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Gradient estimates for a weighted nonlinear equation on complete noncompact manifolds

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Abstract. Ma, Huang and Luo [12] considered $\Delta u+cu^{\alpha}=0(\alpha<0)$ with $\mathrm{Ric}_{ij}\geq -Kg_{ij}$, and obtained some gradient estimates. In the present paper, we investigate the weighted nonlinear equation $\Delta_f u+cu^{-\alpha}=0$ with $\mathrm{Ric}_f^N\geq -K$, where f is a smooth real-valued function on a complete noncompact Riemannian manifold (M^n,g), $\alpha>0$ and c are two real constants, and we achieve some gradient estimates for positive solutions of this weighted nonlinear equation. The results posed in this paper can be regarded as a natural generalization of the results in [12].

1. Introduction

Let f be a smooth real-valued function on a complete noncompact Riemannian manifold (M^n,g). The f-Laplacian (see [13], [14]) is given by $\Delta_f=\Delta-\nabla f\nabla$. It is known that Δ_f is self-adjoint, while $d\mu=e^{-f}dV_g$ is a naturally associated measure on M^n. The definition of the N-Bakry–Emery Ricci tensor is as follows:

$$\mathrm{Ric}_f^N=\mathrm{Ric}+\nabla^2 f-\frac{1}{N}df\otimes df,$$

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where ∇^2 denotes the Hession and Ric denotes the Ricci tensor, $0 \leq N \leq \infty$, $N = 0$ if and only if the function $f = 0$. When $N = \infty$, it is called the ∞-Bakry–Emery Ricci tensor:

$$\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f.$$

In particular, when $\mathrm{Ric}_f = \lambda g$, it is called a gradient Ricci soliton which is closely related to Ricci flow.

Ma, Huang and Luo [12] and Yang [19] obtained some gradient estimates for positive solutions of the equation as follows:

$$\Delta u + cu^{-\alpha} = 0, \tag{1.1}$$

in a complete noncompact Riemannian manifold M, where $\alpha > 0$, c are two real numbers. Equation (1.1) has an important position (see [6], [9]). In this paper, we study the weighted nonlinear equation

$$\Delta_f u + cu^{-\alpha} = 0 \tag{1.2}$$

in M, where $\alpha > 0$, c are two real numbers and f is a smooth real-valued function on M. For gradient estimates, there are many interesting results, one can see [3], [4], [7], [8], [10], [11], [18] for details.

Following Brighton's argument in [1] by choosing a test function $u^\varepsilon (\varepsilon \neq 0)$, there are many papers to get gradient estimates in this way, see [12] and the others. By choosing this test function $u^\varepsilon (\varepsilon \neq 0)$, we also obtain the gradient estimates about (1.2) with $\mathrm{Ric}_f^N \geq -K$, while Zhang and Ma [21] obtained the gradient estimates about (1.2) by choosing $h = \log u$ (the idea was originated by Cheng and Yau [5]). We get the following result firstly.

Theorem 1.1. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric}_f^N \geq -K$ in the metric ball $B_p(2R)$, where $-K = -K(2R)$, $K(2R) \geq 0$, $R > 0$. Assume that u is a positive solution to (1.2) with α, c satisfying one of the following two conditions:*

(1) *When $c > 0$, we have $\alpha > 0$;*

(2) *When $c < 0$,*

$$0 < \alpha < \frac{1}{2[n + N + 2 + \sqrt{(n+N)^2 + 5(n+N) + 3}]}, \quad n \geq 3,$$

then we have

$$|\nabla u(x)| \leq C(n, N, \alpha) M \sqrt{2K - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2 - 3c_1^2}{R^2}}, \tag{1.3}$$

where $M = \sup_{B_p(2R)} u(x)$, $C(n, N, \alpha) > 0$.

Remark 1.1. Comparing with [12], our result in Theorem 1.1 extends the gradient estimate of $\Delta u + cu^{\alpha} = 0$ $(\alpha < 0)$ with $\mathrm{Ric}_{ij} \geq -Kg_{ij}$ to the case of $\Delta_f u + cu^{-\alpha} = 0$ $(\alpha > 0)$ with $\mathrm{Ric}_f^N \geq -K$, where $-K = -K(2R)$, $K(2R) \geq 0$, $R > 0$.

Letting $R \to \infty$ in (1.3), we obtain

Corollary 1.2. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric}_f^N \geq -K$, where $K \geq 0$ is a constant. Suppose that u is a positive solution to* (1.2) *with α, c satisfying conditions* (1) *or* (2) *in Theorem 1.1, then we get*

$$|\nabla u| \leq C(n, N, \alpha) M \sqrt{2K}, \tag{1.4}$$

where $M = \sup_M u(x)$, $C(n, N, \alpha) > 0$.

Remark 1.2. Corollary 1.2 extends the global estimate [12, Corollary 1.5] of $\Delta u + cu^{\alpha} = 0$ $(\alpha < 0)$ with $\mathrm{Ric}_{ij} \geq -Kg_{ij}$ to the global estimate of $\Delta_f u + cu^{-\alpha} = 0$ $(\alpha > 0)$ with $\mathrm{Ric}_f^N \geq -K$, where $K \geq 0$ is a constant.

In particular, when $K = 0$, that is $\mathrm{Ric}_f^N \geq 0$, we have that any positive solution to (1.2) must be constant.

When $c = 0$ in (1.2), it degenerates an f-harmonic function, that is,

$$\Delta_f u = \Delta u - \nabla f \nabla u = 0.$$

To a smooth positive harmonic function, Schoen and Yau [20] studied gradient estimates of the equation above on complete Riemannian manifolds in 1994. To a smooth positive f-harmonic function, Chen and Chen [4] investigated gradient estimates of it with $\mathrm{Ric}_f \geq -H$, $(H \geq 0)$. It is an interesting work to study gradient estimates of f-harmonic functions with $\mathrm{Ric}_f^N \geq -K$, $(K \geq 0)$.

Theorem 1.3. *Suppose that (M^n, g) is a complete noncompact Riemannian manifold and $\mathrm{Ric}_f^N \geq -K$ in the metric ball $B_p(2R)$, where $-K = -K(2R)$, $K(2R) \geq 0$, $R > 0$. Then, for any positive f-harmonic function u, we have*

$$|\nabla u(x)| \leq M \sqrt{\frac{n+N}{(\varepsilon-1)^2-(n+N)(\varepsilon^2-\varepsilon)}} \left[2K + \frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2} + \Big(2 + \frac{(n+N)(\varepsilon-1)^2}{(\varepsilon-1)^2-(n+N)(\varepsilon^2-\varepsilon)}\Big) \frac{c_1^2}{R^2} \right]^{\frac{1}{2}}, \tag{1.5}$$

where $M = sup_{B_p(2R)} u(x)$, $\varepsilon \in (0, 1)$.

By choosing $R \to \infty$ in (1.5), the following global estimate of positive f-harmonic functions is established.

Corollary 1.4. *Suppose that (M^n, g) is a complete noncompact Riemannian manifold and $\mathrm{Ric}_f^N \geq -K$, $(K \geq 0)$. Then, for any positive f-harmonic function u, we have*

$$|\nabla u| \leq 2M\sqrt{\frac{2(n+N-1)K}{n+N}}, \tag{1.6}$$

where $M = sup_M u(x)$.

Remark 1.3. When $\mathrm{Ric}_f^N \geq 0$, we also have that any positive f-harmonic function must be constant here.

In particular, Li [10] studied the diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ and obtained many results of it, including if $\mathrm{Ric}_{m,n}(L) \geq 0$, then every positive solution (and bounded solution) of $Lu = 0$ must be constant (see [10, Theorem 2.2]).

Similarly, Ruan [16] also obtained this Liouville property of L-harmonic function on M (see [16, Corollary 1.2]).

2. Proof of Theorem 1.1.

In this subsection, we give the proof of Theorem 1.1.

Let $h = u^\varepsilon$, where ε is a nonzero constant that is to be determined. From (1.2), we get

$$\begin{aligned}\Delta_f h &= \Delta h - \nabla f \nabla h = \varepsilon(\varepsilon-1)u^{\varepsilon-2}|\nabla u|^2 + \varepsilon u^{\varepsilon-1}\Delta_f u \\ &= \varepsilon(\varepsilon-1)u^{\varepsilon-2}|\nabla u|^2 - c\varepsilon u^{\varepsilon-\alpha-1} = \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^2}{h} - c\varepsilon u^{\varepsilon-\alpha-1} \\ &= \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^2}{h} - c\varepsilon h^{\frac{\varepsilon-\alpha-1}{\varepsilon}},\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\nabla h \nabla \Delta_f h &= \nabla h \nabla \left(\frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^2}{h} - c\varepsilon h^{\frac{\varepsilon-\alpha-1}{\varepsilon}}\right) \\ &= \frac{\varepsilon-1}{\varepsilon h}\nabla h \nabla |\nabla h|^2 - \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h^2} - c\varepsilon\frac{\varepsilon-\alpha-1}{\varepsilon}\frac{|\nabla h|^2}{h}h^{\frac{\varepsilon-\alpha-1}{\varepsilon}} \\ &= \frac{\varepsilon-1}{\varepsilon h}\nabla h \nabla |\nabla h|^2 - \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h^2} - c(\varepsilon-\alpha-1)h^{\frac{\varepsilon-\alpha-1}{\varepsilon}}\frac{|\nabla h|^2}{h}.\end{aligned} \tag{2.2}$$

Substituting (2.1), (2.2) together into the Bochner formula [17] for h, we have:

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 &\geq \frac{1}{n+N}|\Delta_f h|^2 + \langle \nabla h, \nabla(\Delta_f h)\rangle + \mathrm{Ric}_f^N(\nabla h, \nabla h)\\ &= \frac{1}{n+N}\left[\frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^2}{h} - c\varepsilon h^{\frac{\varepsilon-\alpha-1}{\varepsilon}}\right]^2 + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2\\ &\quad - \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h^2} - c(\varepsilon-\alpha-1)h^{\frac{\varepsilon-\alpha-1}{\varepsilon}}\frac{|\nabla h|^2}{h} + \mathrm{Ric}_f^N(\nabla h, \nabla h)\\ &= \left[\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{|\nabla h|^4}{h^2} + \frac{c^2\varepsilon^2}{n+N}h^{2(\frac{\varepsilon-\alpha-1}{\varepsilon})} + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2\\ &\quad - \left[\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\right]h^{\frac{\varepsilon-\alpha-1}{\varepsilon}}\frac{|\nabla h|^2}{h} + \mathrm{Ric}_f^N(\nabla h, \nabla h). \quad (2.3)\end{aligned}$$

Lemma 2.1. *Assume that u is a positive solution to (1.2) and $\mathrm{Ric}_f^N \geq -K$ in the metric ball $B_p(2R)$, where $-K = -K(2R)$, $K(2R) \geq 0$, $R > 0$. Denote $h = u^\varepsilon$ with $\varepsilon \neq 0$. When $c > 0$ and $\alpha > 0$, then there exists $\varepsilon \in (0,1)$ such that*

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq \left[\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{|\nabla h|^4}{h^2} + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2 - K|\nabla h|^2. \quad (2.4)$$

Proof. From (2.3), when $c > 0$, $\alpha > 0$, for $\varepsilon \in (0,1)$,

$$-c\left[\frac{2(\varepsilon-1)}{n+N} + (\varepsilon-\alpha-1)\right] \geq 0,$$

so we have (2.4). Lemma 2.1 is proved. □

Lemma 2.2. *Assume that u is a positive solution to (1.2) and $\mathrm{Ric}_f^N \geq -K$ in the metric ball $B_p(2R)$, where $-K = -K(2R)$, $K(2R) \geq 0$, $R > 0$. Denote $h = u^\varepsilon$ with $\varepsilon \neq 0$. If $c < 0$, and for a fixed α, there exist two positive constants ε, δ such that*

$$c\left[\frac{2(\varepsilon-1)}{n+N} + (\varepsilon-\alpha-1)\right] \geq 0 \quad (2.5)$$

and

$$\frac{c^2\varepsilon^2}{n+N} - \frac{1}{\delta}\left[\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\right] \geq 0, \quad (2.6)$$

then we have

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 &\geq \left\{\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} - \delta\left[\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\right]\right\}\frac{|\nabla h|^4}{h^2}\\ &\quad + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2 - K|\nabla h|^2. \quad (2.7)\end{aligned}$$

PROOF. Fix a point q. Suppose that there exists a constant $\delta > 0$ such that $h^{\frac{\varepsilon-\alpha-1}{\varepsilon}} \leq \delta\frac{|\nabla h|^2}{h}$, according to (2.5), then (2.3) becomes

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 &\geq \left\{\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} - \delta\Big[\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\Big]\right\}\frac{|\nabla h|^4}{h^2} \\ &\quad + \frac{c^2\varepsilon^2}{n+N}h^{\frac{2(\varepsilon-\alpha-1)}{\varepsilon}} + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2 + \mathrm{Ric}_f^N(\nabla h, \nabla h) \\ &\geq \left\{\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} - \delta\Big[\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\Big]\right\}\frac{|\nabla h|^4}{h^2} \\ &\quad + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2 - K|\nabla h|^2.\end{aligned}$$

Conversely, if $h^{\frac{\varepsilon-\alpha-1}{\varepsilon}} \geq \delta\frac{|\nabla h|^2}{h}$ at q, according to (2.5), (2.6), then (2.3) becomes

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 &\geq \left[\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{|\nabla h|^4}{h^2} + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2 + \mathrm{Ric}_f^N(\nabla h, \nabla h) \\ &\quad + \left[\frac{c^2\varepsilon^2}{n+N} - \frac{1}{\delta}\Big(\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\Big)\right]\delta^2\frac{|\nabla h|^4}{h^2} \\ &\geq \left\{\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} - \delta\left[\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\right]\right\}\frac{|\nabla h|^4}{h^2} \\ &\quad + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2 + \mathrm{Ric}_f^N(\nabla h, \nabla h) \\ &\geq \left\{\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} - \delta\left[\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\right]\right\}\frac{|\nabla h|^4}{h^2} \\ &\quad + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2 - K|\nabla h|^2.\end{aligned}$$

In both cases, we can find that (2.7) holds always. Lemma 2.2 is proved. □

According to the maximum principle, we only need to ensure that $\frac{|\nabla h|^4}{h^2}$ has a positive coefficient in (2.4), and (2.7) can get the upper bound of the term $|\nabla h|$. To get this goal, we prove the following proposition.

Proposition 2.3. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric}_f^N \geq -K$, where $K \geq 0$ is a constant. Suppose that u is a positive solution to (1.2). If α and c satisfy one of the following two conditions:*

(i) *when $c > 0$, $\alpha > 0$;*

(ii) *when $c < 0$, $0 < \alpha < \frac{1}{2[n+N+2+\sqrt{(n+N)^2+5(n+N)+3}]}$, $n \geq 3$,*

then we have

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq \tilde{C}_1(n,N,\alpha)\frac{|\nabla h|^4}{h^2} + \tilde{C}_2(n,N,\alpha)\frac{\nabla h}{h}\nabla|\nabla h|^2 - K|\nabla h|^2, \tag{2.8}$$

where $\tilde{C}_1(n,N,\alpha) > 0$, $\tilde{C}_2(n,N,\alpha) < 0$.

PROOF. Next, we prove the above results in two cases.

Case 1. $c > 0$, $\alpha > 0$.
From Lemma 2.1, we have (2.4), that is

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq \left[\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{|\nabla h|^4}{h^2} + \frac{\varepsilon-1}{\varepsilon h}\nabla h\nabla|\nabla h|^2 - K|\nabla h|^2.$$

We find that $\tilde{C}_1(n,N,\alpha) = \frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} > 0$, $\tilde{C}_2(n,N,\alpha) = \frac{\varepsilon-1}{\varepsilon} < 0$ at this situation.

Case 2. $c < 0$, $0 < \alpha < \frac{1}{2[n+N+2+\sqrt{(n+N)^2+5(n+N)+3}]}$, $n \geq 3$.
According to Lemma 2.2, ε and δ need to be selected appropriately to ensure the following inequality:

$$\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} - \delta\left[\frac{2c(\varepsilon-1)}{n+N} + c(\varepsilon-\alpha-1)\right] > 0. \tag{2.9}$$

When $c < 0$, under the assumption of (2.5), inequality (2.6) becomes

$$\delta \geq \frac{\frac{2(\varepsilon-1)}{n+N} + (\varepsilon-\alpha-1)}{\frac{c\varepsilon^2}{n+N}} \tag{2.10}$$

and (2.9) becomes

$$\delta < \frac{\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}}{c\left[\frac{2(\varepsilon-1)}{n+N} + (\varepsilon-\alpha-1)\right]}. \tag{2.11}$$

In order to ensure we can choose a positive δ, from (2.10) and (2.11), we need to choose an ε satisfying

$$\frac{\frac{2(\varepsilon-1)}{n+N} + (\varepsilon-\alpha-1)}{\frac{c\varepsilon^2}{n+N}} < \frac{\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}}{c\left[\frac{2(\varepsilon-1)}{n+N} + (\varepsilon-\alpha-1)\right]}. \tag{2.12}$$

The above inequality (2.12) can be written as

$$\left[\frac{2(\varepsilon-1)}{a}+(\varepsilon-\alpha-1)\right]^2<\frac{\varepsilon^2}{n+N}\left[\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2}-\frac{\varepsilon-1}{\varepsilon}\right], \tag{2.13}$$

where $a=n+N$.

Inequality (2.13) is equivalent to

$$(a^2+5a+3)\varepsilon^2-\Big[2(\alpha+1)a^2+(4\alpha+9)a+6\Big]\varepsilon+\Big[a^2(\alpha+1)^2+4(\alpha+1)a+3\Big]<0. \tag{2.14}$$

To ensure there are some ε suiting for (2.14), we have

$$\begin{aligned}&\Big[2(\alpha+1)a^2+(4\alpha+9)a+6\Big]^2-4\Big[a^2(\alpha+1)^2+4(\alpha+1)a+3\Big](a^2+5a+3)\\&\quad=a^2\Big[4\alpha^2-8\alpha+1-4\alpha^2a-4\alpha a\Big]=(4-4a)\alpha^2-(4a+8)\alpha+1>0.\end{aligned} \tag{2.15}$$

It is necessary that there are some α holding for (2.15) to ensure there are some ε suiting for (2.14), so we have

$$(4a+8)^2-4(4-4a)>0$$

and $(4a+8)^2-4(4-4a)>0$ is right for $\forall a=n+N\geq 3$.

So we can get the roots of (2.15):

$$\alpha_1=\frac{\frac{1}{2}}{a+2-\sqrt{a^2+5a+3}}<0,\qquad \alpha_2=\frac{\frac{1}{2}}{a+2+\sqrt{a^2+5a+3}}>0.$$

So we can choose $\alpha\in\Big(0,\frac{\frac{1}{2}}{a+2+\sqrt{a^2+5a+3}}\Big)$. Then we can obtain the roots of (2.14):

$$\begin{aligned}\varepsilon_1&=\frac{[2(\alpha+1)a^2+(4\alpha+9)a+6]-a\sqrt{4\alpha^2-8\alpha+1-4(\alpha^2+\alpha)a}}{2(a^2+5a+3)},\\ \varepsilon_2&=\frac{[2(\alpha+1)a^2+(4\alpha+9)a+6]+a\sqrt{4\alpha^2-8\alpha+1-4(\alpha^2+\alpha)a}}{2(a^2+5a+3)},\end{aligned}$$

and we can check that $\varepsilon_1>0,\varepsilon_2>0$. We choose

$$\varepsilon:=\tilde{\varepsilon}=\frac{\varepsilon_1+\varepsilon_2}{2}=\frac{2(\alpha+1)a^2+(4\alpha+9)a+6}{2(a^2+5a+3)}. \tag{2.16}$$

We need $\tilde{\varepsilon}-1=\frac{a[(2a+4)\alpha-1]}{2(a^2+5a+3)}<0$ and obtain $\alpha<\frac{1}{2a+4}$. We also need (2.5) is right when (2.16) holds, then we have $\alpha>0$. So we choose $\alpha\in\left(0,\frac{\frac{1}{2}}{a+2+\sqrt{a^2+5a+3}}\right)$ because of $\frac{\frac{1}{2}}{a+2+\sqrt{a^2+5a+3}}<\frac{1}{2a+4}$. In particular, we let

$$\delta:=\tilde{\delta}=\frac{1}{2}\left\{\frac{\frac{2(\tilde{\varepsilon}-1)}{n+N}+(\tilde{\varepsilon}-\alpha-1)}{\frac{c\tilde{\varepsilon}^2}{n+N}}+\frac{\frac{1}{n+N}\frac{(\tilde{\varepsilon}-1)^2}{\tilde{\varepsilon}^2}-\frac{\tilde{\varepsilon}-1}{\tilde{\varepsilon}}}{c\left[\frac{2(\tilde{\varepsilon}-1)}{n+N}+(\tilde{\varepsilon}-\alpha-1)\right]}\right\}. \tag{2.17}$$

Then (2.5), (2.6) and (2.9) are satisfied, and (2.7) becomes

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2&\geq\tilde{C}_1(n,N,\alpha)\frac{|\nabla h|^4}{h^2}+\tilde{C}_2(n,N,\alpha)\frac{\nabla h}{h}\nabla|\nabla h|^2+\mathrm{Ric}_f^N(\nabla h,\nabla h)\\&\geq\tilde{C}_1(n,N,\alpha)\frac{|\nabla h|^4}{h^2}+\tilde{C}_2(n,N,\alpha)\frac{\nabla h}{h}\nabla|\nabla h|^2-K|\nabla h|^2,\end{aligned} \tag{2.18}$$

where $\tilde{C}_1(n,N,\alpha)>0$, $\tilde{C}_2(n,N,\alpha)<0$ are given by

$$\begin{aligned}\tilde{C}_1(n,N,\alpha)&=\frac{1}{2}\left\{\frac{(\tilde{\varepsilon}-1)^2}{a\tilde{\varepsilon}^2}-\frac{\tilde{\varepsilon}-1}{\tilde{\varepsilon}}-\frac{a}{\tilde{\varepsilon}^2}\left[\frac{2(\tilde{\varepsilon}-1)}{a}+\tilde{\varepsilon}-\alpha-1\right]^2\right\},\\\tilde{C}_2(n,N,\alpha)&=\frac{2\alpha a^2+(4\alpha-1)a}{2(\alpha+1)a^2+(4\alpha+9)a+6},\end{aligned}$$

respectively, at this situation, where $a=n+N$. We complete the proof of Proposition 2.3. □

In the following subsection, we will complete the proof of Theorem 1.1. Choosing a cut-off function ξ, it satisfies $\xi(r)\in[0,1]$, $\xi(r)=1$ for $r\leq 1$, $\xi(r)=0$ for $r\geq 2$, and

$$0\geq\xi^{-\frac{1}{2}}\xi'(r)\geq-c_1,\qquad\xi''(r)\geq-c_2,$$

where $c_1>0$, $c_2>0$. Let $\rho(x)=d(x,p)$ be the distance between the point x and the point p in M, and let the function

$$\phi(x)=\xi\left(\frac{\rho(x)}{R}\right).$$

Without loss of generality, we can suppose that ϕ is smooth in $B_p(2R)$ from Calabi [2] or Cheng and Yau [5]. We get

$$\frac{|\nabla\phi|^2}{\phi}\leq\frac{c_1^2}{R^2}. \tag{2.19}$$

From QIAN's results in [15], we get

$$\Delta_f(\rho^2) \le (n+N)\left(1+\sqrt{1+\frac{4K\rho^2}{n+N}}\right).$$

So

$$\begin{aligned}\Delta_f\rho &= \frac{1}{2\rho}\left[\Delta_f(\rho^2)-2|\nabla\rho|^2\right] \le \frac{n+N-2}{2\rho}+\frac{n+N}{2\rho}\left(1+\sqrt{1+\frac{4K\rho^2}{n+N}}\right)\\ &= \frac{n+N-1}{\rho}+\sqrt{(n+N)K},\end{aligned}$$

and

$$\Delta_f\phi = \frac{\xi^{''}(r)|\nabla\rho|^2}{R^2}+\frac{\xi^{'}(r)\Delta_f\rho}{R} \ge -\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2}. \quad (2.20)$$

We define the function $G=\phi|\nabla h|^2$ and will use the maximum principle on it. Suppose that the point $x_0 \in B_p(2R)$ is the maximal value point of G and $G(x_0)>0$ (otherwise the proof is trivial). Then at x_0,

$$\Delta_f G \le 0 \quad \text{and} \quad \nabla|\nabla h|^2 = -\frac{|\nabla h|^2}{\phi}\nabla\phi,$$

and

$$\begin{aligned}0 \ge \Delta_f G &= \phi\Delta_f|\nabla h|^2+|\nabla h|^2\Delta_f\phi+2\nabla\phi\nabla|\nabla h|^2\\ &= \phi\Delta_f|\nabla h|^2+\frac{\Delta_f\phi}{\phi}G-2\frac{|\nabla\phi|^2}{\phi^2}G\\ &\ge 2\phi\left[\tilde{C}_1(n,N,\alpha)\frac{|\nabla h|^4}{h^2}+\tilde{C}_2(n,N,\alpha)\frac{\nabla h}{h}\nabla|\nabla h|^2+\mathrm{Ric}_f^N(\nabla h,\nabla h)\right]\\ &\quad+\frac{\Delta_f\phi}{\phi}G-2\frac{|\nabla\phi|^2}{\phi^2}G\\ &= 2\tilde{C}_1(n,N,\alpha)\frac{G^2}{\phi h^2}-2\tilde{C}_2(n,N,\alpha)\frac{G}{\phi}\nabla\phi\frac{\nabla h}{h}-2KG+\frac{\Delta_f\phi}{\phi}G-2\frac{|\nabla\phi|^2}{\phi^2}G, \quad (2.21)\end{aligned}$$

where, in the second inequality, the estimate (2.18) is used. Multiplying both sides of (2.21) by $\frac{\phi}{G}$ yields

$$2\tilde{C}_1(n,N,\alpha)\frac{G}{h^2} \le 2\tilde{C}_2(n,N,\alpha)\nabla\phi\frac{\nabla h}{h}+2K\phi-\Delta_f\phi+2\frac{|\nabla\phi|^2}{\phi}. \quad (2.22)$$

Using the Cauchy inequality

$$2\tilde{C}_2(n,N,\alpha)\nabla\phi\frac{\nabla h}{h} \le \frac{(\tilde{C}_2(n,N,\alpha))^2}{\tilde{C}_1(n,N,\alpha)}\frac{|\nabla\phi|^2}{\phi} + \tilde{C}_1(n,N,\alpha)\frac{G}{h^2}$$

in (2.22) yields

$$\tilde{C}_1(n,N,\alpha)\frac{G}{h^2} \le 2K\phi - \Delta_f\phi + \left(2 + \frac{(\tilde{C}_2(n,N,\alpha))^2}{\tilde{C}_1(n,N,\alpha)}\right)\frac{|\nabla\phi|^2}{\phi}. \tag{2.23}$$

Hence, for $B_p(R)$, we have

$$\begin{aligned}\tilde{C}_1(n,N,\alpha)G(x) &\le \tilde{C}_1(n,N,\alpha)G(x_0)\\ &\le h(x_0)^2\left[2K + \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}\right.\\ &\qquad \left. + \left(2 + \frac{(\tilde{C}_2(n,N,\alpha))^2}{\tilde{C}_1(n,N,\alpha)}\right)\frac{c_1^2}{R^2}\right].\end{aligned} \tag{2.24}$$

Then, we have

$$\begin{aligned}&\tilde{C}_1(n,N,\alpha)\varepsilon^2u^{2\varepsilon-2}|\nabla u|^2(x)\\ &\le h(x_0)^{2\varepsilon}\left[2K+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2}+\left(2+\frac{(\tilde{C}_2(n,N,\alpha))^2}{\tilde{C}_1(n,N,\alpha)}\right)\frac{c_1^2}{R^2}\right]\\ &= h(x_0)^{2\varepsilon}\left[2K+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2}+\frac{2c_1^2}{R^2}+\left(\frac{(\tilde{C}_2(n,N,\alpha))^2}{\tilde{C}_1(n,N,\alpha)}\right)\frac{c_1^2}{R^2}\right].\end{aligned}$$

It shows that

$$\begin{aligned}|\nabla u|^2(x) &\le \frac{1}{\varepsilon^2}M^2\left\{\frac{(\tilde{C}_2(n,N,\alpha))^2}{(\tilde{C}_1(n,N,\alpha))^2}\frac{c_1^2}{R^2} + \frac{1}{\tilde{C}_1(n,N,\alpha)}\times\right.\\ &\qquad \left.\left[2K + \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2 + 2c_1^2}{R^2}\right]\right\}\\ &\le M^2\left[C(n,N,\alpha)\left(2K+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2+3c_1^2}{R^2}\right)\right],\end{aligned}$$

and hence,

$$|\nabla u(x)| \le C(n,N,\alpha)M\sqrt{2K+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2+3c_1^2}{R^2}}, \tag{2.25}$$

where $M = \sup_{B_p(2R)} u(x)$, $C(n,N,\alpha) > 0$. Theorem 1.1 is proved. □

3. Proofs of Theorem 1.3 and Corollary 1.4.

In this section, we will give the proofs of Theorem 1.3 and Corollary 1.4. Firstly, we give the proof of Theorem 1.3.

PROOF. Let $h = u^{\varepsilon}$. Then

$$\Delta_f h = \Delta h - \nabla f \nabla h = \varepsilon(\varepsilon-1)u^{\varepsilon-2}|\nabla u|^2 + \varepsilon u^{\varepsilon-1}\Delta_f u = \varepsilon(\varepsilon-1)u^{\varepsilon-2}|\nabla u|^2, \quad (3.1)$$

and

$$\begin{aligned}\langle \nabla h, \nabla(\Delta_f h)\rangle &= \langle \nabla h, \nabla(\varepsilon(\varepsilon-1)u^{\varepsilon-2}|\nabla u|^2)\rangle \\ &= \frac{\varepsilon-1}{\varepsilon}\langle \nabla h, \nabla|\nabla h|^2\rangle - \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h^2}. \end{aligned} \quad (3.2)$$

From (3.1) and (3.2), using the Bochner formula on h to the N-Bakry–Emery Ricci tensor (see [17]):

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 &\geq \frac{1}{n+N}|\Delta_f h|^2 + 2\langle \nabla h, \nabla(\Delta_f h)\rangle + \mathrm{Ric}_f^N(\nabla h, \nabla h) \\ &= \frac{1}{n+N}\left[\varepsilon(\varepsilon-1)u^{\varepsilon-2}|\nabla u|^2\right]^2 + \frac{\varepsilon-1}{\varepsilon}\langle \nabla h, \nabla|\nabla h|^2\rangle \\ &\quad - \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h^2} + \mathrm{Ric}_f^N(\nabla h, \nabla h) \\ &= \left[\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{|\nabla h|^4}{h^2} + \frac{\varepsilon-1}{\varepsilon h}\langle \nabla h, \nabla|\nabla h|^2\rangle \\ &\quad + \mathrm{Ric}_f^N(\nabla h, \nabla h). \end{aligned} \quad (3.3)$$

According to the maximum principle, we only need to ensure that $\frac{|\nabla h|^4}{h^2}$ has a positive coefficient in (3.3) such that the upper bound of the term $|\nabla h|$ is achieved. It can be checked that for any $\varepsilon \in (0,1)$,

$$\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} > 0.$$

Similarly to the previous approach in the proof of Theorem 1.1, we also choose ϕ as the cut-off function as before. We define the function $G = \phi|\nabla h|^2$ and will use the maximum principle on it. Suppose that the point $x_0 \in B_p(2R)$ is the maximal value point of G and $G(x_0) > 0$ (otherwise the proof is trivial). Then at x_0,

$$\Delta_f G \leq 0 \quad \text{and} \quad \nabla|\nabla h|^2 = -\frac{|\nabla h|^2}{\phi}\nabla\phi,$$

and

$$\begin{aligned}0 \geq \Delta_f G &= \phi\Delta_f|\nabla h|^2 + \frac{\Delta_f\phi}{\phi}G - 2\frac{|\nabla\phi|^2}{\phi^2}G \\ &\geq 2\phi\Bigg\{\left[\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{|\nabla h|^4}{h^2} + \frac{\varepsilon-1}{\varepsilon h}\langle\nabla h, \nabla|\nabla h|^2\rangle \\ &\qquad + \mathrm{Ric}_f^N(\nabla h, \nabla h)\Bigg\} + \frac{\Delta_f\phi}{\phi}G - 2\frac{|\nabla\phi|^2}{\phi^2}G \\ &= 2\left[\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{G^2}{\phi h^2} - \frac{2(\varepsilon-1)}{\varepsilon}\nabla\phi\frac{\nabla h}{h}\frac{G}{\phi} - 2KG \\ &\quad + \frac{\Delta_f\phi}{\phi}G - 2\frac{|\nabla\phi|^2}{\phi^2}G. \end{aligned} \tag{3.4}$$

Then multiplying by $\frac{\phi}{G}$ both sides of (3.4) gives

$$2\left[\frac{1}{n+N}\frac{(\varepsilon-1)^2}{\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{G}{h^2} \leq \frac{2(\varepsilon-1)}{\varepsilon}\nabla\phi\frac{\nabla h}{h} + 2K\phi - \Delta_f\phi + 2\frac{|\nabla\phi|^2}{\phi}. \tag{3.5}$$

Using the Cauchy inequality

$$\frac{2(\varepsilon-1)}{\varepsilon}\nabla\phi\frac{\nabla h}{h} \leq \frac{\frac{(\varepsilon-1)^2}{\varepsilon^2}}{\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}}\frac{|\nabla\phi|^2}{\phi} + \left[\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{G}{h^2}$$

in (3.5) yields

$$\left[\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{G}{h^2} \leq \left[2 + \frac{\frac{(\varepsilon-1)^2}{\varepsilon^2}}{\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}}\right]\frac{|\nabla\phi|^2}{\phi} + 2K\phi - \Delta_f\phi. \tag{3.6}$$

Hence, for $B_p(R)$, we have

$$\begin{aligned}&\left[\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]G(x) \leq \left[\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]G(x_0) \\ &\leq h(x_0)^2\Bigg\{2K + \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} + \left[2 + \frac{\frac{(\varepsilon-1)^2}{\varepsilon^2}}{\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}}\right]\frac{c_1^2}{R^2}\Bigg\}. \end{aligned} \tag{3.7}$$

Then, we have

$$\left[\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2}-\frac{\varepsilon-1}{\varepsilon}\right]\varepsilon^2u^{2\varepsilon-2}|\nabla u|^2(x)$$
$$\leq u^{2\varepsilon}(x_0)\left[2K+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2}+\left(2+\frac{\frac{(\varepsilon-1)^2}{\varepsilon^2}}{\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2}-\frac{\varepsilon-1}{\varepsilon}}\right)\frac{c_1^2}{R^2}\right].$$

It shows that

$$\begin{aligned}|\nabla u|^2(x)&\leq\left[\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2}-\frac{\varepsilon-1}{\varepsilon}\right]^{-1}\frac{1}{\varepsilon^2}M^2\left[2K+\left(2+\frac{\frac{(\varepsilon-1)^2}{\varepsilon^2}}{\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2}-\frac{\varepsilon-1}{\varepsilon}}\right)\frac{c_1^2}{R^2}\right.\\&\quad\left.+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2}\right]\\&=\frac{n+N}{(\varepsilon-1)^2-(n+N)\varepsilon(\varepsilon-1)}M^2\left[2K+\left(2+\frac{\frac{(\varepsilon-1)^2}{\varepsilon^2}}{\frac{(\varepsilon-1)^2}{(n+N)\varepsilon^2}-\frac{\varepsilon-1}{\varepsilon}}\right)\frac{c_1^2}{R^2}\right.\\&\quad\left.+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2}\right]\\&=\frac{n+N}{(\varepsilon-1)^2-(n+N)\varepsilon(\varepsilon-1)}M^2\left[2K+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2}\right.\\&\quad\left.+\left(2+\frac{(n+N)(\varepsilon-1)^2}{(\varepsilon-1)^2-(n+N)\varepsilon(\varepsilon-1)}\right)\frac{c_1^2}{R^2}\right],\end{aligned}$$

and hence

$$\begin{aligned}|\nabla u|(x)&\leq M\sqrt{\frac{n+N}{(\varepsilon-1)^2-(n+N)(\varepsilon^2-\varepsilon)}}\left[2K+\frac{(n+N-1+\sqrt{(n+N)K}R)c_1+c_2}{R^2}\right.\\&\quad\left.+\left(2+\frac{(n+N)(\varepsilon-1)^2}{(\varepsilon-1)^2-(n+N)(\varepsilon^2-\varepsilon)}\right)\frac{c_1^2}{R^2}\right]^{\frac{1}{2}},\end{aligned}$$

for any $\varepsilon\in(0,1)$, where $M=\sup_{B_p(2R)}u(x)$, so we complete the proof of Theorem 1.3. □

In the following, we give the proof of Corollary 1.4.

PROOF. When choosing $R \to \infty$ in (1.5), we have

$$|\nabla u|(x) \leq M\sqrt{\frac{2K(n+N)}{(\varepsilon-1)^2-(n+N)(\varepsilon^2-\varepsilon)}}.$$

By using the knowledge of quadratic equation in one variable, for any $\varepsilon \in (0,1)$, we get that the equation $(\varepsilon-1)^2-(n+N)(\varepsilon^2-\varepsilon)$ reaches its maximal value when $\varepsilon = \frac{n+N-2}{2n+2N-2}$ and $\Big[(\varepsilon-1)^2-(n+N)(\varepsilon^2-\varepsilon)\Big]_{\max} = \frac{(n+N)^2}{4(n+N-1)}$. Then we get $|\nabla u| \leq 2M\sqrt{\frac{2(n+N-1)K}{n+N}}$, where $M = \sup_M u(x)$. This ends the proof of Corollary 1.4. □

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References

[1] K. BRIGHTON, A Liouville-type theorem for smooth metric measure spaces, *J. Geom. Anal.* **23** (2013), 562–570.

[2] E. CALABI, An extension of Hopf's maximum principle with application to Riemannian geometry, *Duke Math. J.* **25** (1958), 45–46.

[3] L. CHEN and W. CHEN, Gradient estimates for a nonlinear parabolic equation on complete non-compact Riemannian manifolds, *Ann. Global Anal. Geom.* **35** (2009), 397–404.

[4] L. CHEN and W. CHEN, Gradient estimates for positive smooth f-harmonic functions, *Acta Math. Sci. Ser. B (Engl. Ed.)* **30** (2010), 1614–1618.

[5] S. Y. CHENG and S. T. YAU, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure. Appl. Math.* **28** (1975), 333–354.

[6] Z. GUO and J. WEI, Hausdorff dimension of ruptures for solutions of a semilinear elliptic equation with singular nonlinearity, *Manuscripta Math.* **120** (2006), 193–209.

[7] S.-Y. HSU, Gradient estimates for a nonlinear parabolic equation under Ricci flow, *Differential Integral Equations* **24** (2011), 645–652.

[8] G. HUANG and B. MA, Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds, *Arch. Math. (Basel)* **94** (2010), 265–275.

[9] J. Y. LI, Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds, *J. Funct. Anal.* **100** (1991), 233–256.

[10] X.-D. LI, Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds, *J. Math. Pures Appl. (9)* **84** (2005), 1295–1361.

[11] L. MA, Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds, *J. Funct. Anal.* **241** (2006), 374–382.

[12] B. MA, G. HUANG and Y. LUO, Gradient estimates for a nonlinear elliptic equation on complete Riemannian manifolds, *Proc. Amer. Math. Soc.* **146** (2018), 4993–5002.

[13] L. MA and B. LIU, Convexity of the first eigenfunction of the drifting Laplacian operator and its applications, *New York J. Math.* **14** (2008), 393–401.

[14] L. Ma and B. Liu, Convex eigenfunction of a drifting Laplacian operator and the fundamental gap, *Pacific J. Math.* **240** (2009), 343–361.

[15] Z. Qian, A comparison theorem for an elliptic operator, *Potenial Anal.* **8** (1998), 137–142.

[16] Q. Ruan, Harnack inequality and Liouville properties for L-harmonic operator, *J. Math. Anal. Appl.* **329** (2007), 1430–1439.

[17] G. Wei and W. Wylie, Comparison geometry for the Bakry–Emery Ricci tensor, *J. Differential Geom.* **83** (2009), 377–405.

[18] Y. Yang, Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds, *Proc. Amer. Math. Soc.* **136** (2008), 4095–4102.

[19] Y. Y. Yang, Gradient estimates for the equation $\Delta u + cu^{-\alpha} = 0$ on Riemannian manifolds, *Acta. Math. Sin. (Engl. Ser.)* **26** (2010), 1177–1182.

[20] R. Schoen and S.-T. Yau, Lectures on Differential Geometry, *International Press*, *Boston*, 1994.

[21] J. Zhang and B. Ma, Gradient estimates for a nonlinear equation $\Delta_f u + cu^{-\alpha} = 0$ on complete noncompact manifolds, *Commun. Math.* **19** (2011), 73–84.

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