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# Rings in which every element is the sum of a left zero-divisor and an idempotent

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**Abstract.** A ring $R$ is called *left zero-clean* if every element is the sum of a left zero-divisor and an idempotent. This class of rings is a natural generalization of $O$-rings and nil-clean rings. We determine when a skew polynomial ring is a left zero-clean ring. It is proved that a ring $R$ is left zero-clean if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is left zero-clean. It is shown that a commutative ring $R$ is zero-clean if and only if the matrix ring $\mathbb{M}_n(R)$ is zero-clean for every positive integer $n \geq 1$. We characterize the zero-clean matrix rings over fields. We also determine when a $2 \times 2$ matrix $A$ over a field is left zero-clean. A ring is called *uniquely left zero-clean* if every element is uniquely the sum of a left zero-divisor and an idempotent. We completely determine when a ring is uniquely left zero-clean.

## 1. Introduction

Throughout this paper, $R$ will be an associative ring with identity, $\mathrm{U}(R)$ its group of units, $\mathrm{J}(R)$ its Jacobson radical, $\mathrm{Idem}(R)$ its set of idempotents, and $\mathrm{Nil}(R)$ is the set of nilpotent elements of $R$. For $x \in R$, $\mathrm{Ann}_l(x) = \{a \in R : ax = 0\}$ and $\mathrm{Ann}_r(x) = \{a \in R : xa = 0\}$ denote the left annihilator and the right annihilator ideals of $x$ in $R$, respectively. When $\mathrm{Ann}_r(x) \neq 0$, we say $x$ is a left zero-divisor; otherwise it is a non-left zero-divisor. Let $\mathrm{Z}_l(R)$ (resp., $\mathrm{Z}_l^*(R)$) denote the set of left zero-divisors (resp., non-left zero-divisors) of $R$. Similarly, let $\mathrm{Z}_r(R)$ (resp., $\mathrm{Z}_r^*(R)$) denote the set of right zero-divisors (resp., non-right

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zero-divisors) of $R$. A non-zero-divisor element is also known as a *regular element*, see [2]. Thus, for a commutative ring $R$, we will write $\mathrm{Z}_l(R) = \mathrm{Z}_r(R) = \mathrm{Z}(R)$ and $\mathrm{Z}_l^*(R) = \mathrm{Z}_r^*(R) = \mathrm{reg}(R)$, where $\mathrm{reg}(R)$ is the set of regular elements (i.e., non-zero-divisors) of $R$. We write $\mathbb{M}_n(R)$ and $\mathbb{T}_n(R)$ for the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over $R$, respectively. Also, let $A$ be a matrix, we write $\mathrm{tr}(A)$ to denote the trace of $A$, and $\det(A)$ for the determinant of $A$. Moreover, $\mathbb{I}_n$ denotes the $n \times n$ identity matrix.

In [6], Diesl defined a ring element $a \in R$ to be *nil-clean* if it can be written in the form $t + e$ where $t \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$. If every $a \in R$ is nil-clean, $R$ is said to be a nil-clean ring. In this paper, we introduce a class of rings which is a generalization of nil-clean rings. These rings, which will be called *left zero-clean* rings, are defined as rings $R$ in which for every $a \in R$ there exist a left zero-divisor $z$ and an idempotent $e$ such that $a = z + e$. Examples of such rings include nil-clean rings and $O$-rings. We recall that Cohn [5] introduced the term *$O$-ring* for commutative rings with 1, in which every element different from 1 is a zero-divisor. Let us say a few words. Examples of $O$-rings are Boolean rings. In fact, for a commutative ring $R$, $R = \mathrm{Idem}(R) \bigcup \mathrm{Z}(R)$ if and only if $R$ is an $O$-ring. In [5], Cohn showed that there exist $O$-rings which are not Boolean. It was conjectured by Anderson and Badawi that $R = \mathrm{Idem}(R) \bigcup \mathrm{Z}(R)$ if and only if $R$ is a Boolean ring, see [1, p. 1022]. Indeed, Cohn's example indicates that this conjecture is false.

This article consists of 5 sections. Section 1 is the introduction. In Section 2, we investigate some fundamental properties of this class. We show that an abelian ring $R$ is left zero-clean if and only if $a - 1 \in \mathrm{Z}_l(R)$ for each $a \in \mathrm{Z}_l^*(R)$. In this section, we show that a commutative ring $R$ is a zero-clean ring if and only if $\mathrm{reg}(R) \subseteq 1 - \mathrm{Z}(R)$. In Section 3, the behavior of zero-clean rings under some classical ring constructions is studied. We determine when a skew polynomial ring is a left zero-clean ring. Also, it is shown that a ring $R$ is left zero-clean if and only if $\mathbb{T}_n(R)$ is left zero-clean. The aim of Section 4 is to give some results of matrix rings over commutative zero-clean rings. It is shown that a commutative ring $R$ is zero-clean if and only if $\mathbb{M}_n(R)$ is left zero-clean. We also determine when a $2 \times 2$ matrix $A$ over a field is left zero-clean. In the last section, we define and give some characterizations of uniquely left zero-clean rings. Let us recall that a commutative ring $R$ is *weakly présimplifiable* if and only if $\mathrm{Z}(R) \subseteq 1 - \mathrm{reg}(R)$, see [2]. It is shown that a commutative ring $R$ is uniquely zero-clean if and only if $\mathrm{Z}(R) = 1 - \mathrm{reg}(R)$.

## 2. Zero-clean rings

We begin with a formal definition of the central concept of the article.

*Definition 2.1.* An element $a$ of a ring $R$ is called *left zero-clean* if it can be expressed as the sum of a left zero-divisor and an idempotent in $R$. Any equation of the form $a = z + e$ will be called a *left zero-clean decomposition* of $a \in R$, where $z \in \mathrm{Z}_l(R)$ and $e \in \mathrm{Idem}(R)$. A ring $R$ is called *left zero-clean* if every element of $R$ is left zero-clean. Right zero-clean rings are defined similarly. A ring which is both right and left zero-clean is called *zero-clean*.

*Remark 2.2.* It is clear that a left zero-divisor element and 1 are trivially left zero-clean.

The following example shows that there exist left zero-clean rings which are not right zero-clean.

*Example 2.3.* Let $R$ be a $\mathbb{Z}_2$-algebra generated by $x_i$, $i \in \mathbb{N}$, with the relations $x_i x_j = 0$, for all $i < j$. It is easy to verify that $R$ is a left zero-clean ring. On the other hand, $x_1$ has not a right zero-clean decomposition, and hence $R$ is not right zero-clean.

The following example shows that there exists a left zero-clean ring which is neither an $O$-ring nor nil-clean.

*Example 2.4.* Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$. It is straightforward to check that $R$ is a zero-clean ring which is neither an $O$-ring nor a nil-clean ring.

The following example shows that the class of left zero-clean rings is not hereditary on subrings.

*Example 2.5.* Let $R$ be a $\mathbb{Z}_2$-algebra generated by $x_i$, $i \in \mathbb{Z}$, with the relations $x_i x_j = 0$, for all $i < j$. Now, let $S$ be the subring of $R$ generated by $x_k$ where $k \leq -1$. One can easily see $S$ is not a left zero-clean ring.

A ring is called *abelian* if all its idempotents are central. Examples of abelian rings are reduced rings (e.g., strongly regular rings), one-sided duo rings (e.g., commutative rings), and of course, all rings with only trivial idempotents $\{0, 1\}$.

**Theorem 2.6.** *Let $R$ be an abelian ring. The following statements are equivalent:*

(1) *$R$ is a left zero-clean ring.*

(2) *If $a \in \mathrm{Z}_l^*(R)$, then $a - 1 \in \mathrm{Z}_l(R)$.*

Proof. (1) $\Rightarrow$ (2) Assume that $R$ is a left zero-clean ring and $a \in \mathrm{Z}_l^*(R)$. Therefore, $a$ has a left zero-clean decomposition, as $a = z + e$, where $z \in \mathrm{Z}_l(R)$ and $0 \neq e \in \mathrm{Idem}(R)$. If $e = 1$, then $a - 1 = z \in \mathrm{Z}_l(R)$. Now, suppose that $e \neq 1$ and consider $0 \neq z' \in \mathrm{Ann}_r(z)$. We have $a(z'(e-1)) = (z+e)(z'(e-1)) = 0$. Since $a$ is a non-left zero-divisor, $z'(e-1) = 0$. Hence $z'e = z' \neq 0$, and we infer that $a - 1 = z + e - 1$. Now, we have $(a-1)(z'e) = (z+e-1)(z'e) = 0$, which implies that $a - 1$ is a left zero-divisor.

(2) $\Rightarrow$ (1) If $a \in \mathrm{Z}_l^*(R)$, then $(a-1)+1$ is a left zero-clean decomposition of $a$. □

**Corollary 2.7.** *If $R$ is an abelian left zero-clean ring, then $\mathrm{J}(R) \subseteq \mathrm{Z}_l(R)$.*

Proof. Suppose that $a \in \mathrm{J}(R) \setminus \mathrm{Z}_l(R)$. Then $1 - a$ is a unit which has to be a left-zero divisor by Theorem 2.6, a contradiction. □

The following is an easy consequence of Theorem 2.6.

**Corollary 2.8.** *The only zero-clean domain is $\mathbb{Z}_2$.*

Recall that a ring $R$ is *local* if the sum of any two non-units is a non-unit, or equivalently, if the ring has a unique maximal left ideal.

**Proposition 2.9.** *Let $R$ be a left zero-clean ring. Then $\mathrm{Z}_l(R) \subseteq \mathrm{J}(R)$ if and only if $R$ is local.*

Proof. Suppose that $\mathrm{Z}_l(R) \subseteq \mathrm{J}(R)$. Since $\mathrm{J}(R)$ contains no non-trivial idempotent, $R$ has only trivial idempotents. Hence $R$ is an abelian ring. Corollary 2.7 shows that $\mathrm{J}(R) \subseteq \mathrm{Z}_l(R)$. Thus, $\mathrm{Z}_l(R) = \mathrm{J}(R)$ and this means that $\mathrm{Z}_l(R)$ is an ideal. Now, we claim that $R \setminus \mathrm{J}(R) = \mathrm{U}(R)$. Suppose that $r \notin \mathrm{J}(R)$. By hypothesis, we infer that $r$ is a non-left zero-divisor in $R$. Thus, there exists $j \in \mathrm{Z}_l(R) = \mathrm{J}(R)$ where $r = j + 1$. This implies that $r$ is a unit. Thus, we conclude that every non-left zero-divisor is a unit. Note that $\mathrm{Z}_l(R)$ is an ideal, and so the sum of any two non-units in $R$ is a non-unit. This means that $R$ is local. The converse is clear. □

*Remark 2.10.* It is clear that $\mathbb{Z}_6$ is a zero-clean ring, but $\mathbb{Z}_6/3\mathbb{Z}_6$ is not a zero-clean ring. Hence a homomorphic image of a left zero-clean ring need not be left zero-clean. Also, a homomorphic image of a non-left zero-clean ring may be left zero-clean. For example, $\mathbb{Z}$ is not a zero-clean ring, but $\mathbb{Z}_4 \cong \mathbb{Z}/4\mathbb{Z}$ is a zero-clean ring.

This next result needs no proof.

**Lemma 2.11.** *Let* $\{R_i\}_{i\in I}$ *be a family of rings. Then* $\mathrm{Z}_l^*(\prod_I R_i)=\prod_I \mathrm{Z}_l^*(R_i)$ *and* $\mathrm{Z}_r^*(\prod_I R_i)=\prod_I \mathrm{Z}_r^*(R_i)$. *Also,* $\mathrm{Idem}(\prod_I R_i)=\prod_I \mathrm{Idem}(R_i)$.

The following seems interesting.

**Proposition 2.12.** *A direct product of rings is left zero-clean if and only if at least one factor is left zero-clean.*

Proof. Let $\{R_i\}_{i\in I}$ be a family of rings and $R=\prod_I R_i$. First assume that, for each $i\in I$, $R_i$ is not a left zero-clean ring. Thus, for each $i\in I$, there exists $a_i\in R_i$ which is not a left zero-clean element. Take $(a_i)_{i\in I}\in R$. We claim that $(a_i)_{i\in I}$ is not a left zero-clean element of $R$. Suppose that $(a_i)_{i\in I}=(z_i)_{i\in I}+(e_i)_{i\in I}$, where $(z_i)_{i\in I}\in \mathrm{Z}_l(R)$ and $(e_i)_{i\in I}\in \mathrm{Idem}(R)$. Lemma 2.11 shows that there exists $i_k\in I$ such that $z_{i_k}\in \mathrm{Z}_l(R_{i_k})$, and also, for each $i\in I$, $e_i\in \mathrm{Idem}(R_i)$. Therefore, we infer that $a_{i_k}=z_{i_k}+e_{i_k}$, where $z_{i_k}\in \mathrm{Z}_l(R_{i_k})$ and $e_{i_k}\in \mathrm{Idem}(R_{i_k})$, is a left zero-clean decomposition for $a_{i_k}\in R_{i_k}$. This is a contradiction.

Conversely, assume that $R=\prod_I R_i$ is not a zero-clean ring. Thus, there exists $(a_i)_{i\in I}\in R$ which is not a left zero-clean element. Suppose that there exists $i_k\in I$ such that $R_{i_k}$ is a left zero-clean ring and we seek a contradiction. This implies that for $a_{i_k}\in R_{i_k}$, there exist $z_{i_k}\in \mathrm{Z}_l(R_{i_k})$ and $e_{i_k}\in \mathrm{Idem}(R_{i_k})$ such that $a_{i_k}=z_{i_k}+e_{i_k}$. Define $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ as follows:

$$p_{i_k}:=\begin{cases} z_{i_k} & \text{if } i=i_k,\\ a_{ij} & \text{otherwise;}\end{cases} \qquad q_{i_k}:=\begin{cases} e_{i_k} & \text{if } i=i_k,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2.11 shows that $(p_i)_{i\in I}\in \mathrm{Z}_l(R)$, $(q_i)_{i\in I}\in \mathrm{Idem}(R)$ and $(a_i)_{i\in I}=(p_i)_{i\in I}+(q_i)_{i\in I}$. Hence $(a_i)_{i\in I}\in R$ is a left zero-clean element, which is a contradiction. □

**Lemma 2.13.** *Let $R$ be a left zero-clean ring. Then 2 is a left zero-divisor.*

Proof. Suppose that 2 is a non-left zero-divisor. Then there exist $z\in \mathrm{Z}_l(R)$ and $0,1\neq e\in \mathrm{Idem}(R)$ such that $2=z+e$. Therefore $1-e=z-1$, and so $z-1$ is an idempotent. Thus $(z-1)^2=(z-1)$, which implies $(3-z)z=2$, a contradiction. □

**Corollary 2.14.** *$\mathbb{Z}_n$ is a zero-clean ring if and only if $2|n$.*

Proof. It follows from Lemma 2.13 and Proposition 2.12. □

## 3. Zero-clean property under algebraic constructions

Let $R$ be a ring, and $\sigma$ be a ring endomorphism of $R$. Let $R[x;\sigma]$ denote the skew polynomial ring consisting of the polynomials in $x$ with coefficients in $R$ written on the left, with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. $R[[x;\sigma]]$ denotes the skew formal power series ring. It is clear that if we put $\sigma = 1_R$, then we have $R[x] = R[x;\sigma]$ and $R[[x]] = R[[x;\sigma]]$. Next, we will characterize when a skew polynomial ring is left zero-clean.

**Theorem 3.1.** *Let $R$ be a ring, and $\sigma$ be a ring endomorphism of $R$.*

(1) *If $\sigma$ is a ring automorphism of $R$, then $R[x;\sigma]$, $R[[x;\sigma]]$ are never left zero-clean rings.*

(2) *If $R$ is a domain, then $R[x;\sigma]$ is never a left zero-clean ring.*

(3) *If $R$ is not a domain, then the following statements are equivalent:*

(a) *$R[x;\sigma]$ is a left zero-clean ring.*

(b) (i) *$R$ is left zero-clean.*

(ii) *$\sigma$ is not injective.*

(iii) *Every $r \in R$ has a left zero-clean decomposition $r = z + e$ where $z \in \mathrm{Z}_l(R)$, $e \in \mathrm{Idem}(R)$, and $\mathrm{Ann}_r(z) \bigcap \mathrm{Ker}(\sigma) \neq 0$.*

Proof. (1) Suppose that $x = z + e$ where $z \in \mathrm{Z}_l(R[x;\sigma])$ and $e \in \mathrm{Idem}(R[x;\sigma]) = \mathrm{Idem}(R)$. This implies that $-e + x = z$. Since $\sigma$ is a ring automorphism, $-e + x$ is not a left zero-divisor, a contradiction. The proof is similar for $R[[x;\sigma]]$.

(2) It is clear.

(3) $(a) \Rightarrow (b)$ It is clear that $R$ is a left zero-clean ring, since $\mathrm{Idem}(R[x;\sigma]) = \mathrm{Idem}(R)$. If $x$ is a non-left zero divisor, then there exist $z \in \mathrm{Z}_l(R[x;\sigma])$ and $e \in \mathrm{Idem}(R[x;\sigma]) = \mathrm{Idem}(R)$ such that $x - z = e \in \mathrm{Idem}(R)$, which is a contradiction. Hence $x$ is a left zero-divisor and this means that $\sigma$ is not injective. Now suppose that $r \in R$, and consider the element $r + x$ in $R[x;\sigma]$. Therefore, there exist $z \in \mathrm{Z}_l(R[x;\sigma])$ and $e \in \mathrm{Idem}(R)$ such that $r + x = z + e$. Hence $z = (r - e) + x$. Since $z = (r - e) + x$ is a left-zero-divisor in $R[x;\sigma]$, there exists $b \in \mathrm{Ann}_r(r - e) \bigcap \mathrm{Ker}(\sigma)$. This means that $r$ has a left zero-clean decomposition $r = (r - e) + e$ such that $\mathrm{Ann}_r(r - e) \bigcap \mathrm{Ker}(\sigma) \neq 0$.

$(b) \Rightarrow (a)$ Suppose that $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is an element of $R[x;\sigma]$. By our assumption, $a_0$ has a left zero-clean decomposition $a_0 = z + e$ where $z \in \mathrm{Z}1(R)$, $e \in \mathrm{Idem}(R)$, and $\mathrm{Ann}_r(z) \bigcap \mathrm{Ker}(\sigma) \neq 0$. Hence $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has the left zero-clean decomposition $(z + a_1x + a_2x^2 + \cdots + a_nx^n) + e$. Thus $R[x;\sigma]$ is a left zero-clean ring. $\square$

We now turn our attention to formal triangular matrix rings. Before stating the next result, we recall the following lemma that is suitable for our purpose.

**Lemma 3.2** ([7, Theorem 3.3]). *Let $A$, $B$ be rings, and $M = {}_AM_B$ be a bimodule. Suppose $m \neq 0$ and $D = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$ is an element of the formal triangular matrix $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$. Then the following statements are equivalent:*

(1) *$D$ is a left zero-divisor.*

(2) *At least one of the following occurs:*

 (a) *$a \in \mathrm{Z}_l(A)$.*

 (b) *$a \in \mathrm{Z}_l^*(A)$, $b \in \mathrm{Z}_l(B)$, and there exists $0 \neq b'' \in \mathrm{Ann}_r(b)$ such that $mb'' = 0$.*

 (c) *$a \in \mathrm{Z}_l^*(A)$, $b \in \mathrm{Z}_l^*(B)$, and there exists $0 \neq m'' \in M$ such that $am'' = 0$.*

**Theorem 3.3.** *Let $A$, $B$ be rings, $M = {}_AM_B$ be a bimodule, and $M$ be a torsion-free $A$-module. The formal triangular matrix ring $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is left zero-clean if and only if either $A$ or $B$ is left zero-clean.*

Proof. First assume that $A$ is a left zero-clean ring, and $C = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$ is an arbitrary element of $\mathrm{Z}_l^*(R)$. By Lemma 3.2, we infer that $a \in \mathrm{Z}_l^*(A)$, $b \in \mathrm{Z}_l^*(B)$, and $am' \neq 0$ for all $0 \neq m' \in M$. Since $A$ is a left zero-clean ring, there exist $z \in \mathrm{Z}_l(A)$ and $e \in \mathrm{Idem}(A)$ such that $a = z + e$. Take $D = \begin{bmatrix} z & m \\ 0 & b \end{bmatrix}$ and $E = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to see that $D \in \mathrm{Z}_l(R)$, $E \in \mathrm{Idem}(R)$ and $C = D + E$. The proof for $B$ is similar.

Conversely, assume that $R$ is a left zero-clean ring, and $A$,$B$ are not left zero-clean rings simultaneously. There exist $a \in A$ and $b \in B$ such that they are not left zero-clean elements. Now, consider the element $C = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ in $R$. Since $R$ is a zero-clean ring, there exist $D = \begin{bmatrix} c & m \\ 0 & d \end{bmatrix} \in \mathrm{Z}_l(R)$ and $E = \begin{bmatrix} f & -m \\ 0 & g \end{bmatrix} \in \mathrm{Idem}(R)$ such that $C = D + E$. Since $E$ is an idempotent, we have $f \in \mathrm{Idem}(A)$ and $g \in \mathrm{Idem}(B)$. Also, since $D$ is a left zero-divisor element, we infer that either $c \in \mathrm{Z}_l(A)$ or $d \in \mathrm{Z}_l(B)$ by Theorem 3.2 (note that $M$ is a torsion-free $A$-module). Thus, we conclude that either $a$ or $b$ is a left zero-clean element, a contradiction. □

Using Theorem 3.3, an inductive argument gives immediately the following observation.

**Corollary 3.4.** *Let $R$ be a ring and $n \geq 1$. Then $R$ is left zero-clean if and only if $\mathbb{T}_n(R)$ is left zero-clean.*

## 4. Matrix rings over commutative rings

In this section, we will focus our attention on matrix rings. We begin with the following theorem.

**Theorem 4.1.** *Let $R$ be a commutative ring. If $\mathbb{M}_n(R)$ is left zero-clean, then $R$ is zero-clean.*

Proof. Suppose that $\mathbb{M}_n(R)$ is left zero-clean. Consider an element $a \in R$. By hypothesis, there is an idempotent matrix $E$ such that $a\,\mathbb{I}_n - E$ is a left zero-divisor. That means that there is a nonzero vector $v$ in $R^n$ such that $(a\,\mathbb{I}_n - E)v = 0$. In other words, $Ev = av$. Since $E$ is idempotent, this means that $E^2 v = Ev$, implying that $a^2 v = av$. Since $v$ is not zero, it has at least one nonzero entry; call it "$y$". This means that $(a^2 - a)y = 0$. But then this means that $a^2 - a$ is a zero-divisor, which means that either $a$ or $a - 1$ is a zero-divisor. Thus, $a$ is left zero-clean. □

**Theorem 4.2.** *Let $R$ be an abelian ring. If $R$ is left zero-clean, then $\mathbb{M}_n(R)$ is left zero-clean.*

Proof. Let $0 \neq A = [a_{ij}]_{n\times n}$ be an arbitrary element in $\mathbb{M}_n(R)$. Consider two different cases as follows.

*Case 1.* Suppose that $a_{11} \in \mathrm{Z}_l^*(R)$. Define $B := [b_{ij}]_{n\times n}$ such that

$$b_{ij} := \begin{cases} a_{11} - 1 & \text{if } i = 1 \text{ and } j = 1, \\ 0 & \text{if } i \neq 1 \text{ and } j = 1, \\ a_{ij} & \text{if } 1 \leq i \leq n \text{ and } j \neq 1. \end{cases}$$

Note that $a_{11} - 1 \in \mathrm{Z}_l(R)$ by Theorem 2.6. It is easy to see that $B \in \mathrm{Z}_l(\mathbb{M}_n(R))$ and $A - B \in \mathrm{Idem}(\mathbb{M}_n(R))$. Hence $A$ is a left zero-clean element.

*Case 2.* Suppose that $a_{11} \in \mathrm{Z}_l(R)$. Define $C := [c_{ij}]_{n\times n}$ such that

$$c_{ij} := \begin{cases} 1 & \text{if } 2 \leq i = j \leq n, \\ a_{ij} & \text{if } i \neq 1 \text{ and } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $C$ is an idempotent and $A - C$ is a left zero-divisor. Thus $A$ is a left zero-clean element. □

The following combines Theorem 4.1 and the commutative case of Theorem 4.2.

**Corollary 4.3.** *Let $R$ be a commutative ring and $n \geq 1$. Then $R$ is zero-clean if and only if $\mathbb{M}_n(R)$ is zero-clean.*

Breaz *et al.* [3, Theorem 3] have shown that $\mathbb{M}_n(F)$, where $F$ is a field, is a nil-clean ring if and only if $F \cong \mathbb{Z}_2$. Now we are in a position to give the complete characterization of zero-clean matrix rings over fields. The following is a consequence of Corollaries 2.8 and 4.3.

**Corollary 4.4.** *Let $F$ be a field. The following statements are equivalent:*

(1) $F \cong \mathbb{Z}_2$.

(2) *For every positive integer $n$, the matrix ring $\mathbb{M}_n(F)$ is zero-clean.*

(3) *There exists a positive integer $n$ such that the matrix ring $\mathbb{M}_n(F)$ is zero-clean.*

The above result motivates us to ask when a $2\times 2$ matrix over a field $F(\neq \mathbb{Z}_2)$ is a left zero-clean element. Before stating our result, we recall the following lemmas, which are standard facts from linear algebra.

**Lemma 4.5** ([4, Theorem 9.1])**.** *Let $R$ be a commutative ring and $A \in \mathbb{M}_n(R)$. Then $A$ is a zero-divisor if and only if $\det(A) \in \mathrm{Z}(R)$.*

**Lemma 4.6.** *Let $F$ be a field. Then $A \in \mathbb{M}_2(F)$ is a non-trivial idempotent if and only if $\det(A) = 0$ and $\mathrm{tr}(A) = 1$.*

First, we determine when a $2 \times 2$ diagonal matrix over a field $F(\neq \mathbb{Z}_2)$ is a left zero-clean element.

**Theorem 4.7.** *Let $F(\neq \mathbb{Z}_2)$ be a field and $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in \mathbb{M}_2(F)$. Then the following statements are equivalent:*

(1) $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ *is left zero-clean in* $\mathbb{M}_2(F)$.

(2) *Either* $a = d = 0, 1$ *or* $a \neq d$.

Proof. (1) $\Rightarrow$ (2) Suppose that $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is a non-left zero-divisor. If $a = d = 1$, then $A = I$, and hence $A$ is a left zero-clean element. Now, suppose that $a \neq d$. Let

$$D = \begin{bmatrix} a - (1 - (a-1)d(a-d)^{-1}) & -1 + (a-1)d(a-d)^{-1} \\ -(a-1)d(a-d)^{-1} & d - (a-1)d(a-d)^{-1} \end{bmatrix}.$$

Clearly, $\det(A) = 0$, and hence $D \in \mathrm{Z}_l(\mathbb{M}_2(F))$. Also, let

$$E = \begin{bmatrix} 1 - (a-1)d(a-d)^{-1} & 1 - (a-1)d(a-d)^{-1} \\ (a-1)d(a-d)^{-1} & (a-1)d(a-d)^{-1} \end{bmatrix}.$$

Lemma 4.6 shows that $E \in \mathrm{Idem}(\mathbb{M}_2(F))$. One can easily see that $A = D + E$, and so $A$ is a left zero-clean element.

(2) $\Rightarrow$ (1) Suppose that $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ is left zero-clean. If $a = 0, 1$, then $A$ is a trivial idempotent, and we are done. Otherwise, suppose that $a \neq 0, 1$. Then $A$ is a non-left zero-divisor, and it has a left zero-clean decomposition. Suppose that $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} b & -c \\ -d & e \end{bmatrix} + \begin{bmatrix} f & c \\ d & k \end{bmatrix}$ where $M = \begin{bmatrix} b & -c \\ -d & e \end{bmatrix} \in \mathrm{Z}_l(\mathbb{M}_2(F))$ and $N = \begin{bmatrix} f & c \\ d & k \end{bmatrix} \in \mathrm{Idem}(\mathbb{M}_2(F))$. If $N$ is a trivial idempotent, then either $A$ or $A - I$ is a left zero divisor, a contradiction. Hence $N$ is not a trivial idempotent. Thus $\mathrm{tr}(N) = 1$ and $\det(N) = 0$ by Lemma 4.6. Hence $f = 1 - k$ and $(1-k)k - cd = 0$. Now, we conclude that $M = \begin{bmatrix} a - (1-k) & -c \\ -d & a - k \end{bmatrix}$. Therefore, we have $\det(M) = a^2 - a + (1-k)k - cd = 0$, and hence $\det(M) = a^2 - a = 0$, a contradiction. $\square$

Next, we prove that every $2 \times 2$ non-diagonal matrix over a field $F(\neq \mathbb{Z}_2)$ is left zero-clean.

**Theorem 4.8.** *Let* $F$ *be a field and* $F(\neq \mathbb{Z}_2)$.

(1) *If* $c \neq 0$, *then* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is left zero-clean in* $\mathbb{M}_2(F)$.

(2) *If* $b \neq 0$, *then* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is left zero-clean in* $\mathbb{M}_2(F)$.

Proof. (1) Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a non-left zero-divisor in $\mathbb{M}_2(F)$ and $c \neq 0$. Consider three different cases as follows:

*Case 1.* If $(a - d + c - b) \neq 0$, then take $E = \begin{bmatrix} 1-f & 1-f \\ f & f \end{bmatrix}$ where $f = ((a-1) + (1-b)c)(a - d + c - b)^{-1}$.

*Case 2.* If $(a-d+c-b) = 0$ and $(a-d+b-c) \neq 0$, then take $E = \begin{bmatrix} 1-f & f \\ 1-f & f \end{bmatrix}$ where $f = ((a-1) + (1-c)b)(a - d + b - c)^{-1}$.

*Case 3.* If $(a - d + c - b) = (a - d + b - c) = 0$, then we conclude that $a = d$ and $c = b$. Therefore, take $E = \begin{bmatrix} 0 & 0 \\ f & 1 \end{bmatrix}$ where $f = (c^2 - a^2 + a)c^{-1}$.

In each of the above cases, $E$ is an idempotent by Lemma 4.6. Also, $\det(A–E)=0$, and so $A - E$ is a left zero-divisor in $\mathbb{M}_2(F)$. Hence $A = (A - E) + E$ is a left zero-clean decomposition of $A$.

(2) The proof is similar to part (1). □

With Theorems 4.7 and 4.8 at our disposal, we are now ready to determine when a $2 \times 2$ matrix over a field is left zero-clean.

**Corollary 4.9.** *Let* $F(\neq \mathbb{Z}_2)$ *be a field and* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(F)$. *Then the following statements are equivalent:*

(1) *A is not a left zero-clean element.*

(2) $a = d \neq 1, 0$ *and* $c = b = 0$.

## 5. Uniquely zero-clean rings

*Definition 5.1.* An element $a$ in a ring $R$ is called *uniquely left zero-clean* if there is a unique idempotent $e$ such that $a - e$ is a left zero-divisor. We will say that a ring is uniquely left zero-clean if each of its elements is uniquely left zero-clean. Uniquely right zero-clean rings are defined similarly. A ring which is both uniquely right and uniquely left zero-clean is called uniquely zero-clean.

**Lemma 5.2.** (1) *A uniquely left zero-clean ring has only trivial idempotents.*

(2) *If $R$ is a uniquely left zero-clean ring, then $\mathrm{J}(R) \subseteq \mathrm{Z}_l(R)$.*

PROOF. (1) Let $R$ be a uniquely left zero-clean ring and $e \in \mathrm{Idem}(R)$. Since $1 = (1-e)+e$ and $1 = 0+1$ are two zero-clean decompositions of 1, we conclude that $e = 0$ or $e = 1$. Therefore, a uniquely left zero-clean ring has only trivial idempotents.

(2) It follows from Corollary 2.7. □

**Corollary 5.3.** *Let $R$ be a uniquely left zero-clean ring. Then*

(1) *$z = z+0$ is the uniquely left zero-clean decomposition of $z$ for each $z \in \mathrm{Z}_l(R)$.*

(2) *$r = (r-1)+1$ is the uniquely left zero-clean decomposition of $r$ for each $r \in \mathrm{Z}_l^*(R)$.*

**Corollary 5.4.** *$\mathbb{Z}_n$ is uniquely zero-clean if and only if $n$ is a power of 2.*

PROOF. It follows from Corollary 2.14 and Lemma 5.2 (1). □

**Proposition 5.5.** *Let $R$ be a left zero-clean ring and $\mathrm{J}(R) = \mathrm{Z}_l(R)$. Then $R$ is a uniquely left zero-clean ring.*

PROOF. The assumption $\mathrm{J}(R) = \mathrm{Z}_l(R)$ implies that $R$ has only trivial idempotents. Since $R$ is a left zero-clean ring, $r = (r-1)+1$ is the unique left zero-clean decomposition of $r$ for each $r \in \mathrm{Z}_l^*(R)$. Now suppose that $z \in \mathrm{Z}_l(R) = \mathrm{J}(R)$. We claim that $z = z+0$ is the unique left zero-clean decomposition of $z$. Otherwise, $z = (z-1)+1$ is a left zero-clean decomposition of $z$ where $z-1 \in \mathrm{Z}_l(R) = \mathrm{J}(R)$. Thus $1 = z-(z-1) \in \mathrm{J}(R)$, which is a contradiction. Hence $R$ is a uniquely left zero-clean ring. □

In the preceding proposition, the condition $\mathrm{J}(R) = \mathrm{Z}_l(R)$ is not superfluous as follows.

*Example 5.6.* It is clear that, for $n \geq 2$, $\mathbb{T}_n(\mathbb{Z}_2)$ is a left zero-clean ring by Corollary 3.4. Note that $\mathrm{J}(\mathbb{T}_n(\mathbb{Z}_2)) \subset \mathrm{Z}_l(\mathbb{T}_n(\mathbb{Z}_2))$, but $\mathbb{T}_n(\mathbb{Z}_2)$ is not a uniquely left zero-clean ring by Lemma 5.2 (1).

*Remark 5.7.* Following [6], a ring $R$ is called *uniquely nil-clean* if, for any $a \in R$, there exists a unique idempotent $e \in R$ such that $a-e \in R$ is nilpotent. It is clear that every Boolean ring is uniquely nil-clean. Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a uniquely nil-clean ring which is not uniquely zero-clean. On the other hand, suppose that $R$ is a $\mathbb{Z}_2$-algebra generated by $x_i$, $i \in \mathbb{N}$, with the relations $x_i x_j = 0$, for all $i < j$. Then $R$ is a uniquely left zero-clean ring, but it is not uniquely nil-clean.

We get the following characterizations for uniquely left zero-clean rings.

**Theorem 5.8.** *Let $R$ be a ring. The following statements are equivalent:*

(1) *$R$ is a uniquely left zero-clean ring.*

(2) (a) *If $z \in \mathrm{Z}_l(R)$, then $z-1 \in \mathrm{Z}_l^*(R)$.*

(b) *If $r \in \mathrm{Z}_l^*(R)$, then $r-1 \in \mathrm{Z}_l(R)$.*

(3) $\mathrm{Z}_l(R) = 1 - \mathrm{Z}_l^*(R)$.

Proof. (1) $\Rightarrow$ (2) (a) If $z \in \mathrm{Z}_l(R)$, then $z = (z-1)+1$ is not a left zero-clean decomposition $z$ by Corollary 5.3. This means that $z-1 \in \mathrm{Z}_l^*(R)$.

(b) If $r \in \mathrm{Z}_l^*(R)$, then $r = (r-1)+1$ is the unique left zero-clean decomposition of $r$ by Corollary 5.3. This means that $r-1 = z \in \mathrm{Z}_l(R)$.

(2) $\Rightarrow$ (1) First, we claim that $R$ has only trivial idempotents. Suppose that $0, 1 \neq e \in \mathrm{Idem}(R)$. Since $e \in \mathrm{Z}_l(R)$, we get $e-1 \in \mathrm{Z}_l^*(R)$, a contradiction. Thus $R$ has only trivial idempotents. Now, suppose that $z \in \mathrm{Z}_l(R)$. Then $z = (z-1)+1$ is not a left zero-clean decomposition, because $z-1 \in \mathrm{Z}_l^*(R)$. This means that $z = z+0$ is the uniquely left zero-clean decomposition of $z$. It is also easy to see that $r = (r-1)+1$ is the uniquely left zero-clean decomposition of $r$ for each $r \in \mathrm{Z}_l^*(R)$. Hence $R$ is a uniquely left zero-clean ring.

(2) $\Leftrightarrow$ (3) It is clear. □

Following [2], a commutative ring $R$ is called *weakly présimplifiable* if, for $x, y \in R$, $x = xy$ implies $x = 0$ or $y$ is regular (i.e., non-zero-divisor). Anderson and Chun [2, Theorem 6] have shown that a commutative ring $R$ is weakly présimplifiable if and only if $\mathrm{Z}(R) \subseteq 1 - \mathrm{reg}(R)$. It is natural to ask: When does the inclusion $\mathrm{reg}(R) \subseteq 1 - \mathrm{Z}(R)$ hold? With the help of Theorem 2.6, we make the following simplifying observation.

**Corollary 5.9.** *Let $R$ be a commutative ring. The following statements are equivalent:*

(1) *$R$ is a zero-clean ring.*

(2) $\mathrm{reg}(R) \subseteq 1 - \mathrm{Z}(R)$.

The following is an immediate consequence of Theorem 5.8.

**Corollary 5.10.** *Let $R$ be a commutative ring. The following statements are equivalent:*

(1) *$R$ is a uniquely zero-clean ring.*

(2) $\mathrm{Z}(R) = 1 - \mathrm{reg}(R)$.

With Corollary 5.10, this proves that every commutative uniquely zero-clean ring is weakly présimplifiable. However, the converse is false, by the following example.

*Example 5.11.* Clearly, $\mathbb{Z}_2$ is a weakly présimplifiable ring. By [2, Theorem 18(2)], the polynomial ring $\mathbb{Z}_2[x]$ is weakly présimplifiable, while $\mathbb{Z}_2[x]$ is not a uniquely zero-clean ring by Theorem 3.1(1).

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