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On Poincaré compactification and the projectivization of polynomial vector fields

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Abstract. The main purpose of this note is to clarify two commonly used notions, the Poincaré compactification and the projectivization, in the study of polynomial vector fields in $\mathbf{C}^n$. We show that these two notions are indeed equivalent.

1. Introduction

Poincaré compactification of real polynomial vector fields was introduced by Poincaré [6] to study their behavior at infinity. Through central projection, a polynomial vector field on $\mathbf{R}^n$ induces a vector field on $\mathbf{S}^n$, the n-dimensional sphere, and the infinity of the original system corresponds to the equator on $\mathbf{S}^n$ (see, e.g., [1]).

As pointed out by Lefschetz [4], the Poincaré compactification is of "projective" nature. But since the real projective space $\mathbf{RP}^n$ is not orientable for n even, there is an intrinsic ambiguity for vector fields on it (cf. [1]). Thus the orientable double cover of $\mathbf{RP}^n$, the sphere $\mathbf{S}^n$, is a better choice for this construction. However, the complex projective space $\mathbf{CP}^n$ is orientable for any n. Therefore $\mathbf{CP}^n$ is the natural choice for the "Poincaré compactification" for polynomial vector fields in $\mathbf{C}^n$. Of course, we do not have the "central projection" any more as in the real case. So the construction needs to be first carried out algebraically and then given a geometric meaning.

On the other hand, there is also the notion of *projectivization* for polynomial vector fields in $\mathbf{C}^n$, which is essentially the extension to homogeneous polynomial

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vector fields in $\mathbf{C}^{n+1}$. Poincaré compactification is defined through Pfaffian forms, while the projectivization is defined through homogeneous vector fields. However, we will show that both notions induce the same foliation on the projective space, which is our main result.

Theorem 1. *Poincaré compactification and the projectivization of polynomial vector fields in* $\mathbf{C}^n$ *are equivalent.*

In Section 2, we first establish the equivalence in dimension two. The higher-dimensional case is treated in Section 3.

2. Poincaré compactification and the projectivization of polynomial vector fields in $\mathbf{C}^2$

Consider a polynomial vector field in $\mathbf{C}^2$

$$\mathcal{X} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y},$$

where p and q are complex polynomials of degree $\leq d = \max\{\deg p, \deg q\}$.

There are two other equivalent representations of the vector field $\mathcal{X}$, one by the differential equation

$$\dot{x} = p(x,y), \quad \dot{y} = q(x,y),$$

and the other by the Pfaffian form

$$\omega = q(x,y)dx - p(x,y)dy = 0. \tag{1}$$

For the Poincaré compactification of $\mathcal{X}$, we work with the Pfaffian form.

Setting $(x,y) = (\frac{X}{Z}, \frac{Y}{Z})$, (1) is transformed to

$$\frac{1}{Z^{d+2}}[(XQ(X,Y,Z) - YP(X,Y,Z))dZ + Z(P(X,Y,Z)dY - Q(X,Y,Z)dX)] = 0, \tag{2}$$

where $P(X,Y,Z) = Z^d p(\frac{X}{Z}, \frac{Y}{Z})$ and $Q(X,Y,Z) = Z^d q(\frac{X}{Z}, \frac{Y}{Z})$. Let $p(x,y) = p_d(x,y) + p_{d-1}(x,y) + \cdots$ and $q(x,y) = q_d(x,y) + q_{d-1}(x,y) + \cdots$ be the homogeneous expansion of $p(x,y)$ and $q(x,y)$, where $p_k(x,y)$ and $q_k(x,y)$ are homogeneous of degree k, $0 \leq k \leq d$. If $xq_d(x,y) - yp_d(x,y) \equiv 0$, we say that $\mathcal{X}$ is *dicritical.* Otherwise we say that $\mathcal{X}$ is *non-dicritical.*

If $\mathcal{X}$ is non-dicritical, we multiply (2) by Z^{d+2} and get

$$(XQ(X,Y,Z) - YP(X,Y,Z))dZ + Z(P(X,Y,Z)dY - Q(X,Y,Z)dX) = 0. \quad (3)$$

If $\mathcal{X}$ is dicritical, we can write

$$P(X,Y,Z) = XR(X,Y,Z) + Z\tilde{P}(X,Y,Z),$$
$$Q(X,Y,Z) = YR(X,Y,Z) + Z\tilde{Q}(X,Y,Z),$$

where $R(X,Y,Z)$, $\tilde{P}(X,Y,Z)$ and $\tilde{Q}(X,Y,Z)$ are homogeneous of degree $d-1$. Multiplying (2) by Z^{d+1}, we get

$$(X\tilde{Q}(X,Y,Z) - Y\tilde{P}(X,Y,Z))dZ + (P(X,Y,Z)dY - Q(X,Y,Z)dX) = 0. \quad (4)$$

A homogeneous one-form $\Omega = A(X,Y,Z)dX + B(X,Y,Z)dY + C(X,Y,Z)dZ$ on $\mathbf{C}^3$ defines a co-dimension one foliation on $\mathbf{CP}^2$ if and only if Ω vanishes on the Euler vector field $V = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}$ identically, i.e., $XA + YB + ZC \equiv 0$ (see [2]). Thus the *Poincaré compactification* of $\mathcal{X}$ is given by (3) if $\mathcal{X}$ is non-dicritical, and by (4) if $\mathcal{X}$ is dicritical (cf. [5]). In the homogeneous coordinates $[X : Y : Z]$ on $\mathbf{CP}^2$, the infinity for the original system $\mathcal{X}$ corresponds to the line at infinity $\{Z = 0\}$.

Next, we consider the *projectivization* of $\mathcal{X}$, which is the extension of $\mathcal{X}$ to a homogeneous vector field $\tilde{\mathcal{X}}$ in $\mathbf{C}^3$ (cf. [3]). Write

$$\frac{\partial}{\partial x} = \frac{Z}{2X}\left(X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y} - Z\frac{\partial}{\partial Z}\right), \quad \frac{\partial}{\partial y} = \frac{Z}{2Y}\left(-X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} - Z\frac{\partial}{\partial Z}\right).$$

Then we can rewrite $\mathcal{X}$ as

$$\frac{1}{2XYZ^{d-1}}\left[YP\left(X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y} - Z\frac{\partial}{\partial Z}\right) + XQ\left(-X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} - Z\frac{\partial}{\partial Z}\right)\right]. \quad (5)$$

Note that the projectivization of $\mathcal{X}$ is well-defined up to adding a multiple of the Euler vector field. If $\mathcal{X}$ is non-dicritical, then after multiplying (5) by $2XYZ^{d-1}$, adding $(YP + XQ)V$ and dividing by $2XY$, we get

$$\tilde{\mathcal{X}} = P(X,Y,Z)\frac{\partial}{\partial X} + Q(X,Y,Z)\frac{\partial}{\partial Y}. \quad (6)$$

If $\mathcal{X}$ is dicritical, then after multiplying (5) by $2XYZ^{d-2}$, adding $(Y\tilde{P} + X\tilde{Q})V$ and dividing by $2XY$, we get

$$\tilde{\mathcal{X}} = \tilde{P}(X,Y,Z)\frac{\partial}{\partial X} + \tilde{Q}(X,Y,Z)\frac{\partial}{\partial Y} - R(X,Y,Z)\frac{\partial}{\partial Z}. \quad (7)$$

To get the projectivized vector field, we consider the compactification of $\mathbf{C}^3$ to $\mathbf{CP}^3$ and the hyperplane at infinity. Both (6) and (7) are homogeneous vector fields in $\mathbf{C}^3$ leaving the hyperplane at infinity invariant, and thus each induces a projectivized vector field on $\mathbf{CP}^2$.

Consider an arbitrary homogeneous vector field

$$F(X,Y,Z)\frac{\partial}{\partial X}+G(X,Y,Z)\frac{\partial}{\partial Y}+H(X,Y,Z)\frac{\partial}{\partial Z}$$

in $\mathbf{C}^3$, which can be equivalently written as

$$\dot{X}=F(X,Y,Z),\quad \dot{Y}=G(X,Y,Z),\quad \dot{Z}=H(X,Y,Z).$$

In Pfaffian form, the above is equivalent to

$$F(X,Y,Z)dZ-H(X,Y,Z)dX=0,\quad G(X,Y,Z)dZ-H(X,Y,Z)dY=0, \tag{8}$$

or

$$F(X,Y,Z)dY-G(X,Y,Z)dX=0,\quad H(X,Y,Z)dY-G(X,Y,Z)dZ=0, \tag{9}$$

or

$$G(X,Y,Z)dX-F(X,Y,Z)dY=0,\quad H(X,Y,Z)dX-F(X,Y,Z)dZ=0. \tag{10}$$

Set $(X,Y,Z)=(\frac{U}{T},\frac{V}{T},\frac{W}{T})$, with $[U:V:W:T]$ being the homogeneous coordinates of $\mathbf{CP}^3$. Then (8) transforms as

$$F(TdW-WdT)-H(TdU-UdT)=0,\quad G(TdW-WdT)-H(TdV-VdT)=0.$$

Thus in the affine chart $\{W=1\}$ and restricting to the hyperplane at infinity $\{T=0\}$, we get

$$\dot{U}=F(U,V,1)-UH(U,V,1),\quad \dot{V}=G(U,V,1)-VH(U,V,1). \tag{11}$$

Similarly, (9) transforms as

$$F(TdV-VdT)-G(TdU-UdT)=0,\quad H(TdV-VdT)-G(TdW-WdT)=0.$$

Thus in the affine chart $\{V=1\}$ and restricting to the hyperplane at infinity $\{T=0\}$, we get

$$\dot{U}=F(U,1,W)-UG(U,1,W),\quad \dot{W}=H(U,1,W)-WG(U,1,W). \tag{12}$$

And (10) transforms as

$$G(TdU-UdT)-F(TdV-VdT)=0,\quad H(TdU-UdT)-F(TdW-WdT)=0.$$

Thus in the affine chart $\{U=1\}$ and restricting to the hyperplane at infinity $\{T=0\}$, we get

$$\dot V=G(1,V,W)-VF(1,V,W),\quad \dot W=H(1,V,W)-WF(1,V,W). \tag{13}$$

Combining (11), (12) and (13), we see that the projectivized vector field induced by $F\frac{\partial}{\partial X}+G\frac{\partial}{\partial Y}+H\frac{\partial}{\partial Z}$ is given in the Pfaffian form by

$$(UG-VF)dW+(WF-UH)dV+(VH-WG)dU=0. \tag{14}$$

In the non-dicritical case, we have $F=P$, $G=Q$ and $H=0$. Thus (14) reads as

$$(UQ-VP)dW+W(PdV-QdU)=0,$$

which is the same as (3).

In the dicritical case, we have $F=\tilde P$, $G=\tilde Q$ and $H=-R$. Thus (14) reads as

$$(U\tilde Q-V\tilde P)dW+(W\tilde P+UR)dV-(W\tilde Q+VR)dU=0.$$

Since $W\tilde P+UR=P$ and $W\tilde Q+VR=Q$, it is the same as (4).

The above discussion establishes the equivalence between Poincaré compactification and the projectivization of polynomial vector fields in $\mathbf{C}^2$.

3. Poincaré compactification and the projectivization of polynomial vector fields in $\mathbf{C}^n$

Since the higher dimensional case is essentially the same, we only describe it briefly below.

Let $\mathcal{X}=\sum_{i=1}^n p_i(z)\frac{\partial}{\partial z_i}$ be a polynomial vector field in $\mathbf{C}^n$, where $p_i(z)$ are polynomials in $z=(z_1,\ldots,z_n)$ of degree $\le d=\max\{\deg p_1,\ldots,\deg p_n\}$. The vector field can be represented equivalently as a set of differential equations

$$\dot z_i=p_i(z),\quad i=1,\ldots,n,$$

or a set of Pfaffian forms

$$\omega_{ij}=p_i(z)dz_j-p_j(z)dz_i=0,\quad 1\le i\ne j\le n.$$

Of course, the Pfaffian system is redundant and a non-redundant system is given by a set of $n-1$ Pfaffian forms with fixed i, for any $1 \le i \le n$.

Let $p_i(z) = p_{i,d}(z) + p_{i,d-1}(z) + \cdots$ be the homogeneous expansion of $p_i(z)$, with $p_{i,k}(z)$ homogeneous of degree k, $0 \le k \le d$. Set $z = \frac{Z}{Z_0}$ with $Z = (Z_1, \ldots, Z_n)$, $P_i(Z_0, Z) = Z_0^d p_i(\frac{Z}{Z_0})$, $P_{i,d}(Z) = Z_0^d p_{i,d}(\frac{Z}{Z_0})$ and $\tilde{P}_i(Z_0, Z) = \frac{P_i(Z_0,Z) - P_{i,d}(Z)}{Z_0}$.

Let C be the algebraic variety on the hyperplane at infinity $\{Z_0 = 0\}$ defined by the set of equations (cf. [7])

$$Z_i P_{j,d}(Z) - Z_j P_{i,d}(Z) = 0, \quad 1 \le i \ne j \le n.$$

If the dimension of C is k, $0 \le k \le n-1$, then we say that $\mathcal{X}$ is *k-dicritical.* If $k = n-1$, we say that $\mathcal{X}$ is *dicritical.* If $0 \le k < n-1$, we say that $\mathcal{X}$ is *indicritical*, and in particular if $k = 0$, we say that $\mathcal{X}$ is *non-dicritical.*

If $Z_i P_{j,d}(Z) - Z_j P_{i,d}(Z) \ne 0$, we set

$$\Omega_{ij} = (Z_i P_j(Z_0, Z) - Z_j P_i(Z_0, Z))dZ_0 + Z_0(P_i(Z_0, Z)dZ_j - P_j(Z_0, Z)dZ_i),$$

otherwise we set

$$\Omega_{ij} = (Z_i \tilde{P}_j(Z_0, Z) - Z_j \tilde{P}_i(Z_0, Z))dZ_0 + (P_i(Z_0, Z)dZ_j - P_j(Z_0, Z)dZ_i).$$

The *Poincaré compactification* of $\mathcal{X}$ is given in Pfaffian form by

$$\Omega_{ij} = 0, \quad 1 \le i \ne j \le n. \tag{15}$$

For the *projectivization* $\tilde{\mathcal{X}}$ of $\mathcal{X}$, we write

$$\frac{\partial}{\partial z_i} = \frac{Z_0}{2Z_i} \left(Z_i \frac{\partial}{\partial Z_i} - \sum_{0 \le j \ne i \le n} Z_j \frac{\partial}{\partial Z_j} \right).$$

If $\mathcal{X}$ is indicritical, then

$$\tilde{\mathcal{X}} = \sum_{i=1}^{n} P_i(Z_0, Z) \frac{\partial}{\partial Z_i}, \tag{16}$$

and if $\mathcal{X}$ is dicritical, then we can write $P_{i,d}(Z) = Z_i R(Z)$ and get

$$\tilde{\mathcal{X}} = \sum_{i=1}^{n} \tilde{P}_i(Z_0, Z) \frac{\partial}{\partial Z_i} - R(Z) \frac{\partial}{\partial Z_0}. \tag{17}$$

Similar computation as in the two-dimensional case shows that projectivization (16) or (17) is equivalent to Poincaré compactification (15). This completes the proof of Theorem 1. □

Remark 2. In summary, the projectivization of a polynomial vector field in $\mathbf{C}^n$ is a homogeneous polynomial vector field in $\mathbf{C}^{n+1}$, which induces a projectivized vector field on $\mathbf{CP}^n$ as the Poincaré compactification.

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