

Some Pexider-type generalizations of the symmetrized multiplicative functional equation on monoids

By BRUCE EBANKS (Louisville)

Abstract. The symmetrized multiplicative functional equation on a semigroup is $f(xy) + f(yx) = 2f(x)f(y)$, and it is known that such a function must be multiplicative if the co-domain is a field of characteristic different from 2. Here we consider some generalizations including the fully Pexiderized equation $f(xy) + g(yx) = h(x)k(y)$ for four unknown functions f, g, h, k . This equation has been solved on groups; here we solve it on monoids. Other related functional equations are also treated.

1. Introduction

Let S be a semigroup and K a (for us commutative) field. A function $f : S \rightarrow K$ is said to be *multiplicative* if

$$f(xy) = f(x)f(y) \quad \text{for all } x, y \in S,$$

and the equation is known as the (Cauchy) multiplicative functional equation. The *symmetrized multiplicative functional equation* is

$$f(xy) + f(yx) = 2f(x)f(y) \quad \text{for all } x, y \in S. \tag{1}$$

Clearly, the first equation implies the second. If one assumes that K is a field of characteristic different from 2, then the second equation also implies the first. Indeed, the following is [3, Theorem 3.21].

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Proposition 1. *Let S be a semigroup and K a field with $\text{char}(K) \neq 2$. Then $f : S \rightarrow K$ satisfies (1) if and only if f is multiplicative.*

The main focus of the present investigation is the fully Pexiderized version

$$f(xy) + g(yx) = h(x)k(y), \quad x, y \in S, \quad (2)$$

of (1), in which we have four unknown functions $f, g, h, k : S \rightarrow K$. This equation has been treated by ELFEN, PERKINS and SAHOO [2] for the case when S is a group.

Here we assume that the semigroup S is a *monoid*, which means there is an *identity* element $e \in S$ (that is, $ex = xe = x$ for all $x \in S$). Our co-domain K is a field with characteristic unequal to 2. Let $K^* := K \setminus \{0\}$.

We also need the following notion. If S is a monoid and K is a field, then a function $t : S \rightarrow K$ is said to be *central* if $t(xy) = t(yx)$ for all $x, y \in S$. Note that every multiplicative function from S to K is central since K is commutative.

The outline of the paper is as follows. The next section contains our main result, Theorem 3, which gives the general solution of (2). Then follows a section with some additional related results. One provides the solution of

$$f(xy) + g(yx) = h(x) + k(y) + \beta h(x)k(y), \quad x, y \in S,$$

under the assumption that $\beta \neq 0$. Another solves the functional equation

$$f(xy) + f(yx) = g(x)h(y) + k(y), \quad x, y \in S,$$

in the case that h is non-constant.

2. General solution of (2)

The following is an improved version of [2, Theorem 4]. We remove the assumptions that f, g , and h are nonzero functions.

Theorem 2. *Let S be a monoid, let K be a field with $\text{char}(K) \neq 2$, and suppose $f, g, h : S \rightarrow K$. Then the triple (f, g, h) satisfies*

$$f(xy) + g(yx) = h(x)h(y), \quad x, y \in S, \quad (3)$$

if and only if there exist a multiplicative function $m : S \rightarrow K$ with $m \neq 0$, a central function $t : S \rightarrow K$, and a constant $\alpha \in K$ such that

$$f = \frac{\alpha^2}{2}m + t, \quad g = \frac{\alpha^2}{2}m - t, \quad h = \alpha m. \quad (4)$$

PROOF. It is easily checked that the forms in (4) fulfill (3). For the converse, first suppose $h = 0$. Then (3) becomes

$$f(xy) + g(yx) = 0, \quad x, y \in S.$$

With $y = e$ this yields $g = -f$, so we have $f(xy) - f(yx) = 0$. Thus f is central and we have (4) with $\alpha = 0$.

For the rest of the proof, we assume $h \neq 0$ and follow the general outline of the proof of [2, Theorem 4], finding the forms of $f - g$ and $f + g$ independently and then combining them.

First we find $f - g$. Since the right-hand side of (3) is symmetric in x and y , we have

$$f(xy) + g(yx) = f(yx) + g(xy), \quad x, y \in S.$$

This means the function $t : S \rightarrow K$ defined by

$$t := \frac{f - g}{2}$$

is central, and $f - g = 2t$.

To find $f + g$, we put $y = e$ in (3), arriving at

$$f(x) + g(x) = h(x)h(e), \quad x \in S. \quad (5)$$

If $h(e) = 0$, then we see that $g = -f$, and (3) becomes

$$f(xy) - f(yx) = h(x)h(y), \quad x, y \in S.$$

Interchanging x and y here and comparing the result, we get

$$0 = 2h(x)h(y), \quad x, y \in S.$$

Since $\text{char}(K) \neq 2$, that contradicts our assumption that $h \neq 0$. Therefore $h(e) \neq 0$.

Interchanging x and y in (3) and adding, we arrive at

$$(f + g)(xy) + (f + g)(yx) = 2h(x)h(y), \quad x, y \in S. \quad (6)$$

With $\alpha := h(e) \in K^*$, using (5) we can rewrite this as

$$\alpha h(xy) + \alpha h(yx) = 2h(x)h(y), \quad x, y \in S,$$

which means the function h/α satisfies (1). Hence, by Proposition 1 we have $h = \alpha m$ for some multiplicative function $m : S \rightarrow K$. Moreover $m \neq 0$, since $h \neq 0$. Referring to (5) once more, we see that

$$f + g = \alpha h = \alpha^2 m, \quad x \in S.$$

Combining the forms of $f - g$ and $f + g$, we find that f and g are as claimed. $\square$

Our main result is the following generalization of [2, Theorem 5] to monoids. We again remove the assumptions that our unknown functions are nonzero. Recall that (2) is the fully Pexiderized symmetrized multiplication equation

$$f(xy) + g(yx) = h(x)k(y), \quad x, y \in S.$$

Theorem 3. *Let S be a monoid, K a field with $\text{char}(K) \neq 2$, and suppose $f, g, h, k : S \rightarrow K$. The solutions (f, g, h, k) of (2) fall into three classes:*

(i) *$h = 0$, k is arbitrary, and there exists a central function $t : S \rightarrow K$ such that*

$$f = -g = t.$$

(ii) *h is arbitrary, $k = 0$, and there exists a central function $t : S \rightarrow K$ such that*

$$f = -g = t.$$

(iii) *There exist a multiplicative function $m : S \rightarrow K$ with $m \neq 0$, a central function $t : S \rightarrow K$, and constants $\lambda, \alpha \in K^*$ such that*

$$f = \frac{\lambda\alpha}{2}m + t, \quad g = \frac{\lambda\alpha}{2}m - t, \quad h = \alpha m, \quad k = \lambda m. \quad (7)$$

PROOF. It is easy to check that the functions of the forms specified above satisfy equation (2). For the converse, consider first the case that $h = 0$ or $k = 0$. In either case, the other function (k or h respectively) is arbitrary. As in the proof of the previous theorem we get $f = -g = t$ for some central function $t : S \rightarrow K$, so we are in class (i) or (ii).

For the remainder of the proof, we assume that $h \neq 0$ and $k \neq 0$. Putting $y = e$ in (2) yields

$$f(x) + g(x) = h(x)k(e), \quad x \in S. \quad (8)$$

We claim that $k(e) \neq 0$. Indeed, suppose $k(e) = 0$. Then (8) shows that $g = -f$ and (2) gives

$$f(xy) = f(yx) + h(x)k(y), \quad x, y \in S.$$

Computing $f(xyz)$ two ways, we have, on the one hand,

$$f(x \cdot yz) = f(yzx) + h(x)k(yz) = f(zxy) + h(y)k(zx) + h(x)k(yz),$$

and on the other hand,

$$f(xy \cdot z) = f(zxy) + h(xy)k(z).$$

Hence

$$h(y)k(zx) + h(x)k(yz) = h(xy)k(z), \quad x, y, z \in S.$$

Putting $z = e$ here, we get

$$h(y)k(x) + h(x)k(y) = h(xy)k(e) = 0, \quad x, y \in S. \quad (9)$$

Since $h \neq 0$, we can choose $y = y_0$ such that $h(y_0) \neq 0$, and with this obtain that $k = \delta h$ for $\delta = -k(y_0)/h(y_0) \in K$. Moreover $\delta \in K^*$, since $k \neq 0$. Thus (9) reduces to

$$2h(x)h(y) = 0, \quad x, y \in S.$$

Since $\text{char}(K) \neq 2$, this contradicts $h \neq 0$.

Therefore $k(e) \neq 0$. Also, (2) with $x = e$ gives $f + g = h(e)k$. Comparing this with (8), we get

$$h = \frac{h(e)}{k(e)}k.$$

Since $h \neq 0$, this equation shows that $h(e) \neq 0$, so we have $k = \delta h$ for $\delta = k(e)/h(e) \in K^*$.

With this representation for k , equation (2) can now be written as

$$f(xy) + g(yx) = \delta h(x)h(y), \quad x, y \in S.$$

Thus the triple $(\frac{1}{\delta}f, \frac{1}{\delta}g, h)$ satisfies equation (3). By Theorem 2, we conclude that there exist a multiplicative function $m : S \rightarrow K$ with $m \neq 0$, a central function $t' : S \rightarrow K$, and a constant $\alpha \in K^*$ (since $h \neq 0$) such that

$$\frac{1}{\delta}f = \frac{\alpha^2}{2}m + t', \quad \frac{1}{\delta}g = \frac{\alpha^2}{2}m - t', \quad h = \alpha m.$$

Finally, defining $t : S \rightarrow K$ and $\lambda \in K^*$ by $t := \delta t'$ and $\lambda := \delta \alpha$ we have (7), as claimed. $\square$

3. Further results

In [2, Theorem 9], the authors also considered the functional equation

$$f(xy) + g(yx) = 2h(x) + 2h(y) + 2\beta h(x)h(y)$$

for three unknown functions f, g, h and a nonzero constant β , solving it on groups. Here we consider the more general equation

$$f(xy) + g(yx) = h(x) + k(y) + \beta h(x)k(y) \quad (10)$$

for four unknown functions and solve it on monoids.

Theorem 4. *Let S be a monoid, K a field with $\text{char}(K) \neq 2$, let $\beta \in K^*$ and $f, g, h, k : S \rightarrow K$. The solutions (f, g, h, k) of (10) fall into three classes:*

- (i) *$h = -1/\beta$, k is arbitrary, and there exists a central function $t : S \rightarrow K$ for which*

$$f = -g - \frac{1}{\beta} = t.$$

- (ii) *h is arbitrary, $k = -1/\beta$, and there exists a central function $t : S \rightarrow K$ for which*

$$f = -g - \frac{1}{\beta} = t.$$

- (iii) *There exist a multiplicative function $m : S \rightarrow K$ with $m \neq 0$, a central function $t : S \rightarrow K$, and constants $\lambda, \alpha \in K^*$ such that*

$$f = \beta \frac{\lambda\alpha}{2} m + t, \quad g = \beta \frac{\lambda\alpha}{2} m - t - \frac{1}{\beta}, \quad h = \alpha m - \frac{1}{\beta}, \quad k = \lambda m - \frac{1}{\beta}. \quad (11)$$

PROOF. First suppose that either $h = -1/\beta$ or $k = -1/\beta$. In either case, the other function of the pair (h, k) is arbitrary and our functional equation (10) becomes

$$f(xy) + g(yx) = -\frac{1}{\beta}.$$

With $y = e$ we get $g = -f - 1/\beta$, and the equation reduces further to $f(xy) - f(yx) = 0$, showing that f is central. Thus we are in solution class (i) or (ii).

For the rest of the proof, we assume that $h \neq -1/\beta \neq k$ and write (10) in the form

$$1 + \beta f(xy) + \beta g(yx) = [1 + \beta h(x)][1 + \beta k(y)], \quad x, y \in K.$$

Defining

$$f' := \beta f, \quad g' := 1 + \beta g, \quad h' := 1 + \beta h, \quad k' := 1 + \beta k,$$

we see that (f', g', h', k') is a solution of (2), with $h' \neq 0$ and $k' \neq 0$. Hence, by Theorem 3 we have

$$f' = \frac{\lambda'\alpha'}{2} m + t', \quad g' = \frac{\lambda'\alpha'}{2} m - t', \quad h' = \alpha' m, \quad k' = \lambda' m$$

for some multiplicative function $m : S \rightarrow K$ with $m \neq 0$, a central function $t' : S \rightarrow K$, and constants $\lambda', \alpha' \in K^*$. Defining $\lambda := \lambda'/\beta$, $\alpha := \alpha'/\beta$, and $t := t'/\beta$, we have (11).

Conversely, it is easy to check that the forms listed in (i)–(iii) are solutions of (10). $\square$

From this, one easily gets the following consequence.

Corollary 5. *Let S be a monoid, K a field with $\text{char}(K) \neq 2$, $f, g, h, k : S \rightarrow K$, and $\gamma, \delta \in K$. The solutions (f, g, h, k) of*

$$f(xy) + g(yx) = \gamma h(x) + \delta k(y) + h(x)k(y), \quad x, y \in S,$$

fall into three classes:

- (i) $h = -\delta$, k is arbitrary, and there exists a central function $t : S \rightarrow K$ for which

$$f = -g - \gamma\delta = t.$$

- (ii) h is arbitrary, $k = -\gamma$, and there exists a central function $t : S \rightarrow K$ for which

$$f = -g - \gamma\delta = t.$$

- (iii) There exist a multiplicative function $m : S \rightarrow K$ with $m \neq 0$, a central function $t : S \rightarrow K$, and constants $\lambda, \alpha \in K^*$ such that

$$f = \frac{\lambda\alpha}{2}m + t, \quad g = \frac{\lambda\alpha}{2}m - t - \gamma\delta, \quad h = \alpha m - \delta, \quad k = \lambda m - \gamma.$$

PROOF. The proof is a simple application of Theorem 3 in the cases $\gamma = 0$ and $\delta = 0$, and that of Theorem 4 in the case $\gamma\delta \neq 0$. The details are left to the reader. $\square$

Our final result concerns the functional equation

$$f(xy) + f(yx) = g(x)h(y) + l(y), \quad x, y \in S, \quad (12)$$

which is a symmetrized version of equation (1.2) in [1] that was treated there on non-abelian groups.

Theorem 6. *Suppose S is a monoid, K is a field with $\text{char}(K) \neq 2$, and $f, g, h, l : S \rightarrow K$ are functions with h nonconstant. The solutions (f, g, h, l) of (12) fall into two classes:*

- (i) There exist constants $\delta, \gamma \in K$ such that $f = \delta$, $g = \gamma$, h is arbitrary (nonconstant), and $l = 2\delta - \gamma h$.
- (ii) There exist a nonconstant multiplicative function $m : S \rightarrow K$, and constants $\beta, \alpha \in K^*$ and $\delta, \gamma \in K$ such that

$$f = \frac{\beta\alpha}{2}m + \delta, \quad g = \beta(m - 1) + \gamma, \quad h = \alpha m, \quad l = 2\delta + (\beta - \gamma)h.$$

PROOF. To begin, put $y = e$, respectively $x = e$ in (12), getting $h(e)g + l(e) = 2f = g(e)h + l$, so that

$$l = h(e)g - g(e)h + l(e).$$

With this we rewrite (12) as

$$f(xy) + f(yx) = [g(x) - g(e)]h(y) + h(e)g(y) + l(e), \quad x, y \in S. \quad (13)$$

Since the left side is symmetric in x and y , so is the right side, hence we arrive at

$$[g(x) - g(e)][h(y) - h(e)] = [g(y) - g(e)][h(x) - h(e)], \quad x, y \in S. \quad (14)$$

By the hypothesis, there exists a $y_0 \in S$ such that $h(y_0) \neq h(e)$. With $y = y_0$ in (14), we get

$$g - g(e) = \lambda[h - h(e)] \quad (15)$$

for some constant $\lambda \in K$. Now (13) becomes

$$f(xy) + f(yx) = \lambda h(x)h(y) + 2\delta, \quad x, y \in S, \quad (16)$$

where $2\delta := -\lambda h(e)^2 + h(e)g(e) + l(e)$.

Suppose first that $\lambda = 0$. From (16) (with $y = e$) we see that $f = \delta$, and from (15) we get that $g = g(e) =: \gamma$. Now (12) reduces to $2\delta = \gamma h + l$, from which solution class (i) follows.

If, on the other hand, $\lambda \neq 0$, then we write (16) as

$$f'(xy) + f'(yx) = h(x)h(y), \quad x, y \in S,$$

where $f' := (f - \delta)/\lambda$. Thus by Theorem 2 (taking $g = f := f'$), we have

$$f' = \frac{\alpha^2}{2}m, \quad h = \alpha m,$$

for some constant $\alpha \in K$ and multiplicative function $m : S \rightarrow K$ with $m \neq 0$. Furthermore, since h is nonconstant, we require that m be nonconstant and $\alpha \neq 0$. Defining $\beta := \lambda\alpha$ ($\neq 0$), we see that f and h are as claimed in solution (ii). Computing g from (15) we find that $g = \beta(m - 1) + \gamma$, where $\gamma := g(e)$. Finally, we calculate from (12) that

$$l(y) = \beta\alpha m(x)m(y) + 2\delta - [\beta(m(x) - 1) + \gamma]\alpha m(y) = 2\delta + (\beta - \gamma)\alpha m(y),$$

and we are in solution class (ii).

The converse is a simple verification. □

We close with a few words about the assumptions $\beta \neq 0$ in Theorem 4 and h is nonconstant in Theorem 6. The treatments of (10) in the case $\beta = 0$ and of (12) in the case h is constant are complicated by the fact that non-central solutions then arise. Indeed, both functional equations then contain as a special case the *symmetrized additive equation*

$$f(xy) + f(yx) = 2f(x) + 2f(y), \quad x, y \in S,$$

which has non-central solutions. (For further discussion of this equation, we refer the reader to [3, Section 2.4].) For example, consider the Heisenberg group H_3 consisting of all matrices of the form

$$\begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$$

for $u, v, w \in \mathbb{R}$ under matrix multiplication. On this group, the symmetrized additive equation has non-central solution

$$f \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} = 2w - uv.$$

References

- [1] B. R. EBANKS, General solution of a simple Levi-Civita functional equation on non-abelian groups, *Aequationes Math.* **85** (2013), 359–378.
- [2] H. H. ELFEN, A. PERKINS and P. K. SAHOO, A few simple Levi-Civita functional equations on groups, *Elem. Math.* **73** (2018), 109–121.
- [3] H. STETKÆR, Functional Equations on Groups, *World Scientific, Singapore*, 2013.

BRUCE EBANKS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KENTUCKY 40292
USA

E-mail: ebanks1950@gmail.com

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