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An asymptotic formula concerning Lehmer numbers

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. Let $L_n,\ n = 0, 1, 2, \dots$, be a Lehmer sequence defined by $L_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ for n odd and $L_n = (\alpha^n - \beta^n)/(\alpha^2 - \beta^2)$ for n even, where $(\alpha + \beta)^2 = A$ and $\alpha\beta = -B$ are fixed rational integers and $|\alpha| \geq |\beta|$. Let m be an integer > 1 and define the sequence (M_n) of integers by $M_n = L_{mn}/L_n$ for $n > 0$. We prove that

$$\frac{\log |M_1 \cdot M_2 \cdots M_N|}{\log[M_1, M_2, \dots, M_N]} = \frac{m-1}{6(1-w)(m - \prod\limits_{p|m} \frac{p}{p+1})}\pi^2 + O\left(\frac{\log N}{N}\right)$$

for sufficiently large N, where $w = \log((A, B))/2 \cdot \log |\alpha|$ and $[M_1, M_2, \dots]$ denotes the least common multiple of $M_1, M_2, \dots$. This result is a generalization and an improvement of a formula given by J. P. Bézivin.

1. Introduction

Let $R_n,\ (n = 0, 1, 2, \dots)$, be a second order linear recursive sequence of integers defined by

$$R_n = C \cdot R_{n-1} + D \cdot R_{n-2} \quad (n > 1),$$

where $R_0 = 0,\ R_1 = 1$ and C, D are given non-zero integer parameters with $C^2 + 4D \neq 0$. If γ and δ are the roots of the polynomial $x^2 - Cx - D$ then, as it is well known,

$$R_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$$

for any $n \geq 0$. We assume $|\gamma| \geq |\delta|$ and γ/δ is not a root of unity, i.e. the sequence is not degenerate. For $C = D = 1$ the sequence is the Fibonacci sequence and we denote it by u_n.

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For the Fibonacci sequence Y. V. Matiyasevich and R. K. Guy [6] proved that

$$\lim_{n\to\infty}\sqrt{\frac{6\cdot\log(u_1\cdot u_2\cdots u_n)}{\log[u_1,u_2,\dots,u_n]}}=\pi\,,$$

where $[u_1, u_2, \dots]$ denotes the least common multiple of the numbers u_1, $u_2, \dots$. For general second order recurrences with $(C, D) = 1$, P. Kiss and F. Mátyás [4] obtained a similar result with error term which was improved, for any C, D, by S. Akiyama [1] showing that

$$\left(\frac{6(1-w')\cdot\log|R_1\cdot R_2\cdots R_N|}{\log[R_1,R_2,\dots,R_N]}\right)^{1/2}=\pi+0\left(\frac{1}{\log N}\right) \tag{1}$$

for any sufficiently large N, where $w'=\dfrac{\log((C^2,D))}{2\cdot\log|\gamma|}$.

J. P. Bézivin [2] investigated another type of sequence. Let $m > 1$ be a given integer and let G_n, $n = 0, 1, 2, \dots$, be a sequence defined by

$$G_n=\frac{R_{mn}}{R_n}=\frac{\gamma^{mn}-\delta^{mn}}{\gamma^n-\delta^n}\,.$$

It is easy to see that the terms of this sequence are also integers and that the terms satisfy a linear recurrence relation of order m. Bézivin proved that if $(C, D) = 1$, then

$$\lim_{n\to\infty}\frac{\log|G_1\cdot G_2\cdots G_n|}{\log[G_1,G_2,\dots,G_n]}=\frac{(m-1)\cdot\prod\limits_{p|m}(1-\frac{1}{p^2})}{6\cdot\sum\limits_{\substack{d|m\\ d>1}}\varphi(d)\cdot\varphi(\frac{m}{d})\cdot\frac{d}{m}}\pi^2, \tag{2}$$

where φ is Euler's function.

In this paper we extend Bézivin's result to more general sequences and give an error term for the limit.

Let A and B be non-zero integers with $A + 4B \neq 0$ and denote by α, β the roots of the polynomial $x^2 - \sqrt{A}\,x - B$. The sequence L_n, $(n = 0, 1, 2, \dots)$, defined by

$$L_n=\begin{cases}\dfrac{\alpha^n-\beta^n}{\alpha-\beta} & \text{for } n \text{ odd}\\[2ex] \dfrac{\alpha^n-\beta^n}{\alpha^2-\beta^2} & \text{for } n \text{ even}\end{cases} \tag{3}$$

is called a Lehmer sequence, or sequence of Lehmer numbers, with parameters A, B. It can be seen that the terms of this sequence are integers since

$(\alpha+\beta)^2$ and $\alpha\beta$ are integers. Furthermore $L_n \neq 0$ for $n > 0$ if α/β is not a root of unity and L_i is divisible by L_j if $j \mid i$.

We will prove

Theorem. *Let $A \neq 0$, $B \neq 0$ and $m > 1$ be given integers with $A + 4B \neq 0$. Let L_n be a Lehmer sequence with parameters A, B and suppose that α/β is not a root of unity and that $|\alpha| \geq |\beta|$. Further let M_n be a sequence of integers defined by*

$$M_n = \frac{L_{mn}}{L_n} \qquad (n > 0)\,.$$

Then

$$\frac{\log|M_1 \cdot M_2 \cdots M_N|}{\log[M_1, M_2, \ldots, M_N]} = \frac{m-1}{6(1-w)\cdot(m - \prod\limits_{p|m} \frac{p}{p+1})}\pi^2 + O\left(\frac{\log N}{N}\right) \tag{4}$$

for any sufficiently large N, where $w = \dfrac{\log((A,B))}{2\cdot\log|\alpha|}$.

This theorem remains valid if we replace L_n by a general second order linear recurrence.

Corollary. *Let R_n be a non degenerate second order linear recurrence defined by parameters C, D and initial terms $R_0 = 0$ and $R_1 \neq 0$. Let $m > 1$ be a fixed integer and define a sequence M_n of integers by*

$$M_n = \frac{R_{mn}}{R_n} \qquad (n > 0)\,.$$

Then for this sequence M_n estimate (4) also holds with $w = (\log(C^2, D))/(2\cdot\log|\gamma|)$.

Remark 1. We note that, using (4), we can give an asymptotic formula for the number π similar to (1) but with a better error term.

Remark 2. From estimate (4), limit (2) also follows since $w = 0$ if A and B (or C and D) are relatively prime. It can be seen that

$$\left(\sum_{\substack{d|m\\ d>1}} \varphi(d)\cdot\varphi\left(\frac{m}{d}\right)\cdot\frac{d}{m}\right)\cdot\prod_{p|m}\frac{p^2}{p^2-1} = m - \prod_{p|m}\frac{p}{p+1}$$

for any $m > 1$.

2. Auxiliary results concerning Lehmer numbers

In the proof of our Theorem we use some properties of Lehmer numbers. Most of these properties were obtained by D. H. Lehmer in [5].

Let L_n be a non degenerate Lehmer sequence with parameters A and B. Assume that $|\alpha| \geq |\beta|$. If p is a prime and $p \nmid B$, then there are terms in the sequence divisible by p. We denote by $r(p)$ the rank of apparition of p in the sequence L_n, i.e. $r(p) > 0$ is a natural number for which $p \mid L_{r(p)}$ but $p \nmid L_n$ for $0 < n < r(p)$. Let $e(p)$ be the exponent of p for which $p^{e(p)} \mid L_{r(p)}$ but $p^{e(p)+1} \nmid L_{r(p)}$.

Lemma 1. *For any prime p with $p \nmid B$ and any integer $k \geq 0$, $p^{e(p)+k} \mid L_n$ if and only if $p^k \cdot r(p) \mid n$. (See [5]).*

Lemma 2. *For any prime p with $p \nmid B$ we have $r(p) \leq p+1$,* (See [5]).

Lemma 3. *If p is a prime, $p \mid B$ and $p \nmid A$, then $p \nmid L_n$ for any $n > 0$,* (See [5]).

If $n = r(p)$ for some prime p, then we say $p^{e(p)}$ is a *primitive* prime power divisor of the Lucas number L_n. In the following we shall denote the product of the primitive prime power divisors of a Lehmer number L_n by $PP(L_n)$,

$$PP(L_n) = \prod_{r(p)=n} p^{e(p)}.$$

For the primitive part of the Lehmer numbers we have

Lemma 4. *If $(A, B) = 1$ and $n > 12$, then*

$$\log(PP(L_n)) = \varphi(n) \cdot \log|\alpha| + \sum_{t|n} \mu(t) \cdot \log\left|1 - \left(\frac{\beta}{\alpha}\right)^{n/t}\right| + O(\log n)\,,$$

where φ and μ are the Euler and Möbius functions.

Proof. Let $\Phi_n(\alpha, \beta)$ denote the n^{th} cyclotomic polynomial in α and β for any integer $n > 1$ and pair α, β of complex numbers, that is

$$\Phi_n(\alpha, \beta) = \prod_{t|n} (\alpha^{n/t} - \beta^{n/t})^{\mu(t)}\,.$$

From some results of C. L. Stewart (Lemma 6 and 7 in [7]), for $n > 12$ we have

$$PP(L_n) = \lambda_n |\Phi_n(\alpha, \beta)|\,,$$

where $\lambda_n = 1$ or $\lambda_n = 1/P(n/(3,n))$ and $P(N)$ denotes the greatest prime divisor of the natural number N. From these equations

$$\log(PP(L_n)) = \sum_{t|n} \mu(t) \cdot \log|\alpha^{n/t} - \beta^{n/t}| + \log\lambda_n =$$

$$= \sum_{t|n} \mu(t) \cdot \frac{n}{t} \cdot \log|\alpha| + \sum_{t|n} \mu(t) \cdot \log\left|1 - \left(\frac{\beta}{\alpha}\right)^{n/t}\right| + \log\lambda_n$$

follows. It implies the lemma since

$$\log\lambda_n = O(\log n)$$

and, as it is well known,

$$\sum_{t|n} \mu(t) \cdot \frac{n}{t} = \varphi(n)\,.$$

We note that this lemma also follows from the lemmas of [3].

We give an estimate for the product of the terms of the sequence defined in the Theorem.

Lemma 5. *Let M_n be the sequence defined in the theorem. Then*

$$\log|M_1 \cdot M_2 \cdots M_N| = \frac{(m-1)\cdot\log|\alpha|}{2} N^2 + O(N \cdot \log N)$$

for any sufficiently large N.

Proof. From (3) and a result of C.L. Stewart (Lemma 6 in [8])

$$|L_n| = |\alpha|^{n+O(\log n)}$$

follows, for any sufficiently large n. But them

$$\log|M_1 \cdot M_2 \cdots M_N| = \sum_{n=1}^{N} \log\left(|\alpha|^{(m-1)n+O(\log n)}\right) =$$

$$= \log|\alpha| \cdot (m-1)\frac{N(N+1)}{2} + \sum_{n=1}^{N} O(\log n) =$$

$$= \frac{(m-1)\cdot\log|\alpha|}{2} N^2 + O(N \cdot \log N)$$

since

$$\sum_{n=1}^{N} \log(n) = \log(N!) = O(N \cdot \log N)\,.$$

3. An asymptotic formula for Euler's φ function

We establish an estimate concerning Euler's φ function which we need in the proof of our Theorem.

Lemma 6. *For any fixed positive integer m we hawe*

$$\sum_{n\le x}\varphi(mn)=\frac{3m}{\pi^2}\left(\prod_{p\mid m}\frac{p}{p+1}\right)x^2+O(x\cdot\log x)$$

if x is sufficiently large.

Proof. First let $m=p$ where p is a prime. Then

$$\begin{aligned}\sum_{n\le x}\varphi(pn)&=(p-1)\sum_{\substack{n\le x\\ p\nmid n}}\varphi(n)+p\sum_{\substack{n\le x\\ p\mid n}}\varphi(n)=\\ &=p\sum_{n\le x}\varphi(n)-\sum_{\substack{n\le x\\ p\nmid n}}\varphi(n)=p\sum_{n\le x}\varphi(n)-\sum_{n\le x}\varphi(n)+\sum_{\substack{n\le x\\ p\mid n}}\varphi(n)\\ &=(p-1)\sum_{n\le x}\varphi(n)+\sum_{n\le\frac{x}{p}}\varphi(pn)\,.\end{aligned}$$

Continuing this process and using the estimation

$$\sum_{n\le x}\varphi(n)=\frac{3}{\pi^2}x^2+O(x\cdot\log x)\,,\tag{5}$$

with a suitable integer k we get

$$\begin{aligned}\sum_{n\le x}\varphi(pn)&=(p-1)\sum_{i=0}^{k-1}\sum_{n\le\frac{x}{p^i}}\varphi(n)+\sum_{n\le\frac{x}{p^k}}\varphi(pn)=\\ &=\frac{3(p-1)}{\pi^2}\left(\sum_{i=0}^{k-1}\frac{x^2}{p^{2i}}+O\left(\sum_{i=0}^{k-1}\frac{x}{p^i}\cdot\log(x/p^i)\right)\right)+O\left(\frac{x^2}{p^{2k}}\right)\,.\end{aligned}\tag{6}$$

Define k by

$$p^{k-1}\le\sqrt{\frac{x}{\log x}}<p^k.$$

Then for any sufficiently large x

$$\frac{x^2}{p^{2k}}=O(x\cdot\log x)\,,$$

and

$$\sum_{i=0}^{k-1} \frac{x^2}{p^{2i}} = x^2 \cdot \frac{1 - \frac{1}{p^{2k}}}{1 - \frac{1}{p^2}} = \frac{p^2}{p^2 - 1} x^2 + O(x \cdot \log x)$$

and

$$\sum_{i=0}^{k-1} \frac{x}{p^i} \log(x/p^i) = O(x \cdot \log x) .$$

So by (6)

$$\sum_{n \le x} \varphi(pn) = \frac{3(p-1)}{\pi^2} \cdot \frac{p^2}{p^2 - 1} x^2 + O(x \cdot \log x) = \frac{3p}{\pi^2} \cdot \frac{p}{p+1} x^2 + O(x \cdot \log x) \tag{7}$$

which establishes the validity of the lemma for $m = p$.

If m is a prime power, $m = p^e$ with $e \ge 1$, then

$$\begin{aligned} \sum_{n \le x} \varphi(p^e n) &= (p^e - p^{e-1}) \cdot \sum_{\substack{n \le x \\ p \nmid n}} \varphi(n) + p^e \sum_{\substack{n \le x \\ p \mid n}} \varphi(n) = \\ &= p^e \sum_{n \le x} \varphi(n) - p^{e-1} \left(\sum_{n \le x} \varphi(n) - \sum_{\substack{n \le x \\ p \mid n}} \varphi(n) \right) = \\ &= (p^e - p^{e-1}) \sum_{n \le x} \varphi(n) + p^{e-1} \sum_{n \le \frac{x}{p}} \varphi(pn) . \end{aligned} \tag{8}$$

From this, using (7) and (5), we get

$$\begin{aligned} \sum_{n \le x} \varphi(p^e n) &= \frac{3p^{e-1}(p-1)}{\pi^2} x^2 + \frac{3p^{e+1}}{\pi^2 (p+1)} \left(\frac{x}{p} \right)^2 + O(x \cdot \log x) = \\ &= \frac{3p^e}{\pi^2} \cdot \frac{p}{p+1} x^2 + O(x \cdot \log x) . \end{aligned}$$

So the lemma holds if m is a prime power.

Now suppose that the lemma is true for some integer m and let q^e be a prime power for which $q \nmid m$. Then

$$\sum_{n \le x} \varphi(m q^e n) = (q^e - q^{e-1}) \sum_{\substack{n \le x \\ q \nmid n}} \varphi(mn) + q^e \sum_{\substack{n \le x \\ q \mid n}} \varphi(mn) =$$

$$= q^e \sum_{n\le x} \varphi(mn) - q^{e-1} \sum_{n\le x} \varphi(mn) + q^{e-1} \sum_{\substack{n\le x\\ q|n}} \varphi(mn) =$$

$$= (q^e - q^{e-1}) \sum_{n\le x} \varphi(mn) + q^{n-1} \sum_{n\le \frac{x}{q}} \varphi(mqn) =$$

$$= (q^e - q^{e-1}) \sum_{n\le x} \varphi(mn) + \sum_{n\le \frac{x}{q}} \varphi(mq^e n)\,.$$

From this, similarly as above, with an integer k we get

$$\sum_{n\le x} \varphi(mq^e n) = (q^e - q^{e-1}) \cdot \sum_{i=0}^{k-1} \sum_{n\le \frac{x}{q^i}} \varphi(mn) + \sum_{n\le \frac{x}{q^k}} \varphi(mq^e n)\,. \tag{9}$$

If k is determined by $q^{k-1} \le \sqrt{\frac{x}{\log x}} < q^k$, then by our assumption

$$\sum_{i=0}^{k-1} \sum_{n\le \frac{x}{q^i}} \varphi(mn) = \frac{3m}{\pi^2} \left(\prod_{p|m} \frac{p}{p+1} \right) \cdot \sum_{i=0}^{k-1} \frac{x^2}{q^{2i}} + O\left(\sum_{i=0}^{k-1} \frac{x}{q^i} \log \frac{x}{q_i} \right) =$$

$$= \frac{3m}{\pi^2} \left(\prod_{p|m} \frac{p}{p+1} \right) x^2 \cdot \frac{q^2}{q^2-1} + 0(x \cdot \log x)$$

and also

$$\sum_{n\le \frac{x}{q^k}} \varphi(mq^e n) = O\left(\frac{x^2}{q^{2k}} \right) = O(x \cdot \log x)$$

follows. So by (9)

$$\sum_{n\le x} \varphi(mq^e n) = \frac{3mq^e}{\pi^2} \left(\frac{q}{q+1} \cdot \prod_{p|m} \frac{p}{p+1} \right) x^2 + O(x \cdot \log x)$$

from which we get the lemma by mathematical induction.

Lemma 7. *Let $Q \ge 1$ be a given integer. Then*

$$\sum_{\substack{n\le x\\ (Q,n)=1}} \varphi(n) = \frac{3}{\pi^2} \left(\prod_{p|Q} \frac{p}{p+1} \right) x^2 + O(x \cdot \log x)$$

for any sufficiently large x.

Proof. If $Q = p^e$ is a prime power, then by Lemma 6 and the first equality in (8) with $e = 1$ we have

$$\begin{aligned}\sum_{\substack{n\le x\\(Q,n)=1}} \varphi(n) = \sum_{\substack{n\le x\\ p\nmid n}} \varphi(n) = \frac{1}{p-1}\left(\sum_{n\le x}\varphi(pn) - p\cdot \sum_{\substack{n\le x\\ p\mid n}}\varphi(n)\right) =\\
= \frac{1}{p-1}\left(\sum_{n\le x}\varphi(pn) - p\cdot \sum_{n\le \frac{x}{p}}\varphi(px)\right) =\\
= \frac{1}{p-1}\cdot\frac{3}{\pi^2}\left(\frac{p^2}{p+1}x^2 - \frac{p^3}{p+1}\left(\frac{x}{p}\right)^2\right) + O(x\cdot\log x) =\\
= \frac{3}{\pi^2}\cdot\frac{p}{p+1}x^2 + O(x\cdot\log x).\end{aligned}$$

Thus the lemma holds if Q has only one prime factor. From this we can complete the proof by induction on the number of prime divisors of Q, similar to what was done in the proof of Lemma 6.

Lemma 8. *Let $m \ge 1$ and $Q \ge 1$ be integers for which $(m, Q) = 1$. Then*

$$\sum_{\substack{n\le x\\(Q,n)=1}} \varphi(mn) = \frac{3m}{\pi^2}\left(\prod_{p\mid mQ}\frac{p}{p+1}\right)x^2 + O(x\cdot\log x).$$

Proof. First let $Q = q^e$, i.e. Q is a power of a prime q. Then by Lemma 6 we have

$$\begin{aligned}\sum_{\substack{n\le x\\(Q,n)=1}} \varphi(mn) = \sum_{\substack{n\le x\\ q\nmid n}} \varphi(mn) = \sum_{n\le x}\varphi(mn) - \sum_{n\le \frac{x}{q}}\varphi(mqn) =\\
= \frac{3m}{\pi^2}\left(\prod_{p\mid m}\frac{p}{p+1}\right)x^2 - \frac{3mq}{\pi^2}\left(\prod_{p\mid mq}\frac{p}{p+1}\right)\frac{x^2}{q^2} + O(x\cdot\log x) =\\
= \frac{3m}{\pi^2}\left(\prod_{p\mid mq}\frac{p}{p+1}\right)x^2 + O(x\cdot\log x)\end{aligned}$$

and so the lemma is true if Q has only one prime factor. From this the lemma follows by induction on the number of prime factors of Q.

4. Proof of the theorem

Let L_n, $(n = 0, 1, 2, \dots)$ and M_n, $(n = 1, 2, \dots)$ be the sequences mentioned in the statement of the theorem.

If $z = (A, B)$ and $A = zA_1$, $B = zB_1$ with $(A_1, B_1) = 1$, then for the roots of the characteristic polynomial $x^2 - \sqrt{A} \cdot x - B$ of L_n we have

$$\alpha = \frac{\sqrt{A} + \sqrt{A + 4B}}{2} = \sqrt{z} \cdot \frac{\sqrt{A_1} + \sqrt{A_1 + 4B_1}}{2} = \sqrt{z}\,\alpha_1$$

and

$$\beta = \sqrt{z}\frac{\sqrt{A_1} - \sqrt{A_1 + 4B_1}}{2} = \sqrt{z}\beta_1$$

and so by (3)

$$L_n = \begin{cases} \sqrt{z}^{n-1} \cdot L'_n, & \text{for } n \text{ odd} \\ \sqrt{z}^{n-2} \cdot L'_n, & \text{for } n \text{ even} \end{cases}$$

where L'_n is a Lehmer sequenced defined by relatively prime parameters A_1, B_1. For the sequence M_n we get

$$M_n = \frac{L_{mn}}{L_n} = \sqrt{z}^{(m-1)n+\varepsilon} \cdot \frac{L'_{mn}}{L'_n} \tag{10}$$

where $\varepsilon = 0$ or $\varepsilon = -1$ ($\varepsilon = -1$ if m is even and n is odd). Let

$$M'_n = \frac{L'_{mn}}{L'_n} \quad (\text{for} \ \ n = 1, 2, \dots).$$

If $p \mid M'_n$ for an integer $n \geq 1$ and $p > m$, then by Lemmas 1 and 3, $r(p) \mid mn$ and $r(p) \nmid n$, so $p^{e(p)} \mid M'_n$ and ($r(p)$ is of the form $r(p) = d \cdot n'$, where $d \mid m$, $d > 1$. Furthermore if $r(p)$ is of the form $r(p) = d \cdot n'$ with $p > m$, $d \mid m$, $d > 1$, then $p^{e(p)} \mid M'_n$, and if $p^{e(p)+k} \mid M'_n$, for some $n \geq 1$ and $k \geq 0$, then $p^k \mid n$.

Let N be a sufficiently integer and let

$$M(N) = [M'_1, M'_2, \dots, M'_N] = P_1(N) \cdot P_2(N), \tag{11}$$

where

$$P_1(N) = \prod_{\substack{p \mid M(N) \\ p > m}} p^{e(p)+k(p)} = \Bigg(\prod_{\substack{p \mid M(N) \\ p > m}} p^{e(P)}\Bigg) \cdot \Bigg(\prod_{\substack{p \mid M(N) \\ p > m}} p^{k(p)}\Bigg) \tag{12}$$

and

$$P_2(N)=\prod_{\substack{p\mid M(N)\\ p\le m}} p^{f(p)}. \tag{13}$$

First we give an estimation for the logarithm of the first product in (12). By the above mentioned results, using Lemma 4, we have

$$\log\left(\prod_{\substack{p\mid M(N)\\ p>m}} p^{e(p)}\right)=\log\left(\prod_{\substack{d\mid m\\ d>1\\ n\le N}} PP\left(L'_{dn}\right)\right)= \tag{14}$$

$$=(\log|\alpha_1|)\cdot\sum_{\substack{d\mid m\\ d>1\\ n\le N}}\varphi(dn)+\sum_{\substack{d\mid m\\ d>1\\ n\le N}}\sum_{t\mid dn}\mu(d)\cdot\log\left(1-\left(\frac{\beta}{\alpha}\right)^{\frac{dn}{t}}\right)+$$

$$+O\left(\sum_{\substack{d\mid m\\ n\le N}}\log(dn)\right),$$

where we assume that the divisors dn are distinct. Let

$$m=\prod_{i=1}^{s}p_1^{c_i}$$

and let S_1 and S_2 be sets of integers defined by

$$S_1=\{dn: d\mid m,\ d>1,\ (m,n)=1,\ n\le N\}$$

and

$$S_2=\left\{dn: d\mid m,\ d>1,\ p_i^{c_i+1}\mid dn \quad\text{for some}\quad i,\ n\le N\right\}.$$

Then

$$\sum_{\substack{d\mid m\\ d>1\\ n\le N}}\varphi(dn)=\sum_{dn\in S_1}\varphi(dn)+\sum_{dn\in S_2}\varphi(dn) \tag{15}$$

By Lemma 7, using

$$\sum_{d\mid m}\varphi(d)=m, \tag{16}$$

we get

$$\text{(17)} \quad \sum_{dn \epsilon S_1} \varphi(dn) = \left(\sum_{\substack{d|m \\ d>1}} \varphi(d) \right) \cdot \sum_{\substack{n \le N \\ (m,n)=1}} \varphi(n) =$$

$$= \frac{3(m-1)}{\pi^2} \left(\prod_{p|m} \frac{p}{p+1} \right) N^2 + O(N \cdot \log N).$$

Let $dn \in S_2$. Then $dn = p_{i_1}^{c_{i_1}+1} \cdots p_{i_j}^{c_{i_j}+1} \cdot d' \cdot n'$ for some j with $1 \le j \le s$, where $\{p_{i_1}, \dots, p_{i_j}\} \subseteq \{p_1, \dots, p_s\}$, and d' is a divisor of $\dfrac{m}{p_{i_1}^{c_{i_1}} \dots p_{i_j}^{c_{i_j}}} = m'$ $(m', n') = 1$ and $n' \le \dfrac{N}{p_{i_1} \dots p_{i_j}} = N'$. By (16) and Lemma 8 we have

$$\sum_{d'|m'} \left(\sum_{\substack{n' \le N' \\ (m',n')=1}} \varphi \left(p_{i_1}^{c_{i_1}+1} \dots p_{i_j}^{c_{i_j}+1} \cdot d'n' \right) \right) =$$

$$= \left(\sum_{d'|m'} \varphi(d') \right) \left(\frac{3p_{i_1}^{c_{i_1}+1} \dots p_{i_j}^{c_{i_j}+1}}{\pi^2} \left(\prod_{p|m} \frac{p}{p+1} \right) \cdot \frac{N^2}{\left(p_{i_1} \dots p_{i_j}\right)^2} \right) +$$

$$+ O(N \cdot \log N) = \frac{3m}{\pi^2} \cdot \frac{1}{p_{i_1} \cdots p_{i_j}} \cdot \left(\prod_{p|m} \frac{p}{p+1} \right) N^2 + O(N \cdot \log N)$$

and so

$$\text{(18)} \quad \sum_{dn \in S_2} \varphi(dn) =$$

$$= \frac{3m}{\pi^2} \left(\prod_{p|m} \frac{p}{p+1} \right) \left(\sum_{j=1}^{s} \sum_{C_j} \frac{1}{p_{i_1} \dots p_{i_j}} \right) N^2 + O(N \cdot \log N)$$

where C_j denotes the extended summation over all j tuples of primes $p_1, \dots, p_s$. But

$$\text{(19)} \qquad \sum_{j=1}^{s} \sum_{C_j} \frac{1}{p_{i_1} \dots p_{i_j}} = \prod_{p|m} \left(1 + \frac{1}{p} \right) - 1 = \prod_{p|m} \frac{p+1}{p} - 1,$$

thus by (15), (17), (18) and (19) we get

$$\sum_{\substack{d|m\\ d>1\\ n\le N}} \varphi(dn) = \tag{20}$$

$$= \frac{3N^2}{\pi^2}\left((m-1)\left(\prod_{p|m}\frac{p}{p+1}\right)+m-m\cdot\prod_{p|m}\frac{p}{p+1}\right)+O(N\cdot\log N)$$

$$= \frac{3}{\pi^2}\left(m-\prod_{p|m}\frac{p}{p+1}\right)N^2+O(N\cdot\log N).$$

On the other hand

$$O\left(\sum_{\substack{d|n\\ n\le N}}\log dn\right)=O(\log(N!))=O(N\cdot\log N)$$

and using an estimation like one in [3], we get

$$\sum_{\substack{d|m\\ d>1\\ n\le N}}\sum_{t|dn}\mu(t)\cdot\log\left(1-\left(\frac{\beta}{\alpha}\right)^{\frac{dn}{t}}\right)=O(N\cdot\log N)$$

and so by (14) and (20)

$$\log\left(\prod_{\substack{p|M(N)\\ p>m}}p^{e(p)}\right)=\frac{3\cdot\log|\alpha_1|}{\pi^2}\left(m-\prod_{p|m}\frac{p}{p+1}\right)N^2+O(N\cdot\log N) \tag{21}$$

follows.

Now we consider the second product in (12). If $p \mid M(N)$ and $k(p)\ge 1$ for a prime p, then by Lemma 1 there is an integer $n\le mN$ for which $p^{k(p)}r(p)\mid n$. But by Lemma 2$p\ge r(p)-1$ and so $p^{k(p)}r(p)\ge(r(p)-1)\cdot r(p)>mN$ if $r(p)\ge\sqrt{mN}+1$. From this $k(p)=0$ follows for primes p for which $r(p)\ge\sqrt{mN}+1$ and we have

$$\log\left(\prod_{\substack{p|M(N)\\ p>m}}p^{k(p)}\right)=\log\left(\prod_{\substack{r(p)\le\sqrt{mN}\\ p>m}}p^{k(p)}\right)+O(\log N). \tag{22}$$

If $k(p) \geq 1$ and $p^{e(p)+k(p)} \mid L'_n$ for some $0 < n < mN$, then $p^{k(p)} \mid n$ and $k(p) \cdot \log p \leq \log(mN)$. This implies the estimate

$$\log\left(\prod_{\substack{r(p)\leq\sqrt{mN}\\ p>m}} p^{k(p)}\right) \leq \sum_{\substack{p\\ r(p)\leq\sqrt{mN}}} \log mN = (\log mN) \cdot \sum_{r(p)\leq\sqrt{mN}} 1. \tag{23}$$

But there are $O(n/\log n)$ primes p for which $r(p) = n$ and so

$$\sum_{\substack{p\\ r(p)\leq\sqrt{mN}}} 1 = O\left(\sum_{n\leq\sqrt{mN}} \frac{n}{\log n}\right) = O\left(\frac{N}{\log N}\right). \tag{24}$$

From (22), (23) and (24) we get

$$\log\left(\prod_{\substack{p\mid M(N)\\ p>m}} p^{k(p)}\right) = O(N). \tag{25}$$

If p is a prime, $p < m$ and $p^{f(p)} \mid M(N)$ to some exponent $f(p)$, then $p^{f(p)} = O\left(|\alpha|^{mN}\right)$ and $f(p) \cdot \log p = O(N)$. So by (13)

$$\log P_2(N) = O(N) \tag{26}$$

follows.

From (11), (12), (21), (25) and (26) we get the estimate

$$\log M(N) = \frac{3 \cdot \log|\alpha_1|}{\pi^2}\left(m - \prod_{p\mid m} \frac{p}{p+1}\right) N^2 + O(N \cdot \log N)\ .$$

But then by (10) we have

$$\log\left[M_1, M_2, \ldots, M_N\right] = \log M(N) + O(N) =$$
$$= \frac{3 \cdot \log|\alpha_1|}{\pi^2}\left(m - \prod_{p\mid m} \frac{p}{p+1}\right) N^2 + O(N \log N).$$

From this, by Lemma 5, the theorem follows since $\alpha = \sqrt{z}\,\alpha_1$ and hence

$$\frac{\log|\alpha|}{\log|\alpha_1|} = \frac{1}{1 - \dfrac{\log z}{2 \cdot \log|\alpha|}}.$$

The Corollary follows from the theorem since the sequence R_n is almost a Lehmer sequence with parameters $A = C^2$, $B = D$. Multiplication of the terms by R_1 and sometimes by C introduces only $O(N)$ error in our estimations.

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