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On the partitions into squares whose reciprocal sum is one

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Abstract. We show that for all positive integers $n > 1223$, there is a partition of n into squares whose reciprocals sum to 1.

1. Introduction

Graham [2] showed that for every positive integer $n > 77$, there is a partition of n into distinct parts such that the sum of reciprocals of the parts equals to 1. For example,

$$78 = 2 + 6 + 8 + 10 + 12 + 40 \qquad \text{and} \qquad \frac{1}{2} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{40} = 1.$$

More recently, Alekseyev [1] obtained a similar result, namely, for every positive integer $n > 8542$, there is a partition of n into squares of distinct integers whose reciprocals sum to 1. For example, the smallest positive integer (except 1) having such a partition is 49, i.e.,

$$49 = 2^2 + 3^2 + 6^2 \qquad \text{and} \qquad \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

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In this note, we investigate an analogous problem which asks whether there is a partition of n into squares whose sum of reciprocals is 1. In this case, as $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we cannot expect there is such a partition of $n > 1$ into distinct squares. For example, here are the first few positive integers having such partitions:

$$\begin{aligned} 1 &= 1^2 \quad \text{and} \quad \frac{1}{1^2} = 1, \\ 16 &= 4 \times 2^2 \quad \text{and} \quad \frac{4}{2^2} = 1, \\ 66 &= 3 \times 2^2 + 2 \times 3^2 + 6^2 \quad \text{and} \quad \frac{3}{2^2} + \frac{2}{3^2} + \frac{1}{6^2} = 1, \\ 76 &= 3 \times 2^2 + 4 \times 4^2 \quad \text{and} \quad \frac{3}{2^2} + \frac{4}{4^2} = 1. \end{aligned}$$

While there are very sparse examples at the beginning, there is always such a partition for sufficiently large integers.

Theorem 1.1. *For every integer $n > 1223$, there is a sequence of positive integers $(x_1, x_2, \ldots, x_k)$ satisfying that*

(1) $0 < x_1 \leq x_2 \leq \cdots \leq x_k$;

(2) $\sum_{i=1}^{k} x_i^2 = n$;

(3) $\sum_{i=1}^{k} \frac{1}{x_i^2} = 1$.

2. Proof of Theorem 1.1

For a positive integer n, we say a sequence $X = (x_1, x_2, \ldots, x_k)$ represents n if X satisfies the three conditions in Theorem 1.1. We also say n is representable if there is a sequence X which represents n. The following lemma is a key to employ a mathematical induction to prove Theorem 1.1.

Lemma 2.1. *If $X = (x_1, \ldots, x_k)$ represents n, then $4n+12$, $4n+197$, $4n+62$ and $4n + 247$ are also representable.*

Proof. Note that if X represents n, then

(1) $(2, 2, 2) \cup 2X$ represents $4n + 12$;

(2) $(2, 2, 3, 6, 6, 6, 6, 6) \cup 2X$ represents $4n + 197$;

(3) $(2, 2, 3, 3, 6) \cup 2X$ represents $4n + 62$;

(4) $(2, 3, 3, 3, 6, 6, 6, 6, 6, 6) \cup 2X$ represents $4n + 247$. □

For the few thousands cases, we need to find representations of small integers n. However, there are 39285423 partitions of 1000 into squares larger than 1. Therefore, we need to use some structure of the representations. The following lemma is an analogue of [1, Lemma 2] for the special case $d = 2$.

Lemma 2.2. *Suppose that* $\sum_{i=1}^{k} x_i^2 = n$ *and* $\sum_{i=1}^{k} \frac{1}{x_i^2} = s$. *Then, we have*

(1) $k \leq \sqrt{sn}$;

(2) $\frac{1}{s} \leq x_1^2 \leq \sqrt{\frac{n}{s}}$;

(3) $x_k^2 \geq \sqrt{\frac{n}{s}}$.

Proof. For the first inequality, from the power mean inequality, we find that

$$\left(\frac{s}{k}\right)^{-1/2} \leq \left(\frac{n}{k}\right)^{1/2},$$

which implies the desired result. For the second inequality, we observe that

$$\frac{1}{x_1^2} \leq s \leq \frac{k}{x_1^2},$$

which implies the claimed inequality with the fact that $k \leq \sqrt{ns}$. Finally, we note that

$$k \cdot x_k^2 \geq n$$

implies that

$$x_k^2 \geq \frac{n}{k} \geq \sqrt{\frac{n}{s}}.$$

□

Using Lemma 2.2, we observe that

(1) there are representations for all positive integers n between 1224 and 2500;

(2) from 1284 to 2500, there is a representation of n containing 2.

See

`https://github.com/KKLPmath/SquaresWhoseReciprocalsSumToOne`

for the numerics. Using these data, we prove our main result.

Proof of Theorem 1.1. From Lemma 2.1, we first note that it suffices to show that there are representations of n between 1224 and $1224 \times 4 + 246 = 5142$. From 1501 to 2500, there is a representation containing 2, thus we can use the trick which replaces 4 by another sum of squares whose reciprocals sum to $1/4$.

We first note that

$$1004 = 2 \times 4^2 + 3 \times 6^2 + 6 \times 12^2$$

with

$$\frac{2}{4^2} + \frac{3}{6^2} + \frac{6}{12^2} = \frac{1}{4}.$$

Thus, in the representations from 1501 to 2500, by replacing 2 by

$$(4, 4, 6, 6, 6, 12, 12, 12, 12, 12, 12),$$

we find representations from $1501 + 1004 - 4 = 2501$ to $2500 + 1004 - 4 = 3500$.

Secondly, we observe that

$$2004 = 2 \times 4^2 + 5^2 + 6^2 + 7^2 + 2 \times 10^2 + 14^2 + 15^2 + 2 \times 20^2 + 21^2$$

with

$$\frac{2}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{2}{10^2} + \frac{1}{14^2} + \frac{1}{15^2} + \frac{2}{20^2} + \frac{1}{21^2} = \frac{1}{4}.$$

Thus, in the representations from 1501 to 2500, by replacing 2 by

$$(4, 4, 5, 6, 7, 10, 10, 14, 15, 20, 20, 21),$$

we obtain representations from $1501 + 2004 - 4 = 3501$ to $2500 + 2004 - 4 = 4500$.

Finally, we note that

$$3004 = 3^2 + 5^2 + 6^2 + 2 \times 10^2 + 2 \times 12^2 + 5 \times 14^2 + 15^2 + 2 \times 20^2 + 21^2$$

with

$$\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{2}{10^2} + \frac{2}{12^2} + \frac{5}{14^2} + \frac{1}{15^2} + \frac{2}{20^2} + \frac{1}{21^2} = \frac{1}{4}.$$

Thus, in the representations from 1501 to 2500, by replacing 2 by

$$(3, 5, 6, 10, 10, 12, 12, 14, 14, 14, 14, 14, 15, 20, 20, 21),$$

we have representations from $1501 + 3004 - 4 = 4501$ to $2500 + 3004 - 4 = 5500$.

In summary, we have shown that every integer between 1224 and 5142 is representable, and thus by the induction, this completes the proof. □

References

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