n)$ is constant on $(\alpha, \beta) \subset (0, 1]$, then $G(X_n)$ is a singleton and $g(x) = c_0(x)$ a.e. on $[0, 1]$, [11, Th. 7.2].

(A7) The L^2 discrepancy of the block X_n is defined by

$$D^{(2)}(X_n) = \int_0^1 \big(F(X_n, x) - x\big)^2 \mathrm{d}x,$$

which can be expressed as

$$D^{(2)}(X_n) = \frac{1}{3} + \frac{1}{nx_n^2}\sum_{i=1}^n x_i^2 - \frac{1}{nx_n}\sum_{i=1}^n x_i - \frac{1}{2n^2 x_n}\sum_{i,j=1}^n |x_i - x_j|,$$

see [8]. For every increasing sequence x_n of positive integers

$$\lim_{n\to\infty} D^{(2)}(X_n) = 0 \Longleftrightarrow \lim_{n\to\infty} F(X_n, x) = x \tag{3}$$

holds. The left-hand side can be divided into three limits (cf. [7, Th. 1]):

$$\lim_{n\to\infty} D^{(2)}(X_n) = 0 \Longleftrightarrow \begin{cases} \text{(i)} & \lim_{n\to\infty} \frac{1}{nx_n}\sum_{i=1}^n x_i = \frac{1}{2}, \\ \text{(ii)} & \lim_{n\to\infty} \frac{1}{nx_n^2}\sum_{i=1}^n x_i^2 = \frac{1}{3}, \\ \text{(iii)} & \lim_{n\to\infty} \frac{1}{n^2 x_n}\sum_{i,j=1}^n |x_i - x_j| = \frac{1}{3}. \end{cases} \tag{4}$$

(A8)

$$G(X_n) = \{c_0(x)\} \Longleftrightarrow \lim_{n\to\infty} \frac{1}{nx_n}\sum_{i=1}^n x_i = 0, \tag{5}$$

$$G(X_n) = \{c_0(x)\} \Longleftrightarrow \lim_{m,n\to\infty} \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \left|\frac{x_i}{x_m} - \frac{x_j}{x_n}\right| = 0, \tag{6}$$

[11, Th. 7.1].

(A9)

$$c_0(x) \in G(X_n) \Longleftrightarrow \liminf_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 0,$$

[4, Th. 4].

(A10) $\max_{g\in G(X_n)} \int_0^1 g(x)\mathrm{d}x \geq \frac{1}{2}$, [11, Th. 7.1].

(A11) By the Helly theorem and integration by parts of the Riemann–Stieltjes integral, if $F(X_{n_k}, x) \to g(x)$, then

$$\int_0^1 x\mathrm{d}F(X_{n_k}, x) = \frac{1}{n_k} \sum_{i=1}^{n_k} \frac{x_i}{x_{n_k}} \quad \to \quad \int_0^1 x\mathrm{d}g(x) = 1 - \int_0^1 g(x)\mathrm{d}x$$

[9, p. 155].

4. Results

The equivalence (5) gives a characterization of the set X for the case $G(X_n) = \{c_0(x)\}$. One may ask what happens if we assume that the arithmetic mean of the numbers $\frac{x_i}{x_n}$, $i = 1, 2, \ldots, n$ tends to a nonzero real number as $n \to \infty$. We will make use of the following lemma.

Lemma 1. *Let*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = s > 0. \tag{7}$$

Then all distribution functions in $G(X_n)$ are continuous at 1.

Proof. By (A10) and (A11) we have that $s \leq \frac{1}{2}$. Assuming the contrary, let us suppose that there exists a d.f. $g(x)$, which is not continuous at 1. Then, for some $h > 0$ and arbitrary $\varepsilon > 0$, there exists a sequence n_k, $k = 1, 2, \ldots$ such that

$$g(1-\varepsilon) = \lim_{k\to\infty} \frac{\#\{i \leq n_k : \frac{x_i}{x_{n_k}} < 1-\varepsilon\}}{n_k} \leq 1 - h < 1.$$

Namely, there exists an $h > 0$ such that for arbitrary $\varepsilon > 0$, we can find sequences m_k, n_k, $k = 1, 2, \ldots$ with the properties

$$(1-\varepsilon)x_{n_k} \leq x_{m_k} \quad \text{and} \quad \lim_{k\to\infty} \frac{\#\{i : x_{m_k} \leq x_i \leq x_{n_k}\}}{n_k} \geq h. \tag{8}$$

By (7)–(8) for given $\varepsilon > 0$, there exists a k_0 such that for arbitrary $k > k_0$, we have

$$\sum_{i=1}^{m_k} \frac{x_i}{x_{m_k}} > m_k(s-\varepsilon), \tag{9}$$

further

$$\frac{x_{m_k}}{x_{n_k}} \geq (1-\varepsilon), \tag{10}$$

and

$$\frac{\#\{i : x_{m_k} \leq x_i \leq x_{n_k}\}}{n_k} > h - \varepsilon \tag{11}$$

for the sequences m_k, n_k, $k = 1, 2, \ldots$.

To get a contradiction, we shall make use of the observation that there are relatively a "lot" of elements between x_{m_k} and x_{n_k}, it follows that the average of the members of the block X_{n_k} will be greater than $s+\varepsilon$; thus, (7) fails, which is a contradiction. For this, we give a lower bound for

$$\begin{aligned} S(n_k) = \frac{1}{n_k}\sum_{i=1}^{n_k}\frac{x_i}{x_{n_k}} &= \frac{1}{n_k}\left(\sum_{i=1}^{m_k}\frac{x_i}{x_{n_k}} + \sum_{i=m_k+1}^{n_k}\frac{x_i}{x_{n_k}}\right) \\ &= \frac{1}{n_k}\left(\frac{x_{m_k}}{x_{n_k}}\sum_{i=1}^{m_k}\frac{x_i}{x_{m_k}} + \sum_{i=m_k+1}^{n_k}\frac{x_i}{x_{n_k}}\right). \end{aligned}$$

Using (10) together with (9) and by the inequalities $x_i > x_{m_k}$ for $i = m_k + 1, \ldots, n_k$, we get

$$\begin{aligned} S(n_k) &> \frac{1}{n_k}\left((1-\varepsilon)m_k(s-\varepsilon) + (n_k - m_k)\frac{x_{m_k}}{x_{n_k}}\right) \\ &\geq \frac{1}{n_k}\big((1-\varepsilon)m_k(s-\varepsilon) + (n_k-m_k)(1-\varepsilon)\big), \end{aligned}$$

which we can rewrite in the form

$$S(n_k) > (1-\varepsilon) - \frac{m_k}{n_k}(1-\varepsilon)(1-s+\varepsilon),$$

so, by (11)

$$S(n_k) > (1-\varepsilon) + (h-\varepsilon-1)(1-\varepsilon)(1-s+\varepsilon).$$

For $\varepsilon \to 0$, this lower bound tends to $s + h - sh$. Therefore,

$$\liminf_{k\to\infty} \frac{1}{n_k}\sum_{i=1}^{n_k}\frac{x_i}{x_{n_k}} \geq s + h - sh = s + h(1-s) > s,$$

which is a contradiction with (7). □

Theorem 1. *Let $0 < \lambda \le 1$ be a real number. Then $G(X_n) = \{x^\lambda\}$ if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = \frac{\lambda}{\lambda+1}. \tag{12}$$

Proof. The proof of the sufficient part follows from the Helly theorem (see (A11)). We have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = \int_0^1 x \mathrm{d}g(x) = 1 - \int_0^1 g(x)\mathrm{d}x = 1 - \int_0^1 x^\lambda \mathrm{d}x = \frac{\lambda}{\lambda+1}.$$

Now, let us suppose that (12) holds. Let $\tilde{g}(x) \in G(X_n)$ and $F(X_{n_k}, x) \to \tilde{g}(x)$ for $k \to \infty$. Then by (A11),

$$\int_0^1 \tilde{g}(x)\mathrm{d}x = 1 - \int_0^1 x \mathrm{d}\tilde{g}(x) = 1 - \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=1}^{n_k} \frac{x_i}{x_{n_k}} = 1 - \frac{\lambda}{\lambda+1} = \frac{1}{\lambda+1}.$$

It follows from Lemma 1 that all distribution functions in $G(X_n)$ are continuous at 1. If $\tilde{g}(x)$ were constant on some interval $(\alpha, \beta] \subset (0,1]$, then we would have $\tilde{g}(x) = c_0(x)$, but it is impossible according to (5). Therefore, by part (i) of (A6), we get

$$\tilde{g}(x) = x^{\frac{1-\frac{1}{\lambda+1}}{\frac{1}{\lambda+1}}} = x^\lambda. \qquad \square$$

The proved theorem has interesting consequences for the case $\lambda = 1$. Then X_n is uniformly distributed. If (12) holds for $\lambda = 1$ (i.e., part (i) of (4) holds), then $\lim_{n\to\infty} D^{(2)}(X_n) = 0$, for this reason, by (A7) parts (ii) and (iii) of (4) hold too. This consequence of the theorem we formulate in the next corollary.

Corollary 1. *Let*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = \frac{1}{2}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n x_n^2} \sum_{i=1}^{n} x_i^2 = \frac{1}{3} \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{n^2 x_n} \sum_{i,j=1}^{n} |x_i - x_j| = \frac{1}{3}.$$

It is worth mentioning that we did not improve (4), the very nice and deep result of O. Strauch in the general case for which it was originally formulated. In [7], distribution functions for sequences reduced modulo 1 were studied.

We will be looking at sets the block-sequences of which have a single distribution function. According to the previous discussion, this can be only of type $c_0(x)$

or $x^\lambda, \lambda \in (0,1]$. We will show that λ is closely related to how fast the sequence x_n is increasing. In the case $\lambda = 1$ (uniform distribution), the sequence x_n may increase only slowly. With the decrease of λ, x_n increases more rapidly, finally, it increases the fastest in the case of $c_0(x)$.

We will deal with the case $G(X_n) = \{c_0(x)\}$ separately.

Theorem 2. *Let $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$, and let $G(X_n) = \{c_0(x)\}$. Then*

$$\lim_{n\to\infty} \frac{\ln x_n}{\ln n} = \infty. \tag{13}$$

Proof. As the distribution function of X_n is identically equal to 1 for all $x \in (0,1]$, for any $x \in (0,1]$, we have

$$\lim_{n\to\infty} \frac{\#\{i : \frac{x_i}{x_{2n}} < x\}}{2n} = 1.$$

In particular, for a fixed $\varepsilon > 0$, for large enough n we have

$$\frac{\#\{i : \frac{x_i}{x_{2n}} < \varepsilon\}}{2n} > \frac{1}{2},$$

which is possible only if $x_n < \varepsilon x_{2n}$.

What (13) means is that, for an arbitrary $k > 0$, there is a threshold index n_0, so whenever $n > n_0$,

$$x_n > n^k.$$

We will prove this by contradiction. Assume there is a $k > 0$ and infinitely many corresponding natural numbers m with

$$x_m \le m^k. \tag{14}$$

Fix $\varepsilon > 0$ so that

$$\varepsilon < 2^{-3k}. \tag{15}$$

Then there is an n_0 such that $x_n < \varepsilon x_{2n}$ for all $n > n_0$.

Since (14) is satisfied by infinitely many numbers, we can find natural numbers l, m so that

$$x_m \le m^k, \quad 2^l > n_0, \quad \text{and} \quad 2^{2l} \le m < 2^{2l+1}. \tag{16}$$

Multiplying the inequalities

$$x_{2^l} < \varepsilon x_{2^{l+1}}$$
$$x_{2^{l+1}} < \varepsilon x_{2^{l+2}}$$
$$\dots$$
$$x_{2^{2l-1}} < \varepsilon x_{2^{2l}}$$

we get

$$x_{2^l} < \varepsilon^l x_{2^{2l}}, \tag{17}$$

however, (16) implies

$$x_{2^{2l}} \leq x_m \leq m^k < 2^{(2l+1)k}.$$

Moreover, it follows by (17) and (15) that

$$x_{2^l} < \varepsilon^l 2^{(2l+1)k} < 2^{-3lk+(2l+1)k} = 2^{k-lk} \leq 1,$$

thus, $x_{2^l} < 1$, a contradiction. □

When the single distribution function is of the form x^λ, we will use (A3).

Theorem 3. *Let $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$, and $\lambda \in (0,1]$ be a real number. If $G(X_n) = \{x^\lambda\}$, then*

$$\lim_{n\to\infty} \frac{\ln x_n}{\ln n} = \frac{1}{\lambda}. \tag{18}$$

Proof. It suffices to show that for any ε, where $0 < \varepsilon < \frac{1}{\lambda}$, there is an n_0 so that whenever $n > n_0$,

$$n^{\frac{1}{\lambda}-\varepsilon} < x_n < n^{\frac{1}{\lambda}+\varepsilon}. \tag{19}$$

Indeed, let $\varepsilon > 0$ be given. It follows from (2) that for each $\eta > 0$, there is some m so that for all $n \geq m$,

$$2^{\frac{1}{\lambda}}(1-\eta) < \frac{x_{2n}}{x_n} < 2^{\frac{1}{\lambda}}(1+\eta). \tag{20}$$

Repeatedly using the above inequalities, we get

$$2^{\frac{1}{\lambda}}(1-\eta) < \frac{x_{2m}}{x_m} < 2^{\frac{1}{\lambda}}(1+\eta)$$
$$2^{\frac{1}{\lambda}}(1-\eta) < \frac{x_{4m}}{x_{2m}} < 2^{\frac{1}{\lambda}}(1+\eta)$$
$$\dots$$
$$2^{\frac{1}{\lambda}}(1-\eta) < \frac{x_{2^k m}}{x_{2^{k-1} m}} < 2^{\frac{1}{\lambda}}(1+\eta),$$

which multiply to

$$2^{\frac{k}{\lambda}}(1-\eta)^k < \frac{x_{2^k m}}{x_m} < 2^{\frac{k}{\lambda}}(1+\eta)^k, \tag{21}$$

respectively, after replacing k with $k+1$, we get

$$2^{\frac{k+1}{\lambda}}(1-\eta)^{k+1} < \frac{x_{2^{k+1} m}}{x_m} < 2^{\frac{k+1}{\lambda}}(1+\eta)^{k+1}. \tag{22}$$

Fix some $\eta > 0$ for which

$$2^{\varepsilon} > 1+\eta \quad \text{and} \quad 2^{\varepsilon}(1-\eta) > 1, \tag{23}$$

and let m be the corresponding threshold index, i.e., such a number that (20) be true for all $n \geq m$. Let k_0 be the smallest natural number k such that

$$\left(\frac{2^{\varepsilon}}{1+\eta}\right)^k > x_m 2^{\frac{1}{\lambda}}(1+\eta) \quad \text{and} \quad (2^{\varepsilon}(1-\eta))^k > \frac{2^{\frac{1}{\lambda}-\varepsilon}}{1-\eta} m^{\frac{1}{\lambda}} \tag{24}$$

(as the right-hand side is constant, this is possible), moreover, n_0 be $2^{k_0} m$.

Let $n \geq n_0$ be an arbitrary natural number. Then we can find $k \geq k_0$ with

$$2^k m \leq n < 2^{k+1} m.$$

(i) In (19), we will first consider only the case

$$x_n < n^{\frac{1}{\lambda}+\varepsilon},$$

the second inequality works analogously. Since

$$x_n < x_{2^{k+1} m} < x_m 2^{\frac{k+1}{\lambda}}(1+\eta)^{k+1} \quad \text{and} \quad n^{\frac{1}{\lambda}+\varepsilon} \geq (2^k m)^{\frac{1}{\lambda}+\varepsilon},$$

it suffices to show that

$$(2^k m)^{\frac{1}{\lambda}+\varepsilon} > x_m 2^{\frac{k+1}{\lambda}}(1+\eta)^{k+1}$$

holds. This inequality is equivalent to

$$\left(\frac{2^{\varepsilon}}{1+\eta}\right)^k > \frac{x_m 2^{\frac{1}{\lambda}}(1+\eta)}{m^{\frac{1}{\lambda}+\varepsilon}},$$

which follows from (23) and (24).

(ii) The inequality

$$n^{\frac{1}{\lambda}-\varepsilon} < x_n$$

follows from

$$x_m 2^{\frac{k}{\lambda}}(1-\eta)^k > (2^{k+1}m)^{\frac{1}{\lambda}-\varepsilon},$$

which is obtained from

$$x_n > x_{2^k m} \geq x_m 2^{\frac{k}{\lambda}}(1-\eta)^k \quad \text{and} \quad n^{\frac{1}{\lambda}-\varepsilon} < (2^{k+1}m)^{\frac{1}{\lambda}-\varepsilon}.$$

This, in turn, follows from the second part of (23) and (24). □

Corollary 2. *Let* $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$, *and the sequence of blocks* X_n *be uniformly distributed in* $[0,1]$, *i.e.,* $G(X_n) = \{x\}$. *Then*

$$\lim_{n\to\infty} \frac{\ln x_n}{\ln n} = 1.$$

The subsequent examples will demonstrate that the two previous theorems cannot be reversed.

Example. Define the sequence a_n as follows: let $a_n = n!$ for any $n = 1, 2. \dots$. Consider the set

$$X = \{x_1 < x_2 < x_3 < \cdots\} = \bigcup_{n=1}^{\infty} \{a_n + 1, a_n + 2, \dots, a_n + 2^{n-1}\}.$$

The set X shows that the converse of Theorem 2 is false. This set satisfies (13). It is easy to check that

$$\limsup_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i \geq \frac{1}{2},$$

thus, by (5), we have that $G(X_n) \neq \{c_0(x)\}$.

We will now define a set Y which will show that the converse of Theorem 3 is false for $\lambda = 1$. Let

$$Y = \{y_1 < y_2 < \cdots\} = \left(\bigcup_{n=1}^{\infty} [a_{2n-1}, a_{2n}) \cap \mathbb{N}\right) \cup \left(\bigcup_{n=1}^{\infty} [a_{2n}, a_{2n+1}) \cap 2\mathbb{N}\right),$$

where $2\mathbb{N}$ denotes the set of all even positive integers. It is easily shown that $\ln y_n$ is asymptotically equal to $\ln n$. We will prove that the block-sequence (Y_n) is not uniformly distributed.

Let $n \geq 5$ be an arbitrary odd natural number. Denote by m the natural number for which $y_m = n!$. The definition of Y immediately yields that $m < \frac{3}{4}n!$ and $y_{2m} = n! + m$. Thus

$$\frac{y_{2m}}{y_m} = \frac{n! + m}{n!} < \frac{n! + \frac{3}{4}n!}{n!} = \frac{7}{4}. \tag{25}$$

If we suppose that the block-sequence (Y_n) is uniformly distributed, then (2) implies that

$$\lim_{m\to\infty} \frac{y_{2m}}{y_m} = 2.$$

But this contradicts (25). Hence (Y_n) is not uniformly distributed.

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