

## Moments of additive statistics with respect to the Ewens sampling formula

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**Abstract.** The additive semigroup of vectors with non-negative integer coordinates endowed with the Ewens probability measure plays an important role as a probabilistic space for many statistical models. In the present paper, we obtain upper estimates of the power moments of additive statistics defined on the semigroup. The statistics are sums of dependent random variables; however, our results have the form of the Rosenthal and von Bahr–Esseen inequalities. The arguments perfected in probabilistic number theory are adopted in the proofs.

### 1. Introduction

Let  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  be the sets of natural, real and complex numbers, respectively,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , and  $n \in \mathbb{N}$ . Denote by  $\Omega := \mathbb{N}_0^n$  the additive semigroup of vectors  $\bar{s} := (s_1, \dots, s_n)$ , where  $s_j \in \mathbb{N}_0$  and  $1 \leq j \leq n$  and  $\bar{0} = (0, \dots, 0)$  is the zero vector. The partial order defined by  $\bar{s} = (s_1, \dots, s_n) \leq \bar{t} = (t_1, \dots, t_n)$ , meaning that  $s_j \leq t_j$  for each  $1 \leq j \leq n$ , will be essential throughout the paper. Moreover, we say that  $\bar{s}, \bar{t} \in \Omega$  are orthogonal, denoted by  $\bar{s} \perp \bar{t}$ , if  $s_1 t_1 + \dots + s_n t_n = 0$ . Afterwards, we shall use the notation  $\bar{t} \parallel \bar{s}$  to express that  $\bar{t}$  *exactly enters*  $\bar{s}$ . Formally, then  $\bar{t} \leq \bar{s}$  and  $\bar{t} \perp \bar{s} - \bar{t}$ . Using the notation when dealing with functions defined on  $\Omega$ , we come closer to probabilistic number theory which has been developed on the multiplicative semigroup  $\mathbb{N}$  (see [12] and [4]), in which the partial order is defined by division, and the orthogonality of  $m, n \in \mathbb{N}$  means that their greatest common divisor equals 1. The semigroup structures and the partial

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orders in  $\Omega$  and  $\mathbb{N}$  could have played a greater role in developing parallel theories. The advantage of applying this approach has been discussed in the first author's joint talk [11] and in some papers referred to then. The present paper further demonstrates the number-theoretic ideas adopted when estimating moments of functions defined on  $\Omega$ .

For a probability measure, we take that one proposed by EWENS [6] in mathematical genetics. It continues to serve in various statistical models and probabilistic combinatorics (see, for example, [10], [1] or [7]). To present it, we introduce the mapping  $\ell : \Omega \rightarrow \mathbb{N}_0$  by  $\ell(\bar{s}) = 1s_1 + \dots + ns_n$  and denote  $\Omega_n := \ell^{-1}(n) = \{\bar{s} \in \Omega : \ell(\bar{s}) = n\}$ . Set also

$$\Theta(n) := \binom{\theta + n - 1}{n} = [z^n] \frac{1}{(1-z)^\theta} = \frac{n^{\theta-1}}{\Gamma(\theta)} \left(1 + O\left(\frac{1}{n}\right)\right) \quad (1)$$

if  $n \geq 1$ . Here and throughout the paper,  $\theta > 0$  is a fixed parameter,  $\Gamma(z)$  is the Gamma function, and, as usual,  $[z^n]g(z)$  stands for the  $n$ -th coefficient of the power series  $g(z)$  if  $n \in \mathbb{N}_0$ . Then the celebrated Ewens sampling formula defines the probability

$$P_n(\{\bar{s}\}) := \mathbf{1}\{\ell(\bar{s}) = n\} \Theta(n)^{-1} \prod_{j=1}^n \left(\frac{\theta}{j}\right)^{s_j} \frac{1}{s_j!} =: \mathbf{1}\{\ell(\bar{s}) = n\} \Theta(n)^{-1} P(\bar{s}) \quad (2)$$

ascribed for each  $\bar{s} \in \Omega$ . Here  $\mathbf{1}\{\cdot\}$  is the indicator function. Now every mapping  $G : \Omega \rightarrow \mathbb{C}$  becomes a complex-valued r.v., and

$$M_n(G) := \Theta(n)^{-1} \sum_{\bar{s} \in \Omega_n} G(\bar{s}) P(\bar{s}) \quad (3)$$

is its mean value. Let  $M_0(G) = G(\bar{0})$  for every  $G : \Omega \rightarrow \mathbb{C}$ .

It is worth recalling the following property of (2). If  $\xi_j$ ,  $1 \leq j \leq n$ , are mutually independent Poisson random variables (r.vs) with parameters  $\theta/j$  given on some probability space and  $\bar{\xi} := (\xi_1, \dots, \xi_n)$ , then

$$P_n(\{\bar{s}\}) = \Pr(\bar{\xi} = \bar{s} \mid \ell(\bar{\xi}) = n), \quad \bar{s} \in \Omega.$$

This clearly shows the dependence of coordinates  $s_j$ ,  $1 \leq j \leq n$ , under the probability measure  $P_n$ . Despite this, some recent results on the asymptotic behavior as  $n \rightarrow \infty$  of distributions of the linear statistics  $a_{n1}s_1 + \dots + a_{nn}s_n$ , where  $a_{nj} \in \mathbb{R}$  and  $1 \leq j \leq n$ , give general conditions for weak convergence or sharp estimates of the convergence rates. They are mainly formulated in the

terminology of the theory of random permutations; therefore, we now present the connections to the latter.

Let  $\mathbb{S}_n$  denote the symmetric group of permutations  $\sigma$  acting on  $n \geq 1$  letters. Each  $\sigma \in \mathbb{S}_n$  has a unique representation (up to the order) by the product of independent cycles  $\varkappa_i$ :

$$\sigma = \varkappa_1 \cdots \varkappa_w, \quad (4)$$

where  $w = w(\sigma)$  denotes the number of cycles. Denote by  $k_j(\sigma) \geq 0$  the number of cycles in (4) of length  $j$  for  $1 \leq j \leq n$  and  $\bar{k}(\sigma) := (k_1(\sigma), \dots, k_n(\sigma))$ . The latter is called a *cycle vector* of the permutation  $\sigma$ . The *Ewens probability measure*  $\nu_{n,\theta}$  on  $\mathbb{S}_n$  is defined by

$$\nu_{n,\theta}(\{\sigma\}) := \theta^{w(\sigma)} / (\theta(\theta+1) \cdots (\theta+n-1)), \quad \sigma \in \mathbb{S}_n,$$

where  $\theta > 0$  is a parameter. Then the case  $\theta = 1$  corresponds to uniformly sampled permutations. An easy combinatorial argument (see [1]) gives the distribution of the cycle vector and the coincidence:

$$\nu_{n,\theta}(\bar{k}(\sigma) = \bar{s}) = P_n(\{\bar{s}\})$$

if  $\bar{s} \in \Omega_n$ . Thus, when dealing with statistics of random permutations expressed via  $\bar{k}(\sigma)$ , we may examine corresponding statistics of random vectors  $\bar{s} \in \Omega_n$  taken with probabilities (2).

The linear statistics  $a_{1n}k_1(\sigma) + \cdots + a_{nn}k_n(\sigma)$  and, in particular,  $w(\sigma)$  have attracted much attention in the recent investigations. However, so far, the advance in probabilistic number theory has not been adequately followed by the corresponding results in probabilistic combinatorics. For instance, the results exposed in [1, Section 8.5] did not reach the level of their analogs in  $\mathbb{N}$  (compare with [4]). In the recent papers [16] and [3] (see also the references therein), the first author attempted to fill up this gap.

Let us continue by introducing more definitions. A mapping  $F : \Omega \rightarrow \mathbb{C}$ ,  $F(\bar{0}) = 1$ , is called a *multiplicative function* if  $F(\bar{s} + \bar{t}) = F(\bar{s})F(\bar{t})$  holds for every pair  $\bar{s}, \bar{t} \in \mathbb{N}_0^n$  such that  $\bar{s} \perp \bar{t}$ . Denote a generic vector  $\bar{e}_j := (0, \dots, 1, \dots, 0)$ , where the only 1 stands at the  $j$ -th place. Then the multiplicative function  $F$  has the decomposition

$$F(\bar{k}) = \prod_{j \leq n} F(k_j \bar{e}_j) =: \prod_{j \leq n} f_j(k_j). \quad (5)$$

Conversely, given a complex two-dimensional array  $\{f_j(k)\}$ ,  $1 \leq j \leq n$  and  $k \in \mathbb{N}_0$ , satisfying the condition  $f_j(0) \equiv 1$ , by the last equality, we can define

a multiplicative function. If  $f_j(k) = f_j(1) =: f_j$  for all  $k \in \mathbb{N}$  and  $j \leq n$ , the function  $F$  is called *strongly* multiplicative and, similarly, if  $f_j(k) = f_j^k$  and  $0^0 := 1$ , then  $F$  is called *completely* multiplicative. By definition,  $F$  may depend on  $n$ ; moreover, dealing with its values attained on  $\Omega_n$ , the values  $f_j(k)$  are irrelevant if  $jk > n$ . We treat them arbitrarily to make the appearing expressions well and conveniently defined. Denote respectively by  $\mathfrak{M}$ ,  $\mathfrak{M}_s$ , and  $\mathfrak{M}_c$  the sets of just introduced multiplicative functions. We stress that  $P(\bar{s}) \in \mathfrak{M}$  and  $P(k\bar{e}_j) = (\theta/j)^k/k! =: p_j(k)$  if  $k \in \mathbb{N}_0$  and  $1 \leq j \leq n$ .

Similarly, the condition  $H(\bar{s} + \bar{t}) = H(\bar{s}) + H(\bar{t})$ , holding for every pair  $\bar{s}, \bar{t} \in \mathbb{N}_0^n$  such that  $\bar{s} \perp \bar{t}$ , defines an *additive function*  $H : \Omega \rightarrow \mathbb{C}$ . Let us also set  $h_j(k) := H(k\bar{e}_j)$  where  $h_j(0) := 0$ . Now condition  $h_j(k) = kh_j(1)$ ,  $k \in \mathbb{N}_0$  and  $1 \leq j \leq n$ , implies *completely* additive functions. The function  $H$  is also allowed to depend on  $n$ .

The purpose of the present paper is to establish power moment inequalities for a complex-valued additive function  $H(\bar{s})$ . The tail probability estimates for additive functions proposed in [13] and refined in [2], together with a subsequent use of relevant results for sums of independent r.vs, provide an indirect approach to deal with the problem if  $\theta \geq 1$ . The experience (see [14] and [15]) acquired by the first author in using this approach when deriving inequalities for the case  $\theta = 1$  makes us to believe that the direct proof, as exposed below, gives sharper results. This has been evidenced by the authors of [18] and [19] when dealing with the second moment of additive functions defined on general decomposable structures, including permutations sampled according to the Ewens probability. In the second moment estimates, there have been a few attempts to find the optimal constants (see [20] and the references therein).

Denote

$$A_n := \sum_{jk \leq n} h_j(k)p_j(k) \frac{\Theta(n - jk)}{\Theta(n)},$$

$$B_n(\alpha) := B_n(\alpha; H) = \sum_{jk \leq n} |h_j(k)|^\alpha p_j(k) \frac{\Theta(n - jk)}{\Theta(n)},$$

where  $\alpha > 0$ . Here and in the sequel,  $\sum_{jk \leq n}$  means summation over  $j, k \in \mathbb{N}$  such that the product  $jk \leq n$ . Afterwards,  $\ll$  is an analogue of the symbol  $O(\cdot)$  with a constant in it depending at most on  $\theta$  and  $\alpha$ .

**Theorem 1.** *Let  $\theta \geq 1$  be fixed, and  $H$  be an additive function, possibly depending on  $n$ . Then*

$$M_n(|H(\bar{s}) - A_n|^\alpha) \ll \begin{cases} B_n(2)^{\alpha/2} + B_n(\alpha) & \text{if } \alpha \geq 2, \\ B_n(2)^{\alpha/2} & \text{if } 0 \leq \alpha \leq 2, \end{cases}$$

uniformly for all  $n \geq 1$ .

The formulas of moments  $M_n(H^i)$ ,  $i = 1, 2$ , can be found in [18, page 308]. Comparing, we observe that  $A_n = M_n(H) + O(B_n(2)^{1/2})$ . Moreover,  $B_n(2)$  differs from the sum of variances of the summands in  $H$  by a negligible term. The latter has been witnessed in estimate (4) of the same paper [18] for completely additive functions. This motivates our choice of  $A_n$  and  $B_n(\alpha)$ . In many cases, they can be further simplified by taking the main asymptotic terms. Relation (1) is also at our disposition to implement this. For example, for the number  $W(\bar{s})$  of different nonzero coordinates of a vector  $\bar{s}$ , Theorem 1 reduces to the following estimate:

$$M_n(|W(\bar{s}) - \theta \log n|^\alpha) \ll \log^{\alpha/2} n, \quad \alpha \geq 0, \quad n \geq 2. \quad (6)$$

The details are given at the end of the paper.

The inequalities in Theorem 1 have the form of the classical Rosenthal's and von Bahr–Esseen's results in the case of independent summands. The dependence present in our case involves some new facts. For the completely additive function  $\ell(\bar{s})$ , which equals  $n$  on  $\Omega_n$ , Theorem 1 gives only a trivial estimate. To avoid the trivialities, one can first split  $H(\bar{s})$  into the sum of  $\lambda \ell(\bar{s})$  and the remaining additive function  $H_\lambda(\bar{s})$ , choosing  $\lambda \in \mathbb{C}$  so that  $\min_\lambda B_n(\alpha; H_\lambda)$  is attained. Then the first function can be eliminated using the inequality  $|a + b|^\alpha \leq 2^\alpha(|a|^\alpha + |b|^\alpha)$ , and Theorem 1 can be applied for  $H_\lambda(\bar{s})$ .

The authors frankly confess that they have failed to extend the result to the case  $0 < \theta < 1$  when the essential inequality  $\Theta(m) \leq \Theta(n)$  if  $m \leq n$  is reversed. In the proof, we adopt ELLIOTT's [5] argument. The main task is the analysis of the mean value of  $e^{zH(\bar{s})}$ , which is a multiplicative function depending on a complex parameter  $z$ . The needed technique is developed in the next section. The theorem will be proved at the end of the paper.

## 2. Mean values of multiplicative functions

This section is devoted to estimates of the mean values  $M_n(F)$  of mappings  $F : \Omega \rightarrow \mathbb{R}^+$  belonging to more specialized classes. Afterwards,  $\sum_{\ell(\bar{s})=n}$  means the summation over the set of vectors  $\bar{s} \in \Omega_n$ . If  $F \in \mathfrak{M}$ , then (3) and (5) give

the following expressions:

$$\begin{aligned}\Theta(n)M_n(F) &= \sum_{\ell(\bar{s})=n} \prod_{j \leq n} p_j(s_j) f_j(s_j) = [x^n] \prod_{j \geq 1} \left( 1 + \sum_{k=1}^{\infty} p_j(k) f_j(k) x^{jk} \right) \\ &=: [x^n] \prod_{j \geq 1} \chi_j(x; F) =: [x^n] Z(x; F).\end{aligned}\tag{7}$$

Actually, the values  $f_j(k)$  if  $jk > n$ , which do not appear in the quantity on the left-hand side, can be chosen in a convenient way, say, equal to zeros or ones. Here and in the sequel,  $j, k \in \mathbb{N}$ . On the other hand, if  $F \in \mathfrak{M}_c$ , that is, for  $f_j(k) = f_j(1)^k$ ,  $j, k \geq 1$ , there is no need to do so.

Set  $\Pi_0(F) := 1$ , and

$$\Pi_m(F) := \prod_{j \leq m} \left( 1 + \sum_{k=1}^{\infty} p_j(k) f_j(k) \right), \quad 1 \leq m \leq n.$$

From the definitions, one directly obtains the inequality

$$\Sigma_n(F) := \sum_{0 \leq m < n} M_m(F) \Theta(m) \leq \Pi_{n-1}(F)\tag{8}$$

if  $F$  is a non-negative multiplicative function and  $n \in \mathbb{N}$ .

We begin with a convenient identity. Introduce a function  $P^{(j)} \in \mathfrak{M}$  such that  $p_i^{(j)}(k) = p_i(k)$  if  $i \neq j$ , and  $p_i^{(j)}(k) = 0$  if  $i = j$  and  $k \in \mathbb{N}$ . Let

$$\Theta^{(j)}(n) := \sum_{\ell(\bar{s})=n} P^{(j)}(\bar{s}) = [z^n] e^{-\theta z^j/j} (1-z)^{-\theta}.$$

Define the following conditional mean value:

$$M_n^{(j)}(F) := \frac{1}{\Theta^{(j)}(n)} \sum_{\ell(\bar{s})=n} \prod_{i \leq n} p_i^{(j)}(s_i) f_i(s_i) = \frac{1}{\Theta^{(j)}(n)} \sum_{\ell(\bar{s})=n} P^{(j)}(\bar{s}) F(\bar{s}).$$

**Lemma 1.** *Let  $F : \Omega \rightarrow \mathbb{C}$  be a multiplicative function, then*

$$n \Theta(n) M_n(F) = \sum_{jk \leq n} j k f_j(k) p_j(k) \Theta^{(j)}(n-jk) M_{n-jk}^{(j)}(F).$$

*In particular, if  $F \in \mathfrak{M}_c$ , then*

$$n \Theta(n) M_n(F) = \theta \sum_{j \leq n} f_j(1) \Theta(n-j) M_{n-j}(F).$$

PROOF. This is just the identity

$$[x^n](xZ'(x; F)) = [x^n] \left( x \sum_{j \geq 1} \chi'_j(x; F) \prod_{i \geq 1, i \neq j} \chi_i(x; F) \right);$$

nevertheless, we provide an elementary proof exposing the idea used in the sequel. The decomposition of  $\bar{s} \in \Omega_n$  into a sum of orthogonal vectors, namely,

$$\bar{s} = k\bar{e}_j + \bar{t}, \quad 1 \leq k \leq n/j, \quad \bar{e}_j \perp \bar{t},$$

is essential in it. Check that  $k\bar{e}_j \parallel \bar{s}$  and  $t_j = 0$  in this case. Now

$$\ell(\bar{s}) = \sum_{k\bar{e}_j \parallel \bar{s}} \ell(k\bar{e}_j) = \sum_{\substack{jk \leq n, \\ k\bar{e}_j \parallel \bar{s}}} jk = n$$

and

$$n\Theta(n)M_n(F) = \sum_{\ell(\bar{s})=n} F(\bar{s})P(\bar{s}) \sum_{\substack{jk \leq n, \\ k\bar{e}_j \parallel \bar{s}}} jk.$$

Applying the decomposition and changing the summation order, we obtain

$$\begin{aligned} n\Theta(n)M_n(F) &= \sum_{jk \leq n} jk f_j(k) p_j(k) \sum_{\substack{\bar{t}, t_j=0 \\ \ell(\bar{t})=n-jk}} F(\bar{t})P(\bar{t}) \\ &= \sum_{jk \leq n} jk f_j(k) p_j(k) \Theta^{(j)}(n-jk) M_{n-jk}^{(j)}(F). \end{aligned}$$

The lemma is proved.  $\square$

**Corollary 1.** *If  $\theta > 0$  and  $F \geq 0$  is a multiplicative function, then*

$$n\Theta(n)M_n(F) \leq \sum_{jk \leq n} jk f_j(k) p_j(k) \Theta(n-jk) M_{n-jk}(F).$$

PROOF. By the definitions above,  $\Theta^{(j)}(m)M_m^{(j)}(F) \leq \Theta(m)M_m(F)$  for each  $1 \leq j \leq m \leq n$ .  $\square$

**Lemma 2.** *If  $\theta > 0$  and  $F \in \mathfrak{M}$  is such that  $0 \leq f_j(k) \leq K$  for  $k, j \leq n$  and  $n \geq 1$ , then*

$$n\Theta(n)M_n(F) \leq \theta K \Sigma_n(F) \left( 1 + O \left( \frac{K \log(n+1) + 1}{n} \right) \right).$$

PROOF. By Corollary 1,

$$\begin{aligned}
n\Theta(n)M_n(F) &\leq K \sum_{1 \leq m \leq n} m\Theta(n-m)M_{n-m}(F) \sum_{jk=m} p_j(k) \\
&= \theta K \sum_{0 \leq m < n} \Theta(m)M_m(F) \\
&\quad + K \sum_{1 \leq m \leq n} \Theta(n-m)M_{n-m}(F) \left( m \sum_{jk=m} p_j(k) - \theta \right). \quad (9)
\end{aligned}$$

The first sum is just  $\Sigma_n(F)$ . In the second, the term corresponding to  $m = 1$  is zero. For  $m \geq 2$ , we observe that

$$\begin{aligned}
0 \leq m \sum_{jk=m} p_j(k) - \theta &= \frac{\theta^m}{(m-1)!} + \frac{\theta^2}{m} \sum_{\substack{jk=m, \\ j,k \geq 2}} \left( \frac{\theta}{j} \right)^{k-2} \frac{k}{(k-1)!} \\
&\leq \theta \left( \frac{3\theta}{m-1} \right)^{m-1} + \frac{2\theta^2}{m} \sum_{2 \leq k \leq m/2} \left( \frac{\theta k}{m} \right)^{k-2} \frac{1}{(k-2)!} \\
&\leq \frac{\theta}{m} \left( \max_{m \geq 2} \left\{ m \left( \frac{3\theta}{m-1} \right)^{m-1} \right\} + 2\theta e^{\theta/2} \right) =: \frac{\theta C(\theta)}{m},
\end{aligned}$$

where  $C(\theta) > 0$  is a constant depending only on  $\theta$ . Plugging this into (9), we obtain

$$n\Theta(n)M_n(F) \leq \theta K \sum_{0 \leq m < n} \Theta(m)M_m(F) \left( 1 + \frac{C(\theta)}{n-m} \right). \quad (10)$$

This also yields a rough estimate  $m\Theta(m)M_m(F) \ll K\Sigma_m(F)$ ,  $1 \leq m \leq n$ , needed below. Observe that

$$\begin{aligned}
\left( \sum_{1 \leq m \leq n/2} + \sum_{n/2 \leq m < n} \right) \frac{\Theta(m)M_m(F)}{n-m} &\ll \frac{\Sigma_n(F)}{n} + K \sum_{n/2 \leq m < n} \frac{1}{m(n-m)} \Sigma_m(F) \\
&\ll \frac{1 + K \log(n+1)}{n} \Sigma_n(F).
\end{aligned}$$

Inserting this into (10), we complete the proof.  $\square$

In the sequel, we will apply the just proved result in a more convenient form.

**Corollary 2.** *If  $\theta > 0$  and  $F \in \mathfrak{M}$  is such that  $0 \leq f_j(k) \leq 1$  for  $k, j \leq n$  and  $n \geq 1$ , then*

$$M_n(F) \ll \exp \left\{ \theta \sum_{j \leq n} \frac{f_j(1) - 1}{j} \right\}.$$

PROOF. It suffices to apply (1) and the well-known asymptotic formula

$$\sum_{j \leq n} \frac{1}{j} = \log n + \gamma + O\left(\frac{1}{n}\right),$$

where  $\gamma$  denotes the Euler–Mascheroni constant. Indeed, using Lemma 2 and (8), we see that

$$\begin{aligned} M_n(F) &\leq \Gamma(\theta + 1) n^{-\theta} \Pi_{n-1}(F) \left( 1 + O\left(\frac{\log(n+1)}{n}\right) \right) \\ &\leq \Gamma(\theta + 1) \exp \left\{ \theta\gamma + \theta \sum_{j \leq n} \frac{f_j(1) - 1}{j} \right\} \left( 1 + O\left(\frac{\log(n+1)}{n}\right) \right) \\ &\times \prod_{j \leq n} e^{-\theta f_j(1)/j} \left( 1 + \sum_{k=1}^{\infty} \left(\frac{\theta}{j}\right)^k \frac{f_j(k)}{k!} \right). \end{aligned} \quad (11)$$

Applying the inequality

$$|e^x - 1 - x| \leq |x|^2 e^{|x|} \quad (12)$$

if  $x \in \mathbb{R}$ , we easily estimate the last product:

$$\begin{aligned} &\prod_{j \leq n} e^{-\theta f_j(1)/j} \left( 1 + \frac{\theta f_j(1)}{j} + \sum_{k=2}^{\infty} \left(\frac{\theta}{j}\right)^k \frac{f_j(k)}{k!} \right) \\ &\leq \prod_{j \leq n} e^{-\theta f_j(1)/j} \left( 1 + \frac{\theta f_j(1)}{j} + e^{\theta/j} - 1 - \frac{\theta}{j} \right) \\ &\leq \prod_{j \leq n} e^{-\theta f_j(1)/j} \left( e^{\theta f_j(1)/j} + \left(\frac{\theta}{j}\right)^2 e^{\theta/j} \right) \leq \prod_{j \leq n} \left( 1 + \left(\frac{\theta}{j}\right)^2 e^{\theta/j} \right) \ll 1. \end{aligned}$$

This and (11) complete the proof of the corollary.  $\square$

Observing that  $\frac{\Theta(n-j)}{\Theta(n)} \leq 1$  if  $\theta \geq 1$  for all  $j \leq n$ , we also have

$$M_n(F) \ll \exp \left\{ \theta \sum_{j \leq n} \frac{f_j(1) - 1}{j} \frac{\Theta(n-j)}{\Theta(n)} \right\}. \quad (13)$$

*Remarks.* We first stress the parallelism with HALL's [9] paper exploring number-theoretic submultiplicative functions. Inequality (11) also holds for a *submultiplicative* function  $G : \Omega \rightarrow \mathbb{R}$  which by definition satisfies the inequality  $G(\bar{s} + \bar{t}) \leq G(\bar{s})G(\bar{t})$  for all  $\bar{s}, \bar{t} \in \Omega$  if  $\bar{s} \perp \bar{t}$ . For example, such is the statistics  $G(\bar{s}) = l.c.m.\{j : s_j \geq 1\}$ , related to the group-theoretical order of a permutation in the group  $\mathbb{S}_n$ . Here the letters *l.c.m.* stand for the least common multiplier of the indicated natural numbers. This more general case can be dealt with by a repetition of the used argument or by a direct application of Lemma 2. Indeed, given a submultiplicative function  $G : \Omega \rightarrow [0, 1]$ , we have

$$G(\bar{k}) \leq \prod_{j \leq n} G(k_j \bar{e}_j) =: \prod_{j \leq n} g_j(k_j).$$

Further, one can define  $F \in \mathfrak{M}$  so that  $f_j(k) = g_j(k)$  to obtain  $G(\bar{k}) \leq F(\bar{k})$  and a subsequent ability to apply Corollary 2 for the function  $F$ .

Secondly, if  $\theta = 1$  and  $F$  is a multiplicative function satisfying the conditions in Corollary 2 and the values  $f_j(1)$  for  $\varepsilon n < j \leq n$ , where  $0 < \varepsilon < 1$ , are close to 1, one can substitute  $e^\gamma$  in (11) by a smaller quantity (see [17]). Constructing appropriate indicator functions and using Lemma 2 or (11), one can obtain sharp estimates of the probabilities of vectors with a forbidden pattern. Note that the lower estimates have been discussed in [11].

In the next step, we will need some facts about the algebraic structure  $(\mathfrak{G}, *)$ , where  $\mathfrak{G} := \{G : \Omega \rightarrow \mathbb{C}\}$  and  $*$  is the convolution defined as follows:

$$F * G(\bar{t}) := \sum_{\bar{s} \leq \bar{t}} F(\bar{s})G(\bar{t} - \bar{s}), \quad \bar{s}, \bar{t} \in \Omega.$$

Let  $I(\bar{t}) \equiv 1$  and  $E(\bar{t}) = \mathbf{1}\{\bar{t} = \bar{0}\}$  be the indicator function of the subset  $\{\bar{0}\}$ . It is straightforward to check that  $(\mathfrak{G}, *)$  is an Abelian group in which  $E$  serves as the neutral element. The inverse of  $I$  in the group is an analogue of the Möbius function. Let us leave the notation  $\mu$  for it. The latter is a multiplicative function such that  $\mu(\bar{e}_j) = -1$ , and  $\mu(r\bar{e}_j) = 0$  if  $r \geq 2$ , where  $1 \leq j \leq n$ . It is easy to check that the relations  $F = I * G$  and  $G = \mu * F$  are equivalent. Finally, we stress that  $\mathfrak{M}$  is a subgroup in  $\mathfrak{G}$ .

**Lemma 3.** *Let  $\theta \geq 1$  and  $F \in \mathfrak{M}$  be such that  $f_j(1) \geq 1$  for each  $j \leq n$ . Then*

$$M_n(F) \leq \exp \left\{ \sum_{jk \leq n} f_j(k)p_j(k) \frac{\Theta(n - jk)}{\Theta(n)} - \sum_{j \leq n} p_j(1) \frac{\Theta(n - j)}{\Theta(n)} \right\} \quad (14)$$

for all  $n \geq 1$ .

PROOF. Let  $F$  be as in the lemma. Define a function  $G = F * \mu$ . Then  $g_j(1) = f_j(1) - 1 \geq 0$ , and hence  $G(\bar{t})\mu^2(\bar{t}) \geq 0$ . Moreover, if  $\mu^2(\bar{t} + \bar{s}) \neq 0$ , then  $\bar{t} \perp \bar{s}$ . If  $\bar{s} \in \Omega$ , we have

$$\begin{aligned} M_m(F\mu^2) &= \Theta(m)^{-1} \sum_{\ell(\bar{k})=m} F(\bar{k})\mu^2(\bar{k})P(\bar{k}) = \Theta(m)^{-1} \sum_{\substack{\ell(\bar{s}+\bar{t})=m \\ \bar{t} \perp \bar{s}}} G(\bar{t})\mu^2(\bar{s}+\bar{t})P(\bar{s}+\bar{t}) \\ &\leq \Theta(m)^{-1} \sum_{\ell(\bar{t}) \leq m} G(\bar{t})\mu^2(\bar{t})P(\bar{t}) \sum_{\ell(\bar{s})=m-\ell(\bar{t})} P(\bar{s}) \\ &= \sum_{\ell(\bar{t}) \leq m} G(\bar{t})\mu^2(\bar{t})P(\bar{t}) \frac{\Theta(m-\ell(\bar{t}))}{\Theta(m)} \leq \prod_{j \leq m} \left(1 + (f_j(1) - 1)P(\bar{e}_j) \frac{\Theta(m-j)}{\Theta(m)}\right). \end{aligned}$$

In the last step, we used the inequality

$$\frac{\Theta(n-\ell(\bar{s}))}{\Theta(n)} \leq \prod_{j \leq n} \frac{\Theta(n-js_j)}{\Theta(n)}$$

valid for all  $n \geq 1$ ,  $\ell(\bar{s}) \leq n$  and  $\theta \geq 1$ . The latter implies

$$M_m(F\mu^2) \leq \exp \left\{ \sum_{j \leq m} (f_j(1) - 1)p_j(1) \frac{\Theta(m-j)}{\Theta(m)} \right\}$$

if  $0 \leq m \leq n$ .

In the general case, we reorganize the expression of  $M_n(F)$ . We first uniquely split  $\bar{k} = \bar{t} + \bar{s}$  with  $\bar{t} \perp \bar{s}$ , where  $\bar{s} \in (\mathbb{N}_0 \setminus \{1\})^n$ ,  $\bar{t} \in \{0, 1\}^n$ , and  $t_j = 1$  if and only if  $k_j = 1$ . Then, keeping this agreement in the summation carried out in the next few lines, we proceed as follows:

$$\begin{aligned} M_n(F) &= \Theta(n)^{-1} \sum_{\substack{\ell(\bar{t}+\bar{s})=n \\ \bar{t} \perp \bar{s}}} F(\bar{t})P(\bar{t})\mu^2(\bar{t})F(\bar{s})P(\bar{s}) \\ &\leq \sum_{\ell(\bar{s}) \leq n} F(\bar{s})P(\bar{s}) \frac{\Theta(n-\ell(\bar{s}))}{\Theta(n)} \Theta(n-\ell(\bar{s}))^{-1} \sum_{\ell(\bar{t})=n-\ell(\bar{s})} F(\bar{t})\mu^2(\bar{t})P(\bar{t}). \end{aligned}$$

The inner sum was just estimated. Since

$$\begin{aligned} \sum_{\ell(\bar{s}) \leq r} F(\bar{s})P(\bar{s}) \frac{\Theta(r-\ell(\bar{s}))}{\Theta(r)} &\leq \prod_{j \leq r} \left(1 + \sum_{2 \leq k \leq r/j} f_j(k)p_j(k) \frac{\Theta(r-jk)}{\Theta(r)}\right) \\ &\leq \exp \left\{ \sum_{\substack{jk \leq r \\ k \geq 2}} f_j(k)p_j(k) \frac{\Theta(r-jk)}{\Theta(r)} \right\} \end{aligned}$$

for  $0 \leq r \leq n$ , we further obtain

$$\begin{aligned} M_n(F) &\leq \exp \left\{ \sum_{j \leq n} (f_j(1) - 1)p_j(1) \frac{\Theta(n-j)}{\Theta(n)} + \sum_{\substack{j \leq n \\ k \geq 2}} f_j(k)p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} \right\} \\ &= \exp \left\{ \sum_{jk \leq n} f_j(k)p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} - \sum_{j \leq n} p_j(1) \frac{\Theta(n-j)}{\Theta(n)} \right\}. \end{aligned}$$

The lemma is proved.  $\square$

### 3. Moments of an additive function

We now embark on the power moments of a complex-valued additive function  $H(\bar{s})$ . Throughout the section, we assume  $\theta \geq 1$  and allow dependence on  $\theta$  of the constants in  $\ll$  and of  $c, c_1, \dots, C, C_1, \dots$  introduced afterwards. Their dependence on other parameters will be additionally indicated. Let  $A := A_n$  and  $B := (B_n(2))^{1/2}$  be the quantities defined in the Introduction. Define the multiplicative function  $F(\bar{s}) = e^{zH(\bar{s})/B}$ , where  $z \in \mathbb{C}$ , and set  $\varphi_n(z) = e^{-zA/B} M_n(F)$ . Afterwards, we adopt Elliott's [5] argument.

**Lemma 4.** *Assume that  $0 \leq h_j(k) \leq \delta B$  holds for some  $\delta > 0$  and all products  $jk \leq n$ . Then there is a positive constant  $c(\delta)$  such that  $|\varphi_n(z)| \leq c(\delta)$  uniformly in  $z$  if  $|z| \leq 1$  and  $n \geq 1$ .*

PROOF. Inequality (12) gives us

$$\begin{aligned} &\sum_{\substack{j \leq n \\ k \geq 2}} p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} \\ &\leq \sum_{\substack{j \leq n \\ k \geq 2}} p_j(k) \leq \sum_{j \leq n} \sum_{k \geq 2} \left( \frac{\theta}{j} \right)^k \frac{1}{k!} = \sum_{j \leq n} \left( e^{\theta/j} - 1 - \frac{\theta}{j} \right) \ll C. \end{aligned} \quad (15)$$

Applying this and the Cauchy–Schwarz inequality, we have

$$\left| A - \sum_{j \leq n} h_j(1)p_j(1) \frac{\Theta(n-j)}{\Theta(n)} \right| \leq B \left( \sum_{\substack{j \leq n \\ k \geq 2}} p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} \right)^{1/2} \ll B.$$

In the case  $-1 \leq z = r \leq 0$ , we have  $0 \leq F(\bar{s}) \leq 1$ . Thus, (13), the inequality above and  $|e^{-x} - 1 + x| \leq x^2$  if  $x \geq 0$  yield the estimate

$$\begin{aligned} \varphi_n(r) &\ll \exp \left\{ \sum_{j \leq n} (e^{rh_j(1)/B} - 1) p_j(1) \frac{\Theta(n-j)}{\Theta(n)} - \frac{rA}{B} \right\} \\ &\ll \exp \left\{ \sum_{j \leq n} \left( e^{rh_j(1)/B} - 1 - \frac{rh_j(1)}{B} \right) p_j(1) \frac{\Theta(n-j)}{\Theta(n)} \right\} \\ &\leq \exp \left\{ \sum_{j \leq n} \left| \frac{rh_j(1)}{B} \right|^2 p_j(1) \frac{\Theta(n-j)}{\Theta(n)} \right\} \leq \exp \{r^2\} \leq e. \end{aligned}$$

In other words,  $\varphi_n(r) \leq C_1$  for  $-1 \leq z = r \leq 0$ .

If  $0 \leq z = r \leq 1$ , then  $F(\bar{s}) \geq 1$ . Arguing as in Lemma 3, we obtain

$$\begin{aligned} \varphi_n(r) &\leq \exp \left\{ \sum_{jk \leq n} e^{rh_j(k)/B} p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} - \sum_{j \leq n} p_j(1) \frac{\Theta(n-j)}{\Theta(n)} - \frac{rA}{B} \right\} \\ &= \exp \left\{ \sum_{jk \leq n} p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} \left( e^{rh_j(k)/B} - 1 - \frac{rh_j(k)}{B} \right) \right. \\ &\quad \left. + \sum_{jk \leq n} p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} - \sum_{j \leq n} p_j(1) \frac{\Theta(n-j)}{\Theta(n)} \right\} \\ &\leq \exp \left\{ \frac{r^2 e^{r\delta}}{B^2} \sum_{jk \leq n} h_j(k)^2 p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} + \sum_{\substack{jk \leq n \\ k \geq 2}} p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} \right\} \\ &\ll \exp \{r^2 e^{r\delta}\}. \end{aligned}$$

Hence  $\varphi_n(r) \leq C_2(\delta)$  if  $0 \leq z = r \leq 1$ .

In the general case when  $r = \Re z$  and  $|z| \leq 1$ , we have

$$|\varphi_n(z)| \leq \Theta(n)^{-1} \sum_{\ell(\bar{s})=n} P(\bar{s}) \left| e^{z(H(\bar{s})-A)/B} \right| \leq \varphi_n(r) \leq \max\{C_1, C_2(\delta)\}. \quad \square$$

**Lemma 5.** Assume  $H(\bar{s})$  is a complex-valued additive function such that  $|h_j(k)| \leq \delta B$  holds for some  $\delta > 0$  and all products  $jk \leq n$ . Then for each  $\alpha > 0$ , there is a positive constant  $c_1(\alpha, \delta)$  so that the inequality

$$M_n(|H(\bar{s}) - A|^\alpha) \leq c_1(\alpha, \delta) B^\alpha \tag{16}$$

holds for all  $n \geq 1$ .

PROOF. Since the weighted power means

$$(M_n(|H(\bar{s}) - A|^\alpha))^{1/\alpha}$$

do not decrease as  $\alpha$  increases, it will suffice to prove inequality (16) for integer values of  $\alpha$ .

By considering real and imaginary parts separately, we see that there is no loss in generality in assuming that  $H(\bar{s})$  takes only real values, and, indeed, only non-negative real values. For example, we can define additive functions  $H_i(s)$ ,  $i = 1, 2$ , by

$$h_{1,j}(k) := \begin{cases} h_j(k) & \text{if } h_j(k) > 0, \\ 0 & \text{otherwise,} \end{cases} \quad h_{2,j}(k) := \begin{cases} -h_j(k) & \text{if } h_j(k) < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{in} := \sum_{jk \leq n} h_{i,j}(k) p_j(k) \frac{\Theta(n - jk)}{\Theta(n)}, \quad i = 1, 2.$$

Then

$$|H(\bar{s}) - A|^\alpha \leq 2^\alpha \sum_{i=1}^2 |H_i(\bar{s}) - A_{in}|^\alpha.$$

Summing over the vectors  $\bar{s}$  such that  $\ell(\bar{s}) = n$  justifies our assertion.

For every positive integer  $l$ , we calculate the  $l$ -th derivative of  $\varphi_n(z)$  evaluated at  $z = 0$ . Namely,

$$\varphi_n^{(l)}(0) = \frac{\Theta(n)^{-1}}{B^l} \sum_{\ell(\bar{s})=n} P(\bar{s}) (H(\bar{s}) - A)^l = \frac{M_n((H(\bar{s}) - A)^l)}{B^l}.$$

By Cauchy's integral representation theorem,

$$\varphi_n^{(l)}(0) = \frac{l!}{2\pi i} \int_{|z|=1} z^{-l-1} \varphi_n(z) dz,$$

and by Lemma 4,

$$|\varphi_n^{(l)}(0)| \leq \frac{l!}{2\pi} 2\pi \max_{|z|=1} |z^{-l-1} \varphi_n(z)| \leq l! c(\delta).$$

This completes the proof of Lemma 5.  $\square$

Let us remark that those  $j, k, jk \leq n$ , for which  $|h_j(k)| > \delta B$  holds, satisfy

$$\sum_{\substack{jk \leq n \\ |h_j(k)| > \delta B}} p_j(k) \frac{\Theta(n - jk)}{\Theta(n)} \leq \sum_{jk \leq n} p_j(k) \frac{\Theta(n - jk)}{\Theta(n)} \left| \frac{h_j(k)}{\delta B} \right|^2 = \delta^{-2}, \quad (17)$$

and are in this sense few.

**Lemma 6.** *Suppose  $J \subset \{1, 2, \dots, n\}$  and define*

$$L := L(n) = \sum_{j \in J} p_j(1) \frac{\Theta(n - j)}{\Theta(n)}.$$

*Let  $\omega(\bar{s})$  denote the number of nonzero coordinates  $s_j$  such that  $j \in J$  or  $s_j \geq 2$  and  $\alpha \geq 0$ . Then there is a positive constant  $c_2(\alpha)$  such that the inequality*

$$M_m(\omega(\bar{s})^\alpha) \leq c_2(\alpha)(L + 1)^\alpha \quad (18)$$

*holds uniformly for all  $n, m$  if  $1 \leq m \leq n$ .*

PROOF. As we have remarked in the proof of Lemma 5, it will suffice to prove inequality (18) for all integers  $\alpha = l \geq 0$ .

We argue inductively on  $l$ . For  $l = 0$  inequality (18) is trivially valid. Assume that it holds for  $l = 0, 1, \dots, v - 1, v \geq 1$ . Then

$$\begin{aligned} M_m(\omega(\bar{s})^v) &= \Theta(m)^{-1} \sum_{\ell(\bar{s})=m} P(\bar{s}) \omega(\bar{s})^v = \Theta(m)^{-1} \sum_{\ell(\bar{s})=m} P(\bar{s}) \omega(\bar{s})^{v-1} \sum_{k \bar{e}_j \parallel \bar{s}}^* 1 \\ &= \Theta(m)^{-1} \sum_{jk \leq m}^* \sum_{\substack{\ell(\bar{s})=m \\ s_j=k}} P(\bar{s}) \omega(\bar{s})^{v-1}. \end{aligned}$$

Here and further on, the asterisk means that in the case  $k = 1$  the summation is taken only over  $j \in J$ . If  $s_j = k$ , say  $\bar{s} = k \bar{e}_j + \bar{t}$  where  $\bar{e}_j \perp \bar{t}$ , then  $\ell(\bar{t}) = \ell(\bar{s}) - jk$  and  $\omega(\bar{s}) \leq 1 + \omega(\bar{t})$ . According to our induction hypothesis, the inner sum

$$\begin{aligned} \sum_{\substack{\ell(\bar{t})=m-jk \\ \bar{e}_j \perp \bar{t}}} P(\bar{t} + k \bar{e}_j) \omega(\bar{t} + k \bar{e}_j)^{v-1} &\leq P(k \bar{e}_j) \sum_{\ell(\bar{t})=m-jk} P(\bar{t}) (1 + \omega(\bar{t}))^{v-1} \\ &= p_j(k) \sum_{\ell(\bar{t})=m-jk} P(\bar{t}) \sum_{i=0}^{v-1} \binom{v-1}{i} \omega(\bar{t})^i \leq \end{aligned}$$

$$\begin{aligned}
&\leq \Theta(m - jk)p_j(k) \sum_{i=0}^{v-1} \binom{v-1}{i} c_2(i)(L+1)^i \\
&\leq \Theta(m - jk)p_j(k) \max_{0 \leq i \leq v-1} c_2(i)(L+1+1)^{v-1} \\
&\leq C_3(v)\Theta(m - jk)p_j(k)(L+1)^{v-1}.
\end{aligned}$$

Hence, by (15),

$$\begin{aligned}
M_m(\omega(\bar{s})^v) &\leq C_3(v)(L+1)^{v-1} \sum_{jk \leq m}^* \frac{\Theta(m - jk)}{\Theta(m)} p_j(k) \\
&\leq C_3(v)(L+1)^{v-1}(C+L) \leq C_4(v)(L+1)^v.
\end{aligned}$$

Setting  $c_2(v) = C_4(v)$ , we complete the proof of Lemma 6.  $\square$

**Lemma 7.** *Let a complex-valued additive function  $H(\bar{s})$  and  $\delta > 0$  be such that either  $|h_j(k)| > \delta B$  or  $h_j(k) = 0$  is true for each of the products  $jk \leq n$ . Then for each  $\alpha \geq 1$ , there is a positive constant  $c_3(\alpha, \delta)$  so that the inequality*

$$M_n(|H(\bar{s}) - A|^\alpha) \leq c_3(\alpha, \delta)B_n(\alpha) \quad (19)$$

holds for all  $n \geq 1$ .

PROOF. Let  $J$  be the set of indices  $j \leq n$  such that  $h_j(1) \neq 0$ , and let  $L$ ,  $\omega(\bar{s})$  be as in Lemma 6.

By Hölder's inequality, we see that

$$|H(\bar{s})|^\alpha \leq \omega(\bar{s})^{\alpha-1} \sum_{k \in J \setminus \{\bar{s}\}} |h_j(k)|^\alpha.$$

Hence

$$\sum_{\ell(\bar{s})=n} P(\bar{s})|H(\bar{s})|^\alpha \leq \sum_{jk \leq n} |h_j(k)|^\alpha \sum_{\substack{\ell(\bar{s})=n \\ s_j=k}} P(\bar{s})\omega(\bar{s})^{\alpha-1}. \quad (20)$$

Since

$$L \leq \sum_{\substack{jk \leq n \\ |h_j(k)| > \delta B}} p_j(k) \frac{\Theta(n - jk)}{\Theta(n)} \leq \delta^{-2}$$

due to (17), the inner sum on the right-hand side of (20) is by Lemma 6 no more than

$$c_2(\alpha)\Theta(n - jk)p_j(k)(L+1)^\alpha \leq C_5(\alpha, \delta)\Theta(n - jk)p_j(k). \quad (21)$$

Inequalities (20) and (21) show that

$$\Theta(n)^{-1} \sum_{\ell(\bar{s})=n} P(\bar{s}) |H(\bar{s})|^\alpha \leq C_5(\alpha, \delta) \sum_{jk \leq n} |h_j(k)|^\alpha p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} = C_5(\alpha, \delta) B_n(\alpha).$$

Moreover, Hölder's inequality and (17) show that

$$|A|^\alpha \leq \left( \sum_{\substack{jk \leq n \\ |h_j(k)| > \delta B}} p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} \right)^{\alpha-1} B_n(\alpha) \leq \delta^{2(1-\alpha)} B_n(\alpha).$$

Collecting the last two inequalities and using Hölder's inequality once again, we finish the proof of Lemma 7.  $\square$

#### 4. Proof of Theorem 1

We define additive functions  $H^{(i)}(\bar{s})$ ,  $i = 1, 2$ , by

$$h_j^{(1)}(k) := \begin{cases} h_j(k) & \text{if } |h_j(k)| \leq B, \\ 0 & \text{otherwise,} \end{cases} \quad h_j^{(2)}(k) := \begin{cases} h_j(k) & \text{if } |h_j(k)| > B, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_n^{(i)} := \sum_{jk \leq n} h_j^{(i)}(k) p_j(k) \frac{\Theta(n-jk)}{\Theta(n)}, \quad i = 1, 2.$$

Since, for  $A := A_n$  defined in the Introduction,

$$|H(\bar{s}) - A|^\alpha \leq 2^\alpha \sum_{i=1}^2 |H^{(i)}(\bar{s}) - A_n^{(i)}|^\alpha,$$

the desired inequality of Theorem 1 in the case  $\alpha \geq 2$  follows from Lemma 5 applied to the function  $H^{(1)}(\bar{s})$  with  $\delta = 1$ , together with Lemma 7 applied to the function  $H^{(2)}(\bar{s})$  with  $\delta = 1$ .

If  $0 \leq \alpha \leq 2$ , then

$$M_n(|H(\bar{s}) - A|^\alpha)^{1/\alpha} \leq M_n(|H(\bar{s}) - A|^2)^{1/2},$$

and the result follows from the just established inequality.

Theorem 1 is proved.  $\square$

Let us apply it to the additive function

$$H(\bar{s}) = W(\bar{s}) := \sum_{j \leq n} \mathbf{1}\{s_j \geq 1\}$$

and verify (6). Now  $h_j(k) = w_j(k) = \mathbf{1}\{k \geq 1\}$  if  $j \leq n$ ; therefore, the corresponding  $A_n$  and  $B_n(\alpha)$  can be simplified. To see this, applying the singularity analysis (see [8, Chapter VI]), we obtain

$$\begin{aligned} \sum_{2 \leq k \leq n/j} p_j(k) \frac{\Theta(n - jk)}{\Theta(n)} &= \frac{1}{\Theta(n)} [z^n] \frac{e^{\theta z^j/j} - 1 - \theta z^j/j}{(1 - z)^\theta} \\ &= \left( e^{\theta/j} - 1 - \frac{\theta}{j} \right) (1 + o(1)) \ll \frac{1}{j^2} \end{aligned}$$

uniformly in  $j \leq n$ . Hence, by virtue of (1) and  $(1 - x)^{\theta-1} - 1 \ll x$  if  $0 < x \leq 1/2$ ,

$$A_n = B_n(\alpha) = \theta \sum_{j \leq n} \frac{1}{j} \left( 1 - \frac{j}{n} \right)^{\theta-1} + O(1) = \theta \sum_{j \leq n} \frac{1}{j} + O(1) = \theta \log n + O(1).$$

Consequently, Theorem 1 implies the desired estimate (6).

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