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An upper bound for the number of solutions of ternary purely exponential Diophantine equations II

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Abstract. Let a,b,c be fixed pairwise coprime positive integers with $\min\{a,b,c\}>1$. In this paper, by analyzing the gap rule for solutions of the ternary purely exponential Diophantine equation $a^x + b^y = c^z$, we prove that if $\max\{a,b,c\} \geq 10^{62}$, then the equation has at most two positive integer solutions (x, y, z).

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers, respectively. Let a, b, c be fixed pairwise coprime positive integers with $\min\{a, b, c\} > 1$. In this paper, we discuss the number of solutions (x, y, z) of the ternary purely exponential Diophantine equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N}. \tag{1.1}$$

In 1933, K. Mahler [9] used his p-adic analogue of the Thue–Siegel method to prove that (1.1) has only finitely many solutions (x, y, z). His method is ineffective. Later, an effective result for solutions of (1.1) was given in [4] by A. O. Gel'fond. Let $N(a, b, c)$ denote the number of solutions (x, y, z) of (1.1). As a straightforward consequence of an upper bound for the number of solutions of binary S-unit equations due to F. Beukers and H. P. Schlickewei [2], we have $N(a, b, c) \leq 2^{36}$. In nearly two decades, many papers investigated the

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exact values of $N(a,b,c)$. The known results showed that (1.1) has only a few solutions for some special cases (see [8]).

Recently, Y.-Z. Hu and M.-H. Le [5], [6] successively proved that

(i) if a, b, c satisfy certain divisibility conditions and $\max\{a,b,c\}$ is large enough, then (1.1) has at most one solution (x,y,z) with $\min\{x,y,z\} > 1$;

(ii) if $\max\{a,b,c\} > 5 \times 10^{27}$, then $N(a,b,c) \le 3$.

R. Scott and R. Styer [12] proved that if $2 \nmid c$, then $N(a,b,c) \le 2$. The proofs of the first two results use the Gel'fond–Baker method with an elementary approach, and the proof of the last result uses some elementary algebraic number theory methods. In this paper, by analyzing the gap rule for solutions of (1.1) along the approach given in [6], we use another new idea dealing with the existence of three distinct solutions to prove a general result as follows:

Theorem 1.1. *If* $\max\{a,b,c\} \ge 10^{62}$, *then* $N(a,b,c) \le 2$.

Notice that, for any positive integer k with $k>1$, if $(a,b,c)=(2,2^k-1,2^k+1)$, then (1.1) has only two solutions $(x,y,z) = (1,1,1)$ and $(k+2,2,2)$, see [10]. This implies that there exist infinitely many triples (a,b,c) with $N(a,b,c) = 2$. Therefore, in general, $N(a,b,c) \le 2$ should be the best upper bound for $N(a,b,c)$, except for the case $(a,b,c) = (3,5,2)$, where the equation $3^x + 5^y = 2^z$ has only three solutions $(x,y,z) = (1,1,3),(1,3,7),(3,1,5)$, see [11].

2. Preliminaries

Lemma 2.1. *Let t be a real number. If $t \ge 10^{62}$, then $t > 6500^6(\log t)^{18}$, where* log *is used for natural logarithm.*

Let α be a fixed positive irrational number, and let $\alpha = [a_0, a_1, \dots]$ denote the simple continued fraction expansion of α. For any nonnegative integer i, let p_i/q_i be the i-th convergent of α. By [7, Chapter 10], we obtain the following two lemmas immediately.

Lemma 2.2. (i) *The convergents $p_i/q_i (i = 0,1,\dots)$ satisfy*

$$p_{-1} = 1, \quad p_0 = a_0, \quad p_{i+1} = a_{i+1}p_i + p_{i-1},$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_{i+1} = a_{i+1}q_i + q_{i-1}, \quad i \ge 0.$$

(ii)
$$p_0/q_0 < p_2/q_2 < \cdots < p_{2i}/q_{2i} < p_{2i+2}/q_{2i+2} < \cdots < \alpha < \cdots < p_{2i+3}/q_{2i+3} < p_{2i+1}/q_{2i+1} < \cdots < p_3/q_3 < p_1/q_1, \quad i \ge 0.$$

(iii) $1/q_i(q_{i+1}+q_i) < |\alpha - p_i/q_i| < 1/q_iq_{i+1}, \quad i \ge 0.$

Lemma 2.3. *Let p and q be positive integers. If $|\alpha - p/q| < 1/(2q^2)$, then $(p/d)/(q/d)$ is a convergent of α, where $d = \gcd(p,q)$.*

Let u, v, k be fixed positive integers such that $\min\{u,v,k\}>1$ and $\gcd(u,v)=1$.

Lemma 2.4 ([6, Lemma 4.3]). *The equation*

$$u^l + v^m = k, \quad l, m \in \mathbb{N} \tag{2.1}$$

has at most two solutions (l,m).

Lemma 2.5. *Let (l_1, m_1) and (l_2, m_2) be two solutions of (2.1). If $l_1 < l_2$, then $m_1 > m_2$,*

$$\max\{u^{l_2-l_1}, v^{m_1-m_2}\} > \sqrt{k} \tag{2.2}$$

and

$$u^{l_2-l_1} = v^{m_2}t + 1, \quad v^{m_1-m_2} = u^{l_1}t + 1 \tag{2.3}$$

for some $t \in \mathbb{N}$.

Proof. Since

$$u^{l_1} + v^{m_1} = k, \quad u^{l_2} + v^{m_2} = k, \tag{2.4}$$

we have

$$u^{l_1} \equiv -v^{m_1} \pmod{k}, \qquad u^{l_2} \equiv -v^{m_2} \pmod{k}. \tag{2.5}$$

If $l_1 < l_2$ and $m_1 \le m_2$, then from (2.5) we get

$$u^{l_2-l_1} \equiv v^{m_2-m_1} \pmod{k}. \tag{2.6}$$

Since $\gcd(u,v) = 1$ and $\min\{u,v\} > 1$, we have $u^{l_2-l_1} \neq v^{m_2-m_1}$. Hence, by (2.4) and (2.6), we get

$$k > \max\{u^{l_2}, v^{m_2}\} > \max\{u^{l_2-l_1}, v^{m_2-m_1}\} > k, \tag{2.7}$$

a contradiction. Therefore, if $l_1 < l_2$, then $m_1 > m_2$. Moreover, by (2.5), we get $u^{l_2-l_1}v^{m_1-m_2} \equiv 1 \pmod{k}$ and (2.2).

On the other hand, by (2.4), we have

$$u^{l_1}(u^{l_2-l_1} - 1) = v^{m_2}(v^{m_1-m_2} - 1). \tag{2.8}$$

Therefore, since $\gcd(u,v) = 1$, by (2.8), we get (2.3). The lemma is proved. □

Lemma 2.6 ([1])**.** *The equation*

$$u^l - v^m = k, \quad l, m \in \mathbb{N} \tag{2.9}$$

has at most two solutions (l, m).

Lemma 2.7. *Let* (l_1, m_1) *and* (l_2, m_2) *be two solutions of* (2.9). *If* $l_1 < l_2$, *then* $m_1 < m_2$,

$$u^{l_2-l_1} = v^{m_1}t + 1, \quad v^{m_2-m_1} = u^{l_1}t + 1 \tag{2.10}$$

for some $t \in \mathbb{N}$,

$$v^{m_2-m_1} > u^{l_2-l_1} > v^{m_1} \tag{2.11}$$

and

$$v^{m_2-m_1} > k. \tag{2.12}$$

Proof. Since

$$u^{l_1} - v^{m_1} = k, \quad u^{l_2} - v^{m_2} = k, \tag{2.13}$$

if $l_1 < l_2$, then from (2.13) we get $v^{m_2} + k = u^{l_2} > u^{l_1} = v^{m_1} + k$ and $m_1 < m_2$. Hence, by (2.13), we have

$$u^{l_1}(u^{l_2-l_1} - 1) = v^{m_1}(v^{m_2-m_1} - 1), \tag{2.14}$$

whence we obtain (2.10), since $\gcd(u, v) = 1$. Further, by (2.10) and (2.13), we have

$$v^{m_2-m_1} - u^{l_2-l_1} = (u^{l_1} - v^{m_1})t = kt. \tag{2.15}$$

Therefore, by (2.10) and (2.15), we obtain (2.11) and (2.12). The lemma is proved. □

Let $r > 1$, $s > 2$ be fixed coprime positive integers.

Lemma 2.8 ([3])**.** *There exist positive integers* n *such that*

$$r^n \equiv \delta \pmod{s}, \quad \delta \in \{1, -1\}. \tag{2.16}$$

Let n_1 *be the least value of* n *with* (2.16). *Then we have* $r^{n_1} \equiv \delta_1 \pmod{s}$ *and*

$$r^{n_1} = sf + \delta_1, \quad \delta_1 \in \{1, -1\} \tag{2.17}$$

for some $f \in \mathbb{N}$.

A positive integer n *satisfies* (2.16) *if and only if* $n_1 | n$. *Moreover, if* $n_1 | n$, *then* $r^{n_1} - \delta_1 | r^n - \delta$.

Obviously, for any fixed r and s, the corresponding n_1, δ_1 and f are uniquely determined.

Lemma 2.9. *Let t be a positive integer such that $t > 1$ and s is divisible by every prime divisor of t. Let n' be a positive integer satisfying*

$$r^{n'} \equiv \delta' \pmod{st}, \quad \delta' \in \{1, -1\}. \tag{2.18}$$

If s satisfies either $2 \nmid s$ or $4|s$, then $n_1|n'$ and

$$\frac{n'}{n_1} \equiv 0 \quad \left(\bmod \frac{t}{\gcd(t, f)}\right). \tag{2.19}$$

Proof. Notice that $\gcd(r, s) = 1$ and s is divisible by every prime divisor of t. We have $\gcd(r, st) = 1$. Hence, by Lemma 2.8, there exist positive integers n' satisfying (2.18). Further, since $r^{n'} \equiv \delta' \pmod{s}$ by (2.18), we get $n_1|n'$ and

$$n' = n_1 n_2 \tag{2.20}$$

for some $n_2 \in \mathbb{N}$.

Since either $2 \nmid s$ or $4|s$, we have

$$s > 2. \tag{2.21}$$

By (2.17), (2.18) and (2.20), we get

$$\begin{aligned} r^{n'} &\equiv (r^{n_1})^{n_2} \equiv (sf + \delta_1)^{n_2} \equiv \delta_1^{n_2} + n_2 \delta_1^{n_2 - 1} sf \\ &+ \sum_{i=2}^{n_2} \binom{n_2}{i} \delta_1^{n_2 - i} (sf)^i \equiv \delta' \pmod{st}. \end{aligned} \tag{2.22}$$

We see from (2.22) that $\delta_1^{n_2} \equiv \delta' \pmod{s}$. Hence, by (2.21), we get $\delta_1^{n_2} = \delta'$, and by (2.22),

$$f \left(n_2 + \sum_{i=2}^{n_2} \binom{n_2}{i} (\delta_1 sf)^{i-1} \right) \equiv 0 \pmod{t}. \tag{2.23}$$

Further, by (2.23), we obtain

$$n_2 + \sum_{i=2}^{n_2} \binom{n_2}{i} (\delta_1 sf)^{i-1} \equiv 0 \left(\bmod \frac{t}{\gcd(t, f)} \right). \tag{2.24}$$

Let p be a prime divisor of $t/\gcd(t,f)$. Since $p|t$ and s is divisible by every prime divisor of t, we see from (2.24) that $p|n_2$. Let

$$p^{\alpha}\big|\big|n_2, \quad p^{\beta}\big|\big|sf, \quad p^{\gamma}\big|\big|\frac{t}{\gcd(t,f)}, \quad p^{\pi_i}\big|\big|i, \qquad i \geq 2. \tag{2.25}$$

Then, α, β and γ are positive integers. If $p = 2$, then $4\big|s$ and $\beta \geq 2$. Thus, $\pi_i (i \geq 2)$ are nonnegative integers satisfying

$$\pi_i \leq \frac{\log i}{\log p}\begin{cases} cc \leq i-1 < 2(i-1) \leq \beta(i-1), & \text{if } \ p = 2, \\ < i-1, & \text{otherwise.} \end{cases} \tag{2.26}$$

Hence, by (2.25) and (2.26), we have

$$\binom{n_2}{i}(\delta_1 sf)^{i-1} \equiv n_2 \binom{n_2 - 1}{i-1}\frac{(\delta_1 sf)^{i-1}}{i} \equiv 0 \pmod{p^{\alpha+1}} \tag{2.27}$$

for $i \geq 2$.

By (2.25) and (2.27), we get

$$p^{\alpha}\big|\big|n_2 + \sum_{i=2}^{n_2}\binom{n_2}{i}(\delta_1 sf)^{i-1}. \tag{2.28}$$

Further, we see from (2.24), (2.25) and (2.28) that

$$\alpha \geq \gamma. \tag{2.29}$$

Therefore, letting p run through all prime divisors of $t/\gcd(t,f)$, by (2.20), (2.25) and (2.29), we obtain (2.19). The lemma is proved. □

3. Further lemmas on the solutions of (1.1)

Lemma 3.1 ([6, Theorem 2.1]). *All solutions (x,y,z) of (1.1) satisfy $\max\{x,y,z\} < 6500(\log\max\{a,b,c\})^3$.*

Lemma 3.2. *Let (x,y,z) be a solution of (1.1) with $a^{2x} < c^z$. If $b \geq 3$ and $c \geq 16$, then y/z is a convergent of $\log c/\log b$ with*

$$0 < \frac{\log c}{\log b} - \frac{y}{z} < \frac{2}{zc^{z/2}\log b}. \tag{3.1}$$

Proof. Since $\min\{b,c\} > 1$ and $\gcd(b,c)=1$, $\log c/\log b$ is a positive irrational number. Let $d=\gcd(y,z)$. Since $a^{2x} < c^z$, if $d \geq 2$, then from (1.1) we get

$$c^{z/2} > a^x = c^z - b^y = (c^{z/d} - b^{y/d})\sum_{i=0}^{d-1} c^{(d-1-i)z/d} b^{iy/d} > c^{(d-1)z/d} \geq c^{z/2},$$

a contradiction. So we have $d=1$ and $\gcd(y,z)=1$.

Since $a^x < c^{z/2}$, we have $a^x < b^y$. Hence, by (1.1), we get

$$z\log c = \log(b^y(1+\frac{a^x}{b^y})) < y\log b + \frac{a^x}{b^y}. \tag{3.2}$$

Since $a^x < b^y$, by (1.1), we have $c^z < 2b^y$ and

$$\frac{a^x}{b^y} < \frac{2a^x}{c^z} < \frac{2c^{z/2}}{c^z} = \frac{2}{c^{z/2}}. \tag{3.3}$$

Hence, by (3.2) and (3.3), we get

$$0 < z\log c - y\log b < \frac{2}{c^{z/2}}, \tag{3.4}$$

whence we obtain (3.1). On the other hand, since $b \geq 3$ and $c \geq 16$, we have $2/(zc^{z/2}\log b) < 1/(2z^2)$. This implies that $0 < \log c/\log b - y/z < 1/(2z^2)$ by (3.4). Therefore, applying Lemma 2.3, y/z is a convergent of $\log c/\log b$. Thus, the lemma is proved. □

Using the same method as in the proof of Lemma 3.2, we can obtain the following lemma immediately.

Lemma 3.3. *Let (x,y,z) be a solution of (1.1) with $b^{2y} < c^z$. If $a \geq 10^{62}$, then x/z is a convergent of $\log c/\log a$ with*

$$0 < \frac{\log c}{\log a} - \frac{x}{z} < \frac{2}{zc^{z/2}\log a}.$$

Lemma 3.4. *Let (x,y,z) and (x',y',z') be two solutions of (1.1) such that $x > x'$ and $z < z'$. If $c = \max\{a,b,c\} \geq 10^{62}$, then $(y'/d)/(z'/d)$ is a convergent of $\log c/\log b$ with*

$$0 < \frac{\log c}{\log b} - \frac{y'/d}{z'/d} < \frac{2}{z'ac\log b},$$

where $d=\gcd(y',z')$.

Proof. Since $x > x'$ and $z < z'$, if $a^{x'} > b^{y'}$, then we get $2a^{x'} > c^{z'} > c^z > a^x \geq a^{x'+1} \geq 2a^{x'}$, a contradiction. So we have $a^{x'} < b^{y'}$ and

$$z' \log c = \log(b^{y'}(1 + \frac{a^{x'}}{b^{y'}})) < y' \log b + \frac{a^{x'}}{b^{y'}}. \tag{3.5}$$

Since $2b^{y'} > c^{z'}$, we get

$$\frac{a^{x'}}{b^{y'}} < \frac{2a^{x'}}{c^{z'}} = \frac{2}{a^{x-x'}c^{z'-z}} \cdot \frac{a^x}{c^z} < \frac{2}{ac}. \tag{3.6}$$

Hence, by (3.5) and (3.6), we obtain

$$0 < \frac{\log c}{\log b} - \frac{y'}{z'} < \frac{2}{z' ac \log b}. \tag{3.7}$$

If $|\log c/\log b - y'/z'| \geq 1/(2z'^2)$, then from (3.7) we get

$$z' > \frac{1}{4} ac \log b. \tag{3.8}$$

Since $c = \max\{a, b, c\}$, by Lemma 3.1, we have $z' < 6500(\log c)^3$. Since $a \log b \geq \min\{2\log 3, 3\log 2\} > 2$, by (3.8), we get

$$13000(\log c)^3 > c. \tag{3.9}$$

But, since $c \geq 10^{62}$, by Lemma 2.1, (3.9) is false. Therefore, we have

$$\left|\frac{\log c}{\log b} - \frac{y'}{z'}\right| < \frac{1}{2z'^2}. \tag{3.10}$$

Applying Lemma 2.3 to (3.10), we find that $(y'/d)/(z'/d)$ is a convergent of $\log c/\log b$. Thus, the lemma is proved. □

Lemma 3.5. *Let (x, y, z) and (x', y', z') be two solutions of (1.1) such that $y > y'$ and $z \leq z'$. If $a = \max\{a, b, c\} \geq 10^{62}$, then $(x'/d)/(z'/d)$ is a convergent of $\log c/\log a$ with*

$$0 < \frac{\log c}{\log a} - \frac{x'/d}{z'/d} < \frac{2}{z' a \log a}, \tag{3.11}$$

where $d = \gcd(x', z')$.

Proof. The proof of this lemma is similar to Lemma 3.4 . Since $y > y'$ and $z \le z'$, we see from

$$a^x + b^y = c^z, \quad a^{x'} + b^{y'} = c^{z'} \tag{3.12}$$

that

$$x < x', \tag{3.13}$$

$a^{x'} > b^{y'}$ and $2a^{x'} > c^{z'}$. Hence, by the second equality of (3.12), we have

$$z' \log c = \log\left(a^{x'}\left(1 + \frac{b^{y'}}{a^{x'}}\right)\right) < x' \log a + \frac{b^{y'}}{a^{x'}} \tag{3.14}$$

and

$$\frac{b^{y'}}{a^{x'}} < \frac{2b^{y'}}{c^{z'}} = \frac{2}{b^{y-y'}c^{z'-z}} \cdot \frac{b^y}{c^z} < \frac{2}{b^{y-y'}c^{z'-z}}. \tag{3.15}$$

By (3.12), we have $b^y \equiv c^z \pmod{a^x}$ and $b^{y'} \equiv c^{z'} \pmod{a^{x'}}$, whence, by (3.13), we get

$$b^{y-y'}c^{z'-z} \equiv 1 \pmod{a^x}. \tag{3.16}$$

Further, since $y > y'$, we have $b^{y-y'}c^{z'-z} > 1$. Hence, by (3.16), we get

$$b^{y-y'}c^{z'-z} > a^x. \tag{3.17}$$

Therefore, by (3.14), (3.15) and (3.17), we obtain $b^{y'}/a^{x'} < 2/a^x$ and

$$0 < \frac{\log c}{\log a} - \frac{x'}{z'} < \frac{2}{z' a^x \log a}. \tag{3.18}$$

Since $a^x \ge a = \max\{a, b, c\} \ge 10^{62}$, by Lemma 3.1, we can deduce that

$$\frac{2}{z' a^x \log a} < \frac{1}{2z'^2}. \tag{3.19}$$

Thus, by Lemma 2.3, we find from (3.18) and (3.19) that $(x'/d)/(z'/d)$ is a convergent of $\log c/\log a$ with (3.11). The lemma is proved. □

4. The equation $A^X + \lambda B^Y = C^Z$

For any fixed triple (a, b, c), put

$$P(a, b, c) = \{(a, b, c, 1), (c, a, b, -1), (c, b, a, -1)\}. \tag{4.1}$$

Let (A,B,C,λ) be an element of $P(a,b,c)$. Obviously, (1.1) has a solution (x,y,z), which is equivalent to saying that the equation

$$A^X+\lambda B^Y=C^Z,\quad X,Y,Z\in\mathbb{N} \tag{4.2}$$

has the solution

$$(X,Y,Z)=\begin{cases}(x,y,z), & \text{if } (A,B,C,\lambda)=(a,b,c,1),\\ (z,x,y), & \text{if } (A,B,C,\lambda)=(c,a,b,-1),\\ (z,y,x), & \text{if } (A,B,C,\lambda)=(c,b,a,-1).\end{cases}$$

This implies that, for any $(A,B,C,\lambda)\in P(a,b,c)$, the numbers of solutions of (1.1) and (4.2) are equal. Moreover, by Lemma 3.1, we have

Lemma 4.1. *All the solutions (X,Y,Z) of* (4.2) *satisfy* $\max\{X,Y,Z\}<6500(\log\max\{a,b,c\})^3$.

Here and below, we always assume that (1.1) has solutions (x,y,z). Then, for any $(A,B,C,\lambda)\in P(a,b,c)$, (4.2) has solutions (X,Y,Z).

For a fixed element $(A,B,C,\lambda)\in P(a,b,c)$, (4.2) is sure to have a solution (X_1,Y_1,Z_1) such that $Z_1\le Z$, where Z runs through all solutions (X,Y,Z) of (4.2) for this (A,B,C,λ). Since $\gcd(A,C)=1$ and $\min\{A,C\}>1$, by Lemma 2.8, there exist positive integers n such that

$$A^n\equiv\delta\pmod{C^{Z_1}},\quad \delta\in\{1,-1\}. \tag{4.3}$$

Let n_1 be the least value of n with (4.3), and let

$$A^{n_1}\equiv\delta_1\pmod{C^{Z_1}},\quad \delta_1\in\{1,-1\}. \tag{4.4}$$

Then we have

$$A^{n_1}=C^{Z_1}f+\delta_1,\quad f\in\mathbb{N}. \tag{4.5}$$

Obviously, for any fixed triple $(A,B,C,\lambda)\in P(a,b,c)$, the parameters Z_1,n_1,δ_1 and f are uniquely determined.

Lemma 4.2. (4.2) *has at most two solutions (X,Y,Z) with the same value Z.*

Proof. By Lemmas 2.4 and 2.6, we obtain the lemma immediately. □

Lemma 4.3 ([5, Lemma 3.3]). *Let (X,Y,Z) and (X',Y',Z') be two distinct solutions of* (4.2) *with $Z\le Z'$. Then we have $XY'-X'Y\ne 0$ and $A^{|XY'-X'Y|}\equiv(-\lambda)^{Y+Y'}\pmod{C^Z}$.*

Lemma 4.4. *Let* (X_1, Y_1, Z_1) *and* (X_2, Y_2, Z_2) *be two solutions of* (4.2) *such that* $Z_1 < Z_2$ *and* $Z_1 \le Z$, *where* Z *runs through all solutions* (X, Y, Z) *of* (4.2) *for this* (A, B, C, λ). *If* C *satisfies*

$$2 \nmid C \quad \text{or} \quad 4 | C^{Z_1}, \tag{4.6}$$

then

$$\gcd(C^{Z_2 - Z_1}, f) | Y_2, \tag{4.7}$$

where f *is defined as in* (4.5).

Proof. The proof of this lemma is similar to that of Lemma 2.9. Since $A^{X_1} + \lambda B^{Y_1} = C^{Z_1}$, $A^{X_2} + \lambda B^{Y_2} = C^{Z_2}$ and $Z_1 < Z_2$, we have

$$\begin{aligned} A^{X_1 Y_2} &= (-\lambda)^{Y_2} B^{Y_1 Y_2} + C^{Z_1} \sum_{i=1}^{Y_2} \binom{Y_2}{i} (-\lambda B^{Y_1})^{Y_2 - i} C^{Z_1 (i-1)}, \\ A^{X_2 Y_1} &\equiv (-\lambda)^{Y_1} B^{Y_1 Y_2} \pmod{C^{Z_2}}. \end{aligned}$$

Eliminating $B^{Y_1 Y_2}$ from the above two equations, we get

$$\begin{aligned} &\lambda' A^{\min\{X_1 Y_2, X_2 Y_1\}} \left(A^{|X_1 Y_2 - X_2 Y_1|} - (-\lambda)^{Y_1 + Y_2} \right) \\ &\equiv Y_2 B^{Y_1 (Y_2 - 1)} C^{Z_1} + \sum_{i=2}^{Y_2} (-\lambda)^{i+1} \binom{Y_2}{i} B^{Y_1 (Y_2 - i)} C^{Z_1 i} \pmod{C^{Z_2}}, \end{aligned} \tag{4.8}$$

where $\lambda' \in \{1, -1\}$.

By Lemma 4.3, $|X_1 Y_2 - X_2 Y_1|$ is a positive integer. Since $Z_1 < Z_2$, using Lemma 4.3 again, we have

$$A^{|X_1 Y_2 - X_2 Y_1|} \equiv (-\lambda)^{Y_1 + Y_2} \pmod{C^{Z_1}}. \tag{4.9}$$

Therefore, by Lemma 2.8, we get from (4.4), (4.5) and (4.9) that

$$A^{n_1} - \delta_1 \big| A^{|X_1 Y_2 - X_2 Y_1|} - (-\lambda)^{Y_1 + Y_2}$$

and

$$A^{|X_1 Y_2 - X_2 Y_1|} - (-\lambda)^{Y_1 + Y_2} = C^{Z_1} f g \tag{4.10}$$

for some $g \in \mathbb{N}$.

Substituting (4.10) into (4.8), we have

$$\begin{aligned} &\lambda' A^{\min\{X_1 Y_2, X_2 Y_1\}} f g \\ &\equiv Y_2 B^{Y_1 (Y_2 - 1)} + \sum_{i=2}^{Y_2} (-\lambda)^{i+1} \binom{Y_2}{i} B^{Y_1 (Y_2 - i)} C^{Z_1 (i-1)} \pmod{C^{Z_2 - Z_1}}. \end{aligned} \tag{4.11}$$

Let p be a prime divisor of $\gcd(C^{Z_2-Z_1}, f)$. Since $p\big|C$ and $\gcd(B,C)=1$, we see from (4.11) that $p\big|Y_2$. Let

$$p^{\alpha}\big|\big|Y_2, \quad p^{\beta}\big|\big|C^{Z_1}, \quad p^{\gamma}\big|\big|\gcd(C^{Z_2-Z_1}, f), \quad p^{\pi_i}\big|\big|i, \qquad i \geq 2. \tag{4.12}$$

Then, by (4.6), α, β and γ are positive integers with $\beta \geq 2$ if $p=2$ and $\pi_i (i \geq 2)$ are nonnegative integers satisfying (2.26). By (2.26) and (4.12), we have

$$\begin{aligned}\binom{Y_2}{i} B^{Y_1(Y_2-i)}C^{Z_1(i-1)} &\equiv Y_2 \binom{Y_2-1}{i-1} \frac{B^{Y_1(Y_2-i)}C^{Z_1(i-1)}}{i} \\ &\equiv 0 \pmod{p^{\alpha+1}}, \quad i \geq 2. \end{aligned} \tag{4.13}$$

Hence, by (4.12) and (4.13), we get

$$p^{\alpha}\big|\big|Y_2B^{Y_1(Y_2-1)} + \sum_{i=2}^{Y_2}(-\lambda)^{i+1}\binom{Y_2}{i}B^{Y_1(Y_2-i)}C^{Z_1(i-1)}. \tag{4.14}$$

Therefore, since $\gcd(C^{Z_2-Z_1}, f)\big|f$ and $\gcd(C^{Z_2-Z_1}, f)\big|C^{Z_2-Z_1}$, we find from (4.11), (4.12) and (4.14) that α and γ satisfy (2.29). Thus, letting p run through all prime divisors of $\gcd(C^{Z_2-Z_1}, f)$, by (2.29) and (4.12), we obtain (4.7). The lemma is proved. □

Lemma 4.5 ([6, Lemma 4.7]). *Let (X_j, Y_j, Z_j) $(j=1,2,3)$ be three distinct solutions of* (4.2) *with $Z_1 < Z_2 \leq Z_3$. If $C = \max\{a,b,c\}$, then $\max\{a,b,c\} < 5 \times 10^{27}$.*

Lemma 4.6. *Let (X_j, Y_j, Z_j) $(j=1,2,3)$ be three distinct solutions of* (4.2) *with $Z_1 < Z_2 \leq Z_3$. If $C^{Z_2-Z_1} > (\max\{a,b,c\})^{1/2}$ and C satisfies* (4.6), *then $\max\{a,b,c\} < 10^{62}$.*

Proof. Since $Z_2 \leq Z_3$, by Lemma 4.3, we have $X_2Y_3 - X_3Y_2 \neq 0$ and

$$A^{|X_2Y_3 - X_3Y_2|} \equiv (-\lambda)^{Y_2+Y_3} \pmod{C^{Z_2}}. \tag{4.15}$$

Further, since $Z_1 < Z_2$ and C satisfies (4.6), by using Lemma 2.9 with $r = A$, $t = C^{Z_2-Z_1}$, $s = C^{Z_1}$ and $n' = |X_2Y_3 - X_3Y_2|$, we get from (4.4), (4.5) and (4.15) that

$$|X_2Y_3 - X_3Y_2| \equiv 0 \left(\bmod \frac{C^{Z_2-Z_1}}{\gcd(C^{Z_2-Z_1}, f)}\right), \tag{4.16}$$

where f is defined as in (4.5). Recall that $X_2Y_3 - X_3Y_2 \neq 0$. By (4.16), we have

$$|X_2Y_3 - X_3Y_2| \gcd(C^{Z_2-Z_1}, f) \geq C^{Z_2-Z_1}. \tag{4.17}$$

Furthermore, by Lemma 4.4, we have $\gcd(C^{Z_2-Z_1}, f) \leq Y_2$. Hence, we get from (4.17) that

$$Y_2 \,|X_2Y_3 - X_3Y_2| \geq C^{Z_2-Z_1}. \tag{4.18}$$

By Lemma 4.1, we have

$$\begin{aligned} Y_2 \,|X_2Y_3 - X_3Y_2| &< Y_2 \max\{X_2Y_3, X_3Y_2\} \leq (\max\{X_2, Y_2, X_3, Y_3\})^3 \\ &< 6500^3 \,(\log \max\{a, b, c\})^9 \,. \end{aligned} \tag{4.19}$$

Therefore, if $C^{Z_2-Z_1} > (\max\{a, b, c\})^{1/2}$, then from (4.18) and (4.19) we get

$$6500^6 \,(\log \max\{a, b, c\})^{18} > \max\{a, b, c\}. \tag{4.20}$$

Thus, applying Lemma 2.1 to (4.20), we obtain $\max\{a, b, c\} < 10^{62}$. The lemma is proved. □

5. Proof of Theorem 1.1 for $c = \max\{a, b, c\}$

By [12], Theorem 1.1 holds for $2 \nmid c$. Therefore, we just have to consider the case that

$$2 \big| c. \tag{5.1}$$

Since $\gcd(ab, c) = 1$, by (5.1), we have

$$2 \nmid a, \quad 2 \nmid b. \tag{5.2}$$

In this section, we will prove the theorem for the case that

$$c = \max\{a, b, c\} \geq 10^{62}. \tag{5.3}$$

We now assume that (1.1) has three distinct solutions $(x_j, y_j, z_j)\,(j = 1, 2, 3)$ with $z_1 \leq z_2 \leq z_3$. Then, (4.2) has three solutions $(X_j, Y_j, Z_j) = (x_j, y_j, z_j)\,(j = 1, 2, 3)$ for $(A, B, C, \lambda) = (a, b, c, 1)$ with $Z_1 \leq Z_2 \leq Z_3$. By Lemma 4.2, we can remove the case $z_1 = z_2 = z_3$. Since $C = c = \max\{a, b, c\} \geq 10^{62}$, by Lemma 4.5, we can remove the case $z_1 < z_2 \leq z_3$. So we have

$$z_1 = z_2 < z_3. \tag{5.4}$$

Since $z_1 = z_2$ and

$$a^{x_1} + b^{y_1} = a^{x_2} + b^{y_2} = c^{z_1}, \tag{5.5}$$

(2.1) has two solutions $(l,m)=(x_j,y_j)\,(j=1,2)$ for $(u,v,k)=(a,b,c^{z_1})$. Since $(x_1,y_1)\neq(x_2,y_2)$, we may therefore assume that

$$x_1 < x_2. \tag{5.6}$$

Then, by Lemma 2.5, we have

$$y_1 > y_2, \tag{5.7}$$

$$a^{x_2-x_1}=b^{y_2}t_1+1, \quad b^{y_1-y_2}=a^{x_1}t_1+1 \tag{5.8}$$

for some $t_1\in\mathbb{N}$, and

$$\max\{a^{x_2-x_1},b^{y_1-y_2}\}>c^{z_1/2}. \tag{5.9}$$

By the symmetry of a and b in (5.5), we may assume that

$$a^{x_2-x_1}>b^{y_1-y_2}. \tag{5.10}$$

Hence, by (5.3), (5.9) and (5.10), we have

$$a^{x_2-x_1}>c^{z_1/2}\geq\sqrt{c}=\left(\max\{a,b,c\}\right)^{1/2}. \tag{5.11}$$

By (5.6), if $x_3\geq x_2$, then (4.2) has three solutions $(X_j,Y_j,Z_j)=(z_j,y_j,x_j)$ $(j=1,2,3)$ for $(A,B,C,\lambda)=(c,b,a,-1)$ with $Z_1<Z_2\leq Z_3$. Since $C^{Z_2-Z_1}=a^{x_2-x_1}>\left(\max\{a,b,c\}\right)^{1/2}$ by (5.11), using Lemma 4.6, we get from (5.2) that $\max\{a,b,c\}<10^{62}$, which contradicts (5.3). Therefore, we have

$$x_3 < x_2. \tag{5.12}$$

By (5.8) and (5.10), we get $a^{x_2-x_1}>b^{y_1-y_2}=a^{x_1}t_1+1>a^{x_1}$, and by (5.5), $c^{z_1}>a^{x_2}>a^{2x_1}$. This implies that $(x,y,z)=(x_1,y_1,z_1)$ is a solution of (1.1) with $a^{2x}<c^z$. Notice that $b\geq 3$ and $c\geq 16$ by (5.2) and (5.3). Using Lemma 3.2, y_1/z_1 is a convergent of $\log c/\log b$ with

$$0<\frac{\log c}{\log b}-\frac{y_1}{z_1}<\frac{2}{z_1c^{z_1/2}\log b}. \tag{5.13}$$

On the other hand, by (5.4) and (5.12), (x_2,y_2,z_2) and (x_3,y_3,z_3) are two solutions of (1.1) such that $x_2>x_3$ and $z_2<z_3$. Since $c=\max\{a,b,c\}\geq 10^{62}$, by Lemma 3.4, $(y_3/d)/(z_3/d)$ is also a convergent of $\log c/\log b$ with

$$0<\frac{\log c}{\log b}-\frac{y_3/d}{z_3/d}<\frac{2}{z_3ac\log b}, \tag{5.14}$$

where $d=\gcd(y_3,z_3)$.

By (5.4), $(X_1, Y_1, Z_1) = (z_1, y_1, x_1)$ and $(X_3, Y_3, Z_3) = (z_3, y_3, x_3)$ are two distinct solutions of (4.2) for $(A, B, C, \lambda) = (c, b, a, -1)$. Hence, by Lemma 4.3, we have $z_1 y_3 - z_3 y_1 \neq 0$. This implies that y_1/z_1 and $(y_3/d)/(z_3/d)$ are two distinct convergents of $\log c/\log b$. Therefore, by (ii) of Lemma 2.2, we see from (5.13) and (5.14) that

$$\frac{y_1}{z_1} = \frac{p_{2s}}{q_{2s}}, \quad \frac{y_3/d}{z_3/d} = \frac{p_{2t}}{q_{2t}} \tag{5.15}$$

for some non-negative integers s, t with $s \neq t$.

If $s < t$, by (i) and (iii) of Lemma 2.2, then from (5.13) and (5.15) we get

$$\begin{aligned} z_3 \geq \frac{z_3}{d} = q_{2t} \geq q_{2s+2} = a_{2s+2} q_{2s+1} + q_{2s} \geq q_{2s+1} + q_{2s} > \left(q_{2s} \left| \frac{\log c}{\log b} - \frac{p_{2s}}{q_{2s}} \right| \right)^{-1} \\ = \left(z_1 \left(\frac{\log c}{\log b} - \frac{y_1}{z_1} \right) \right)^{-1} > \frac{1}{2} c^{z_1/2} \log b > \frac{\sqrt{c}}{2}. \end{aligned} \tag{5.16}$$

Since $c = \max\{a, b, c\}$, by Lemma 3.1, we have $z_3 < 6500(\log c)^3$. Therefore, by (5.16), we get

$$13000^2 (\log c)^6 > c. \tag{5.17}$$

However, since $c \geq 10^{62}$, by Lemma 2.1, (5.17) is false.

Similarly, if $s > t$, then from (5.14) and (5.15) we get

$$\begin{aligned} z_1 = q_{2s} \geq q_{2t+2} \geq q_{2t+1} + q_{2t} > \left(q_{2t} \left| \frac{\log c}{\log b} - \frac{p_{2t}}{q_{2t}} \right| \right)^{-1} \\ = \left(\frac{z_3}{d} \left(\frac{\log c}{\log b} - \frac{y_3/d}{z_3/d} \right) \right)^{-1} > \frac{1}{2} ac \log b > c. \end{aligned} \tag{5.18}$$

Further, by Lemma 3.1, we have $z_1 < 6500(\log c)^3$. Therefore, by (5.18), we get

$$6500 (\log c)^3 > c. \tag{5.19}$$

By Lemma 2.1, this inequality contradicts the assumption that $c \geq 10^{62}$. Thus, we have $N(a, b, c) \leq 2$ for case (5.3).

6. Proof of Theorem 1.1 for $c \neq \max\{a, b, c\}$

In this section, we will prove Theorem 1.1 for the case that $c \neq \max\{a, b, c\}$. Then, by the symmetry of a and b in (1.1), we may assume that

$$a = \max\{a, b, c\} \geq 10^{62}. \tag{6.1}$$

For any solution (x, y, z) of (1.1), since $a^z > c^z = a^x + b^y > a^x \geq a$ by (6.1), we have

$$z \geq 2. \tag{6.2}$$

Hence, by (5.1) and (6.2), we get

$$4 \big| c^z. \tag{6.3}$$

We now assume that (1.1) has three solutions $(x_j, y_j, z_j)\,(j = 1, 2, 3)$ with $x_1 \leq x_2 \leq x_3$. Then, (4.2) has three solutions $(X_j, Y_j, Z_j) = (z_j, y_j, x_j)\,(j = 1, 2, 3)$ for $(A, B, C, \lambda) = (c, b, a, -1)$ with $Z_1 \leq Z_2 \leq Z_3$. By Lemma 4.2, we can remove the case $x_1 = x_2 = x_3$. Since $C = a = \max\{a, b, c\} \geq 10^{62}$, by Lemma 4.5, we can remove the case $x_1 < x_2 \leq x_3$. So we have

$$x_1 = x_2 < x_3. \tag{6.4}$$

Since $x_1 = x_2$, we have

$$c^{z_1} - b^{y_1} = c^{z_2} - b^{y_2} = a^{x_1}. \tag{6.5}$$

This implies that (2.9) has two solutions $(l, m) = (z_j, y_j)\,(j = 1, 2)$ for $(u, v, k) = (c, b, a^{x_1})$. Since $(z_1, y_1) \neq (z_2, y_2)$, we may assume that

$$z_1 < z_2. \tag{6.6}$$

Then, by Lemma 2.7, we get from (6.6) that

$$y_1 < y_2 \tag{6.7}$$

and

$$b^{y_2 - y_1} = c^{z_1} t_2 + 1, \quad c^{z_2 - z_1} = b^{y_1} t_2 + 1 \tag{6.8}$$

for some $t_2 \in \mathbb{N}$.
By the first equality of (6.8), we have

$$b^{y_2 - y_1} > c^{z_1} > a^{x_1}. \tag{6.9}$$

If $y_3 \geq y_2$, by (6.7), then (4.2) has three solutions $(X_j, Y_j, Z_j) = (z_j, x_j, y_j)$ $(j = 1, 2, 3)$ for $(A, B, C, \lambda) = (c, a, b, -1)$ with $Z_1 < Z_2 \leq Z_3$. However, since $C = b$ is odd and $C^{Z_2 - Z_1} = b^{y_2 - y_1} > a^{x_1} \geq a = \max\{a, b, c\} \geq 10^{62}$, by (5.2), (6.1) and (6.9), we have a contradiction to Lemma 4.6. Therefore, we obtain

$$y_3 < y_2. \tag{6.10}$$

Suppose that $z_3 \ge z_2$. By (6.6), then (4.2) has three solutions $(X_j, Y_j, Z_j) = (x_j, y_j, z_j)\,(j = 1, 2, 3)$ for $(A, B, C, \lambda) = (a, b, c, 1)$ with $Z_1 < Z_2 \le Z_3$. Since $a = \max\{a, b, c\} \ge 10^{62}$, by Lemma 4.6, we obtain

$$C^{Z_2 - Z_1} = c^{z_2 - z_1} < (\max\{a, b, c\})^{1/2} = \sqrt{a}. \tag{6.11}$$

Hence, by the second equality of (6.8) and (6.11), we have

$$b^{y_1} < b^{y_1} t_2 + 1 = c^{z_2 - z_1} < \sqrt{a} \le a^{x_1/2} < c^{z_1/2}. \tag{6.12}$$

This implies that $(x, y, z) = (x_1, y_1, z_1)$ is a solution of (1.1) with $b^{2y} < c^z$. Therefore, by Lemma 3.3, x_1/z_1 is a convergent of $\log c/\log a$ with

$$0 < \frac{\log c}{\log a} - \frac{x_1}{z_1} < \frac{2}{z_1 c^{z_1/2} \log a}. \tag{6.13}$$

On the other hand, by (6.10), (1.1) has two solutions (x_2, y_2, z_2) and (x_3, y_3, z_3) such that $y_2 > y_3$ and $z_2 \le z_3$. Since $a = \max\{a, b, c\} \ge 10^{62}$, by Lemma 3.5, $(x_3/d)/(z_3/d)$ is also a convergent of $\log c/\log a$ with

$$0 < \frac{\log c}{\log a} - \frac{x_3/d}{z_3/d} < \frac{2}{z_3 a \log a}, \tag{6.14}$$

where $d = \gcd(x_3, z_3)$. Further, by Lemma 4.3, x_1/z_1 and $(x_3/d)/(z_3/d)$ are two distinct convergents of $\log c/\log a$. Hence, by (ii) of Lemma 2.2, we see from (6.13) and (6.14) that

$$\frac{x_1}{z_1} = \frac{p_{2s}}{q_{2s}}, \quad \frac{x_3/d}{z_3/d} = \frac{p_{2t}}{q_{2t}} \tag{6.15}$$

for some non-negative integers s, t with $s \ne t$.

Since $a = \max\{a, b, c\}$, by Lemmas 2.2 and 3.1, we get from (6.13), (6.14) and (6.15) that

$$6500\,(\log a)^3 > \left\{ \begin{array}{ll} z_3 \ge \dfrac{z_3}{d} = q_{2t} \ge q_{2s+2} \ge q_{2s+1} + q_{2s} > \left(q_{2s} \left| \dfrac{\log c}{\log a} - \dfrac{p_{2s}}{q_{2s}} \right| \right)^{-1} & \\ \quad = \left(z_1 \left(\dfrac{\log c}{\log a} - \dfrac{x_1}{z_1} \right) \right)^{-1} > \dfrac{1}{2} c^{z_1/2} \log a, & \text{if } \ s < t, \\ & \\ z_1 = q_{2s} \ge q_{2t+2} \ge q_{2t+1} + q_{2t} > \left(q_{2t} \left| \dfrac{\log c}{\log a} - \dfrac{p_{2t}}{q_{2t}} \right| \right)^{-1} & \\ \quad = \left(\dfrac{z_3}{d} \left(\dfrac{\log c}{\log a} - \dfrac{x_3/d}{z_3/d} \right) \right)^{-1} > \dfrac{1}{2} a \log a, & \text{if } \ s > t, \end{array} \right.$$

$$> \frac{1}{2} \sqrt{a} \log a. \tag{6.16}$$

But, since $a \geq 10^{62}$, by Lemma 2.1, (6.16) is false. Therefore, we obtain

$$z_3 < z_2. \tag{6.17}$$

Finally, we can write the known results (6.4), (6.6), (6.7), (6.10) and (6.17) as

$$x_1 = x_2 < x_3, \quad , \max\{y_1, y_3\} < y_2, \quad \max\{z_1, z_3\} < z_2,$$

and complete the proof in the following four cases.

Case (i): $y_3 \leq y_1 < y_2$ and $z_3 \leq z_1 < z_2$.

In this case, since $x_3 > x_1 = x_2$ and

$$a^{x_3} = c^{z_3} - b^{y_3} = a^{x_3 - x_1} c^{z_1} - a^{x_3 - x_1} b^{y_1},$$

we get

$$c^{z_3}(a^{x_3 - x_1} c^{z_1 - z_3} - 1) = b^{y_3}(a^{x_3 - x_1} b^{y_1 - y_3} - 1). \tag{6.18}$$

Since $\gcd(b, c) = 1$, by (6.18), we have

$$a^{x_3 - x_1} b^{y_1 - y_3} = c^{z_3} t_3 + 1 \tag{6.19}$$

for some $t_3 \in \mathbb{N}$.

Hence, we see from (6.19) that $a^{x_3 - x_1} b^{y_1 - y_3} > c^{z_3} = a^{x_3} + b^{y_3} > a^{x_3}$ and

$$b^{y_1 - y_3} > a^{x_1}.$$

This implies that $y_3 < y_1$. Therefore, (4.2) has three solutions $(X_j, Y_j, Z_j)\,(j = 1, 2, 3)$ for $(A, B, C, \lambda) = (c, a, b, -1)$ such that $(X_1, Y_1, Z_1) = (z_3, x_3, y_3)$, $(X_2, Y_2, Z_2) = (z_1, x_1, y_1)$, $(X_3, Y_3, Z_3) = (z_2, x_1, y_2)$ and $Z_1 < Z_2 < Z_3$. However, since $2 \nmid b = C$ and $C^{Z_2 - Z_1} = b^{y_1 - y_3} > a^{x_1} \geq a = \max\{a, b, c\} \geq 10^{62}$, we have a contradiction to Lemma 4.6.

Case (ii): $y_3 \leq y_1 < y_2$ and $z_1 < z_3 < z_2$.

Since

$$c^{z_3} = a^{x_3} + b^{y_3} = c^{z_3 - z_1} a^{x_1} + c^{z_3 - z_1} b^{y_1}, \tag{6.20}$$

we have

$$b^{y_3}(c^{z_3 - z_1} b^{y_1 - y_3} - 1) = a^{x_1}(a^{x_3 - x_1} - c^{z_3 - z_1}),$$

whence we get

$$c^{z_3 - z_1} b^{y_1 - y_3} = a^{x_1} t_4 + 1 \tag{6.21}$$

for some $t_4 \in \mathbb{N}$.

By (6.21), we have $c^{z_3-z_1}b^{y_1-y_3} > a^{x_1}$. This implies that

$$\max\{c^{z_3-z_1}, b^{y_1-y_3}\} > a^{x_1/2} \geq \sqrt{a}. \tag{6.22}$$

If $c^{z_3-z_1} > b^{y_1-y_3}$, by (6.22), then $c^{z_3-z_1} > \sqrt{a}$. Notice that (4.2) has three solutions $(X_j, Y_j, Z_j)\,(j = 1, 2, 3)$ for $(A, B, C, \lambda) = (a, b, c, 1)$ such that $(X_1, Y_1, Z_1) = (x_1, y_1, z_1)$, $(X_2, Y_2, Z_2) = (x_3, y_3, z_3)$, $(X_3, Y_3, Z_3) = (x_2, y_2, z_2)$ and $Z_1 < Z_2 < Z_3$. However, because $4 \mid C^{Z_1}$ by (6.3), and $C^{Z_2-Z_1} = c^{z_3-z_1} > \sqrt{a} = (\max\{a, b, c\})^{1/2}$ with $\max\{a, b, c\} \geq 10^{62}$, we have a contradiction to Lemma 4.6.

Similarly, if $c^{z_3-z_1} < b^{y_1-y_3}$, then $y_3 < y_1$ and $b^{y_1-y_3} > \sqrt{a}$. In this case, (4.2) has three solutions $(X_j, Y_j, Z_j)\,(j = 1, 2, 3)$ for $(A, B, C, \lambda) = (c, a, b, -1)$ such that $(X_1, Y_1, Z_1) = (z_3, x_3, y_3)$, $(X_2, Y_2, Z_2) = (z_1, x_1, y_1)$, $(X_3, Y_3, Z_3) = (z_2, x_2, y_2)$ and $Z_1 < Z_2 < Z_3$. However, since $2 \nmid b = C$, $C^{Z_2-Z_1} = b^{y_1-y_3} > \sqrt{a} = (\max\{a, b, c\})^{1/2}$ and $\max\{a, b, c\} \geq 10^{62}$, we have a contradiction to Lemma 4.6.

Case (iii): $y_1 < y_3 < y_2$ and $z_3 \leq z_1 < z_2$.

Since $x_3 > x_1$, we have

$$a^{x_1} + b^{y_1} = c^{z_1} \geq c^{z_3} = a^{x_3} + b^{y_3} > a^{x_1} + b^{y_1},$$

a contradiction.

Case (iv): $y_1 < y_3 < y_2$ and $z_1 < z_3 < z_2$.

By (6.20), we have

$$a^{x_1}(c^{z_3-z_1} - a^{x_3-x_1}) = b^{y_1}(b^{y_3-y_1} - c^{z_3-z_1}). \tag{6.23}$$

Since $\gcd(a, b) = 1$, we get from (6.23) that

$$b^{y_3-y_1} - c^{z_3-z_1} = a^{x_1}t_5, \tag{6.24}$$

where $t_5 \in \mathbb{Z}, t_5 \neq 0$.

If $t_5 > 0$, then from (6.24) we get $b^{y_3-y_1} > a^{x_1} \geq a$. In this case, (4.2) has three solutions $(X_j, Y_j, Z_j)\,(j = 1, 2, 3)$ for $(A, B, C, \lambda) = (c, a, b, -1)$ such that $(X_1, Y_1, Z_1) = (z_1, x_1, y_1)$, $(X_2, Y_2, Z_2) = (z_3, x_3, y_3)$, $(X_3, Y_3, Z_3) = (z_2, x_2, y_2)$ and $Z_1 < Z_2 < Z_3$. However, since $2 \nmid b = C$, $C^{Z_2-Z_1} = b^{y_3-y_1} > a = \max\{a, b, c\} \geq 10^{62}$, we have a contradiction to Lemma 4.6.

Similarly, if $t_5 < 0$, then from the second equality of (6.24), we get $c^{z_3-z_1} > a^{x_1} \geq a = \max\{a, b, c\} \geq 10^{62}$. In this case, (4.2) has three solutions (X_j, Y_j, Z_j)

$(j = 1,2,3)$ for $(A,B,C,\lambda) = (a,b,c,1)$ such that $(X_1,Y_1,Z_1) = (x_1,y_1,z_1)$, $(X_2,Y_2,Z_2) = (x_3,y_3,z_3)$, $(X_3,Y_3,Z_3) = (x_2,y_2,z_2)$ and $Z_1 < Z_2 < Z_3$. However, since $4 \big| c^z = C^Z$ for any solutions (x,y,z) and (X,Y,Z) of (1.1) and (4.2), respectively, and $C^{Z_2-Z_1} = c^{z_3-z_1} > \max\{a,b,c\} \geq 10^{62}$, we have a contradiction to Lemma 4.6.

Thus, Theorem 1.1 holds for the case $c \neq \max\{a,b,c\}$. To sum up, the theorem is proved. □

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