Publ. Math. Debrecen
46 / 3-4 (1995), 349–372

On characterizations of Sasakian space forms and locally φ-symmetric spaces by φ-geodesic tubes

By M. DJORIĆ (Belgrade)

Abstract. This paper continues the study of the problem to which extent the properties of tubes about φ-geodesics on a Sasakian manifold influence the geometry of the ambient space. We characterize Sasakian space forms and φ-symmetric spaces by analysing the action of the shape operator and the Ricci operator on these tubes.

1. Introduction

In this paper, Sasakian manifolds will be studied by investigating the properties of the extrinsic and intrinsic geometry of tubes on these manifolds. The strong influence by the features of the geometry of small geodesic spheres and tubes on the geometry of the ambient space was ascertained a long time ago. Since then, it has been the topic of numerous investigations and discussions, such as [2]–[16], [20].

We refer to [16] and [23] as the most comprehensive and detailed studies of this relation which include a selection of already known and new results. In particular, these studies have shown that the theory of Jacobi vector fields is very important in the investigation of Riemannian geometry, as this is one of the most convenient methods for analysing the extrinsic and intrinsic geometry of small geodesic spheres and small tubes about curves and submanifolds of a Riemannian manifold.

This technique was used in treating the relation between the curvature of the ambient space and the properties of the shape operator and the Ricci operator of small geodesic spheres on an almost Hermitian [9] and on a Sasakian manifold [10]. This led to several new characterizations of Kähler and nearly Kähler manifolds of constant holomorphic sec-

Work supported in part by Science Foundation of Serbia, project # 0401 .

tional curvature, of locally Hermitian symmetric spaces, of nearly Kähler 3-symmetric spaces, of Sasakian space forms and of locally φ-symmetric spaces. In [11] the author, together with L. Vanhecke, initiated the study of similar problems by considering small tubes on Sasakian manifolds. In particular, the research of small tubes about the flow lines of the characteristic vector field resulted in characterizations of Sasakian space forms and locally φ-symmetric spaces. Later, (in [12]), the author started analysing similar problems considering the tubes about geodesics which cut at all of its points the integral curves of the characteristic vector field ξ orthogonally. These particular geodesics are usually called φ-geodesics and this study has led to characterizations of Sasakian space forms and of locally φ-symmetric spaces too. Several new characterizations of these spaces are the main purpose of this paper. More precisely, we analyse the properties of tubes about φ-geodesics σ on a Sasakian manifold $M(\varphi,\xi,\eta,g)$ by investigating the action of the shape operator and the Ricci operator on the plane $\{\xi,\varphi v\}$ which is parallel along a φ-geodesic γ tangent to v and passing through $\sigma(t)$. In this paper γ is tangent to v while meeting $\dot\sigma(t)$ and $\varphi\dot\sigma(t)$ orthogonally, whereas in [12] γ is tangent to $v=\varphi\dot\sigma(t)$. Recently D. Blair and B. Papantoniou gave in [2] a characterization of Sasakian space forms by means of the Weingarten map on sufficiently small geodesic tubes.

The article is organized in the following way. In Section 2 we recall some general preliminary facts concerning Sasakian manifolds. After focussing on the central role of Fermi coordinates and Jacobi vector fields in Section 3, we compute the explicit formulas for the shape operator of tubes about φ-geodesics on a Sasakian space form in the above described points. Then, in Sections 4 and 5, we treat our main results considering the extrinsic and intrinsic geometry of these tubes.

The author is grateful to L. Vanhecke and P. Bueken for several useful discussions.

2. Sasakian manifolds

A C^∞ manifold M^{2n+1} of dimension $2n+1$, together with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$, is said to be a *contact manifold*. Given such a *contact form* η, it is well known that there exists a unique vector field ξ on M^{2n+1} satisfying $\eta(\xi)=1$ and $d\eta(\xi,X)=0;\xi$ is called the *characteristic vector field* of the contact structure. Moreover, if there exists a tensor field φ of type (1,1) such that

$$\varphi^2=-I+\eta\otimes\xi, \tag{1}$$

then M admits a Riemannian metric g satisfying

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),$$

for any vector fields X, Y. The structure (φ, ξ, η, g) is called a *contact metric structure* and a manifold M^{2n+1} with a contact metric structure is said to be a *contact metric manifold*. If the contact metric structure tensors satisfy

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2}$$

where ∇ denotes the Riemannian connection of g, M is said to be a *Sasakian manifold*. This condition implies

$$\nabla_X \xi = -\varphi X \tag{3}$$

from which it follows that ξ is a Killing vector field. Hence, its integral curves are geodesics called *ξ-geodesics*. Also, vectors which are orthogonal to ξ are called *horizontal vectors*. The curvature tensor

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

of a Sasakian manifold satisfies

$$\begin{aligned} R_{XY}\xi &= \eta(Y)X - \eta(X)Y, \\ R_{X\xi}Y &= g(X,Y)\xi - \eta(Y)X. \end{aligned} \tag{4}$$

We refer the reader to [1] and [24] for more details about the study of these manifolds.

Furthermore, it is natural to consider the notion of a locally φ-symmetric Sasakian space since a locally symmetric Sasakian manifold is a space of constant curvature 1 (Okumura [19]). Here, a Sasakian manifold is said to be *locally φ-symmetric* if $\varphi^2(\nabla_V R)_{XY}Z = 0$ for all horizontal vectors X, Y, Z, V (Takahashi [21]). On the other hand, it is easy to see from (3) that a geodesic which is orthogonal to ξ at one point, remains orthogonal to it. Such geodesics are called *φ-geodesics* and Takahashi used them to define special local diffeomorphisms s_m and afterwards to give one very helpful characterization of locally φ-symmetric spaces (see Theorem A (i)). A local diffeomorphism s_m, $m \in M$ is said to be a *φ-geodesic symmetry* if its domain $\mathcal{U}$ is such that, for every φ-geodesic $\gamma(s)$ for which $\gamma(0)$ lies in the intersection of $\mathcal{U}$ with the integral curve of ξ through m,

$$(s_m \circ \gamma)(s) = \gamma(-s)$$

for all s with $\gamma(\pm s) \in \mathcal{U}$, s being the arc length.

We now present some further properties of Sasakian manifolds that will be useful for the work below.

Let $\mathcal{U}$ be a neighborhood in a Sasakian manifold M on which ξ is regular (see [1] for a definition of regularity). In [18] it is proved that each manifold $\mathcal{U} = \tilde{\mathcal{U}}/\xi$, which is the base space of the local fibration $\tilde{\mathcal{U}} \xrightarrow{\pi} \mathcal{U}$, is a Kähler manifold $(\mathcal{U}, G, J)$ with the Kähler structure (G, J) on $\mathcal{U}$. If $X^*, Y^* \dots$ denote the horizontal lifts of $X, Y \dots \in \mathcal{X}(\mathcal{U})$, (the Lie algebra of smooth vector fields on $\mathcal{U}$), with respect to the connection form η, then we have [18]

$$(JX)^* = \varphi X^* \tag{5}$$

$$(\bar{\nabla}_X Y)^* = \nabla_{X^*} Y^* - \eta(\nabla_{X^*} Y^*)\xi, \tag{6}$$

where $\bar{\nabla}$ denotes the Riemannian connection of $(\mathcal{U}, G)$. From this we get

$$\begin{aligned} (\bar{R}_{XY} Z)^* &= R_{X^*Y^*} Z^* + g(\varphi X^*, Z^*)\varphi Y^* - g(\varphi Y^*, Z^*)\varphi X^* \\ &\quad + 2g(\varphi X^*, Y^*)\varphi Z^*, \end{aligned} \tag{7}$$

$$\bar{\rho}(X, Y)^* = \rho(X^*, Y^*) + 2g(X^*, Y^*), \tag{8}$$

$$\bar{\tau}^* = \tau + 2n, \tag{9}$$

where $\bar{R}$ denotes the Riemannian curvature tensor of $(\mathcal{U}, G)$ and $\bar{\rho}, \bar{\tau}$ (resp., ρ, τ) are the Ricci tensor (of type (0,2)) and the scalar curvature of $(\mathcal{U}, G)$ (resp., of $(\tilde{\mathcal{U}}, g)$). Further, we have

$$(\bar{\nabla}_X \bar{R})_{YZVW} \circ \pi = (\nabla_{X^*} R)_{Y^*Z^*V^*W^*}, \tag{10}$$

$$(\bar{\nabla}_X \bar{\rho})_{YZ} \circ \pi = (\nabla_{X^*} \rho)_{Y^*Z^*}, \tag{11}$$

for $X, Y, Z, V, W \in \chi(\mathcal{U})$.

We shall need the following results:

Theorem A [21]**.** *A Sasakian manifold is locally φ-symmetric if and only if*

(i) s_m is a local automorphism of the Sasakian structure (g, ξ, η, φ) for each $m \in M$; or

(ii) each base manifold $\mathcal{U}$ of a local fibration is locally Hermitian symmetric.

Related to (ii) we have

Theorem B (see for example [20])**.** *A Kähler manifold $(\mathcal{U}, G, J)$ is locally Hermitian symmetric if and only if*

$$(\bar{\nabla}_X \bar{R})_{X\,JX\,X\,JX} = 0$$

for all $X \in \mathcal{X}(\mathcal{U})$.

Corresponding to this theorem and (ii) in Theorem A we have, by using (10),

Theorem C [6]. *A Sasakian manifold is locally φ-symmetric if and only if*

$$(\nabla_X R)_{X\,\varphi X\,X\,\varphi X} = 0$$

for all vector fields X orthogonal to ξ.

Next, we give the definition of a special class of locally φ-symmetric spaces, namely that of *Sasakian space forms*. A plane section of the tangent space at a point of M is called a *φ-section* if it is spanned by horizontal vectors X and φX. The sectional curvature of a φ-section is called a *φ-sectional curvature*. If a Sasakian manifold has constant φ-sectional curvature c, i.e., independent of X, then its curvature tensor is given by

$$\begin{aligned} R_{XY}Z = \frac{c+3}{4}\{g(X,Z)Y - g(Y,Z)X\} + \frac{c-1}{4}\{\eta(Y)\eta(Z)X \\ - \eta(X)\eta(Z)Y - g(Z,\varphi Y)\varphi X + g(Z,\varphi X)\varphi Y \\ - 2g(X,\varphi Y)\varphi Z - g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{12}$$

and such a manifold is called a *Sasakian space form*. Note that c is automatically globally constant for $\dim M \geq 5$. Here and in the rest of the paper we shall suppose, if necessary, that M is a connected manifold. The Sasakian space forms have been classified completely (see for example [1], [24]) and locally there are three classes according to $c+3>0$, $c+3=0$ and $c+3<0$.

The following characterization of Sasakian manifolds of constant φ-sectional curvature was given by Tanno:

Theorem D [22]. *A connected Sasakian manifold M of dimension ≥ 5 is a Sasakian space form if and only if, for every horizontal vector X, $R_{X\varphi X}X$ is co-linear with φX.*

This theorem is similar to the following one in Kähler geometry:

Theorem E [22]. *A connnected Kähler manifold (M,G,J) of dimension ≥ 4 is a complex space form (that is, it has constant holomorphic sectional curvature) if and only if, for any vector $X, R_{X\,JX}X$ is proportional to JX.*

Moreover, these two theorems may be related to the following one which follows at once using (5) and (7):

Theorem F [18]. *A connected Sasakian manifold has constant φ-sectional curvature if and only if the holomorphic sectional curvature of each base manifold is constant.*

3. Geometry of tubes

Suppose $\sigma : [a, b] \to M$ is a smooth, embedded curve in a connected n-dimensional Riemannian manifold (M, g) of class C^∞ and denote by $\sigma^\perp$ the normal bundle of σ and by $\exp_\sigma$ the exponential map of this normal bundle, i.e.,

$$\exp_\sigma(\sigma(t), V) = \exp_{\sigma(t)} V$$

for any $t \in [a, b]$ and all $V \in \sigma(t)^\perp$ where $\sigma(t)^\perp$ denotes the fiber of $\sigma^\perp$ over $\sigma(t)$. The set

$$\mathcal{U}_\sigma(r) = \{\exp_{\sigma(t)} V \mid V \in \sigma(t)^\perp,\ \|V\| < r,\ t \in [a, b]\}$$

is said to be the *(open) tubular neighborhood* or the *(open) solid tube* of radius r about σ. Since $[a, b]$ is compact and since $\sigma : [a, b] \to M$ is an embedding, we shall always assume that the radius r of this tubular neighborhood is smaller than the distance from σ to its nearest focal point. In this case, the exponential map $\exp_\sigma$ is a diffeomorphism between $\mathcal{U}_\sigma(r)$ and the so-called (open) solid tube $N_\sigma(r)$ of radius r about the zero section of the normal bundle $\sigma^\perp$ of σ, that is,

$$N_\sigma(r) = \bigcup_{t \in [a,b]} \{U \in \sigma(t)^\perp \mid \|U\| < r\}.$$

Consequently, for small $s > 0$ the set

$$\mathcal{P}_\sigma(s) = \{p \in \mathcal{U}_\sigma(r) \mid d(\sigma, p) = s\}$$

is a smooth hypersurface in M called the *tubular hypersurface* or just the *tube* of radius s about σ. If σ is a geodesic on M, the tubes $\mathcal{P}_\sigma$ are called *geodesic tubes* of M about σ.

To study the geometry of tubular neighborhoods and tubes, we use a special type of coordinate systems, the so-called Fermi coordinate systems [15], [16], [23], which may be introduced as follows. Suppose that $\sigma : [a, b] \to M$ is a unit speed curve and let $\{f_i,\ i = 1, \ldots, n\}$ be an orthonormal basis of $T_{\sigma(a)}M$ such that $f_1 = \dot\sigma(a)$. Further, let F_1 be the unit tangent field $\dot\sigma$ and $F_2, \ldots, F_n$ the normal vector fields along σ which are parallel with respect to the normal connection $\nabla^\perp$ of the normal bundle $\sigma^\perp$ and such that $F_i(a) = f_i,\ i = 2, \ldots, n$. (Note that, in the special case where σ is a geodesic, this translation is just the parallel translation with respect to the Levi Civita connection ∇.) Then the *Fermi coordinate* *system* $(x_1, \dots, x_n)$ with respect to $\sigma(a)$ and $\{F_1, \dots, F_n\}$ is defined by

$$
\begin{aligned}
&x_1\left(\exp_{\sigma(t)}\left(\sum_{j=2}^{n} t_j F_j\right)\right) = t - a,\\
&x_i\left(\exp_{\sigma(t)}\left(\sum_{j=2}^{n} t_j F_j\right)\right) = t_i, \quad i = 2, \dots, n.
\end{aligned} \tag{13}
$$

The extrinsic geometry of the tube $\mathcal{P}_\sigma(r)$ is determined by its shape operator S^σ and we shall now show that the use of Jacobi vector fields along geodesics orthogonal to the curve σ leads to a useful expression for this shape operator. In general, σ is not a geodesic, but for our purposes it is enough to assume that $\mathcal{P}_\sigma$ is a geodesic tube. Further, let $p = \exp_{\sigma(t)}(rv)$, $v \in \sigma(t)^\perp$, $\|v\| = 1$ be a point of $\mathcal{P}_\sigma(r)$ and suppose $\gamma : s \to \exp_{\sigma(t)}(sv)$ is the unit speed geodesic connecting $\sigma(t)$ and p. We specify the frame field $\{F_1, \dots, F_n\}$ along σ such that $F_1(t) = \dot{\sigma}(t)$ and $F_2(t) = \gamma'(0) = v$, and denote by $\{E_1, \dots, E_n\}$ the frame field along γ obtained by parallel translation of $\{F_1(t), \dots, F_n(t)\}$ with respect to the Levi Civita connection ∇. Next, let $Y_1, Y_2, \dots, Y_n$ be the Jacobi vector fields along γ satisfying the initial conditions

$$
\begin{cases}
Y_1(0) = F_1(t), & Y_1'(0) = \left(\nabla_{\gamma'} \frac{\partial}{\partial x_1}\right)(\sigma(t)),\\
Y_i(0) = 0, & Y_i'(0) = F_i(t), \quad i = 3, \dots, n.
\end{cases} \tag{14}
$$

Then it is easy to see that these Jacobi vector fields are related to the basic vector fields $\frac{\partial}{\partial x_i}$ of the Fermi coordinate system with respect to $\sigma(a)$ and $\{F_1, \dots, F_n\}$ as follows:

$$
\begin{cases}
Y_1(s) = \frac{\partial}{\partial x_1}(\gamma(s)),\\
Y_i(s) = s\frac{\partial}{\partial x_i}(\gamma(s)), \quad i = 3, \dots, n.
\end{cases} \tag{15}
$$

Now, define an automorphism-valued function $B : s \to B(s)$ by

$$
Y_i(s) = (BE_i)(s), \quad i = 1, 3, \dots, n. \tag{16}
$$

Then it follows that B satisfies the Jacobi equation

$$
B'' + R \circ B = 0 \tag{17}
$$

with initial conditions

$$
B(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B'(0) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \tag{18}
$$

where $R = R_{\gamma'} \cdot \gamma'$. Further, since $\frac{\partial}{\partial r}$ is a unit normal vector of $\mathcal{P}_\sigma(r)$ at $p = \exp_{\sigma(t)}(rv)$ the *shape operator* S^σ of $\mathcal{P}_\sigma(r)$ at p is defined by

$$(S^\sigma X)(p) = \left(\nabla_X \frac{\partial}{\partial r}\right)(p) \tag{19}$$

for any vector X tangent to $\mathcal{P}_\sigma(r)$ at p. Now, applying this definition to the special tangent vectors Y_i, $i = 1, 3, \ldots, n$ (which form a basis for the tangent space to $\mathcal{P}_\sigma(r)$ at p) and using (16), the expression for the shape operator $S^\sigma(p)$ takes the form

$$S^\sigma(p) = (B' B^{-1})(r). \tag{20}$$

From (17) and the initial conditions (18) one can compute a power series expansion for B in terms of the arc length s along γ. Using (19), one then derives power series expansions for the components of the shape operator $S^\sigma_{\alpha\beta}(p) = g(S^\sigma E_\alpha, E_\beta)(p)$, $\alpha, \beta = 1, 3, \ldots, n$. In what follows we shall write R_{xywz} for the value of the curvature tensor field on tangent vectors x, y, w, z at some point $m \in M$. (Since R is a tensor field, we have $(R_{XYWZ})_m = R_{xywz}$ for any vector fields X, Y, W, Z such that $X_m = x$, and so forth.) We have

$$\begin{cases} S^\sigma_{11}(p) = -rR_{1v1v} - \frac{r^2}{2}\nabla_v R_{1v1v} + O(r^3), \\ S^\sigma_{1\alpha}(p) = -\frac{r}{2}R_{1v\alpha v} - \frac{r^2}{3}\nabla_v R_{1v\alpha v} + O(r^3), \\ S^\sigma_{\alpha\beta}(p) = \frac{1}{r}\delta_{\alpha\beta} - \frac{r}{3}R_{\alpha v\beta v} - \frac{r^2}{4}\nabla_v R_{\alpha v\beta v} + O(r^3), \end{cases} \tag{21}$$

where $\alpha, \beta = 3, \ldots, n$ and R denotes the Riemannian curvature tensor of the ambient space M. As a notational matter, we have posed here $R_{\alpha v\beta v} = R_{F_\alpha v F_\beta v}(\sigma(t))$ and $\nabla_v R_{\alpha v\beta v} = (\nabla_v R)_{F_\alpha v F_\beta v}(\sigma(t))$ for $\alpha, \beta = 3, \ldots, n$. It should be noted that for these expressions the spaces $\{\gamma'(0)\}^\perp$ and $\{\gamma'(r)\}^\perp$ are identified via the parallel orthonormal basis. For more details we refer to [13], [15], [16] and [23].

The *Ricci operator* Q^σ of the tube $\mathcal{P}_\sigma(r)$ contains information about the intrinsic geomery of the tube $\mathcal{P}_\sigma(r)$ and using the method of Jacobi vector fields and the Gauss equation one obtains its power series expansion. This has been done in [13] and we shall need the formulas for the components of the *Ricci tensor* ρ^σ of the geodesic tube $\mathcal{P}_\sigma(r)$ at $p = \exp_{\sigma(t)}(rv)$:

$$\rho^\sigma_{11}(p) = \rho_{11} - (n-1)R_{1v1v} + r\left(\nabla_{v\rho_{11}} - \frac{n}{2}\nabla_v R_{1v1v}\right)$$
$$+ r^2\Big(\frac{1}{2}\nabla^2_{vv}\rho_{11} - \frac{n+1}{6}\nabla^2_{vv}R_{1v1v} + \frac{1}{3}R_{1v1v}\rho_{vv}$$

$$- \frac{n-1}{3} R_{1v1v}^2 - \frac{n+1}{12} \sum_{\lambda=3}^{n} R_{1v\lambda v}^2 \Big) + O(r^3),$$

$$\rho_{1\alpha}^\sigma(p) = \rho_{1\alpha} - \frac{n-1}{2} R_{1v\alpha v} + r \left(\nabla_v \rho_{1\alpha} - \frac{n}{3} \nabla_v R_{1v\alpha v} \right)$$
$$+ r^2 \Big(\frac{1}{2} \nabla_{vv}^2 \rho_{1\alpha} - \frac{n+1}{8} \nabla_{vv}^2 R_{1v\alpha v} + \frac{1}{6} R_{1v\alpha v} \rho_{vv}$$
$$- \frac{3n-5}{24} R_{1v1v}^2 R_{1v\alpha v} - \frac{n+1}{24} \sum_{\lambda=3}^{n} R_{1v\lambda v} R_{\alpha v\lambda v} \Big) + O(r^3),$$

$$\rho_{\alpha\beta}^\sigma(p) = \frac{n-3}{r^2} \delta_{\alpha\beta} + \left(\rho_{\alpha\beta} - \frac{n-1}{3} R_{\alpha v\beta v} - \frac{1}{3} \rho_{vv} \delta_{\alpha\beta} - \frac{2}{3} R_{1v1v} \delta_{\alpha\beta} \right)$$
$$+ r \left(\nabla_v \rho_{\alpha\beta} - \frac{n}{4} R_{\alpha v\beta v} - \frac{1}{4} \nabla_v \rho_{vv} \delta_{\alpha\beta} - \frac{1}{4} \nabla_v R_{1v1v} \delta_{\alpha\beta} \right)$$
$$+ r^2 \Bigg(\frac{1}{2} \nabla_{vv}^2 \rho_{\alpha\beta} - \frac{n+1}{10} \nabla_{vv}^2 R_{\alpha v\beta v} + \frac{1}{9} R_{\alpha v\beta v} \rho_{vv} + \frac{2}{9} R_{1v1v} R_{\alpha v\beta v}$$
$$- \frac{n-1}{20} R_{1v\alpha v} R_{1v\beta v} - \frac{n+1}{45} \sum_{\gamma=3}^{n} R_{\alpha v\gamma v} R_{\beta v\gamma v} - \frac{1}{10} \nabla_{vv}^2 \rho_{vv} \delta_{\alpha\beta}$$
$$- \frac{1}{15} \nabla_{vv}^2 R_{1v1v} \delta_{\alpha\beta} - \frac{1}{3} R_{1v1v}^2 \delta_{\alpha\beta} - \frac{2}{15} \sum_{\gamma=3}^{n} R_{1v\gamma v}^2 \delta_{\alpha\beta}$$
$$- \frac{1}{45} \sum_{\gamma,\mu=3}^{n} R_{\gamma v\mu v}^2 \delta_{\alpha\beta} \Bigg) + O(r^3), \tag{22}$$

where $\alpha, \beta = 3, \dots, n$ and ρ denotes the Ricci tensor of M.

Next, we consider the special case of Sasakian space forms and we study Jacobi vector fields on these manifolds in order to obtain the expression of the shape operator at some special points of tubes along φ-geodesics in these spaces. A similar technique was used in [3] where the Jacobi vector fields were used to study the shape operator of a geodesic sphere, in [7] where these fields were used in the treatment of local symmetries with respect to φ-geodesics and in [12] where they were used in a related study of the shape operator of tubes about φ-geodesics.

Let m be a point on a Sasakian manifold M^{2n+1} with structure tensors (φ, ξ, η, g) and of constant φ-sectional curvature c. Further, let γ be a φ-geodesic, parametrized by arc length s, through $m = \gamma(0)$ with initial velocity vector $\gamma'(0) = v$. Hereafter we shall also write $\gamma'(s) = v$ at any point of γ. For a horizontal vector field V the Jacobi equation

$$\nabla_V \nabla_V X + R_{VX} V = 0 \tag{23}$$

for a given Sasakian space form M^{2n+1} becomes by virtue of (12)

$$\nabla_V\nabla_V X + \frac{c+3}{4}(X - g(V,X)V) - \frac{c-1}{4}(\eta(X)\xi + 3g(V,\varphi X)\varphi V) = 0. \tag{24}$$

To solve (24), we choose an orthonormal basis $\{e_1, \ldots, e_{2n+1}\}$ at m adapted to our constructions, i.e., such that: $e_1 = v$, $e_{2n-1} = u$, $e_{2n} = \xi$ and $e_{2n+1} = \varphi v$, (where u is a horizontal vector orthogonal to v and φv) and we denote by $\{E_1, \ldots, E_{2n+1}\}$ the orthonormal basis along γ obtained by parallel translation of the vectors e_i along γ. Since it follows from the Sasakian condition (2) and its consequence (3) that the two-plane $\{\xi, \varphi v\}$ is parallel along γ, we obtain easily along γ

$$\begin{cases} E_{2n} = \varphi v \sin s + \xi \cos s, \\ E_{2n+1} = \varphi v \cos s - \xi \sin s, \end{cases} \tag{25}$$

s being the arc length from m along γ.

Further, from our construction of the frame field $\{E_1, \ldots, E_{2n+1}\}$ it is easily seen that any vector field X orthogonal to the φ-geodesic γ can be written as

$$X = \sum_{a=2}^{2n-2} l_a E_a + l_{2n-1}E_{2n-1} + l_{2n}E_{2n} + l_{2n+1}E_{2n+1} \tag{26}$$

and hence we easily see that (24) is equivalent to the following system of differential equations:

$$l_a'' + \frac{c+3}{4} l_a = 0, \quad a = 2, \ldots, 2n-1, \tag{27}$$

$$\begin{cases} l_{2n}'' + l_{2n} + (c-1)\sin s(l_{2n}\sin s + l_{2n+1}\cos s) = 0, \\ l_{2n+1}'' + l_{2n+1} + (c-1)\cos s(l_{2n}\sin s + l_{2n+1}\cos s) = 0. \end{cases} \tag{28}$$

The solutions of the $2n-2$ equations (27) are standard and it can be shown that the equations (28) lead to two other equations where one is of the same form as (27) and the other is still more elementary. We refer to [3], [6] and [7] for more details about these solutions, emphasizing that one has to consider three cases according to $c+3 > 0$, $c+3 < 0$ and $c+3 = 0$.

In the further text we shall need the solutions of the Jacobi differential equation along a φ-geodesic γ satisfying the following initial conditions:

$$X_i(0) = 0, \qquad X_i'(0) = e_i, \; i = 2, \ldots, 2n-2, 2n, 2n+1, \tag{29}$$

$$(30) \qquad X_{2n-1}(0) = u, \quad X'_{2n-1}(0) = 0.$$

We shall therefore give these special solutions for the three cases, where we use the notation $k = \sqrt{c+3}$ if $c+3 > 0$ and $k = \sqrt{-(c+3)}$ if $c+3 < 0$. See [7] for details. First, we compute the Jacobi vector fields $X_i(s)$, $i = 2, \dots, 2n-2$ along γ with the initial conditions (29) and for the special orthonormal basis $\{E_1(s), \dots, E_{2n+1}(s)\}$ established at the beginning of this section. Using the previous formulas we obtain

	$c+3<0$	$c+3=0$	$c+3>0$
$X_i(s)$	$\frac{2}{k}\sinh\frac{k}{2}sE_a(s)$	$sE_a(s)$	$\frac{2}{k}\sin\frac{k}{2}sE_a(s)$

for $i = 2, \dots, 2n-2$. Next, consider the Jacobi vector field $X_{2n-1}(s)$ along γ satisfying the initial conditions (30). From the previous formulas we get

	$c+3<0$	$c+3=0$	$c+3>0$
$X_{2n-1}(s)$	$\cosh\frac{k}{2}sE_{2n-1}(s)$	$E_{2n-1}(s)$	$\cos\frac{k}{2}sE_{2n-1}(s)$

Finally, we consider the Jacobi vector fields $X_{2n}(s)$ and $X_{2n+1}(s)$ along γ satisfying the initial conditions (29) and from the previous formulas we obtain

$$X_{2n}(s) = (\rho\sin s + \lambda\cos s)E_{2n}(s) + (\rho\cos s - \lambda\sin s)E_{2n+1}(s),$$
$$X_{2n+1}(s) = (\nu\sin s - \rho\cos s)E_{2n}(s) + (\nu\cos s + \rho\sin s)E_{2n+1}(s)$$

where

	$c+3<0$	$c+3=0$	$c+3>0$
ρ	$\frac{2}{k^2}(\cosh ks - 1)$	s^2	$-\frac{2}{k^2}(\cos ks - 1)$
λ	$-\frac{4}{k^3}\sinh ks - \frac{c-1}{k^2}s$	$-\frac{2}{3}s^3 + s$	$\frac{4}{k^3}\sin ks + \frac{c-1}{k^2}s$
ν	$\frac{1}{k}\sinh ks$	s	$\frac{1}{k}\sin ks.$

Further, consider the tube $\mathcal{P}_\sigma(r)$ of radius r about the φ-geodesic σ embedded in M. Let γ denote the unit-speed geodesic meeting σ orthogonally at $m = \sigma(t)$ and tangent to a horizontal vector v such that v is also orthogonal to φu at m where $u = \dot{\sigma}$ at m. We close this section by giving the explicit formulas for the shape operator S^σ at $p = \exp_{\sigma(t)}(rv)$

of the sufficiently small geodesic tubes about φ-geodesics when the ambient space M is a Sasakian space form. For this purpose we suppose that $\{E_1, \ldots, E_{2n+1}\}$ is an orthonormal frame field along γ defined as above and use relations (15)–(20) and the previously computed Jacobi vector fields. An easy calculation shows that the shape operator S^σ can be represented by the quasi-diagonal matrix

$$S^\sigma(p) = \begin{pmatrix} A(r) & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & A(r) & 0 & 0 & 0 \\ 0 & \ldots & 0 & B(r) & 0 & 0 \\ 0 & \ldots & 0 & 0 & C(r) & D(r) \\ 0 & \ldots & 0 & 0 & D(r) & E(r) \end{pmatrix} \tag{31}$$

with respect to the basis $\{E_2, \ldots, E_{2n-1}, \xi, \varphi v\}$. The explicit expressions for the entries are as follows:

Case 1. $c + 3 = 0$

$$A(r) = \frac{1}{r}, \qquad B(r) = 0, \qquad C(r) = \frac{3}{r(r^2+3)},$$
$$D(r) = -\frac{r^2}{r^2+3}, \quad E(r) = \frac{4r^2+3}{r(r^2+3)};$$

Case 2. $c + 3 > 0$

$$A(r) = \frac{k}{2}\cot\frac{kr}{2}, \qquad B(r) = -\frac{k}{2}\tan\frac{kr}{2},$$
$$C(r) = \frac{k^3}{\omega}\sin kr, \qquad D(r) = -\frac{k^2-4}{\omega}(kr\sin kr + 2\cos kr - 2),$$
$$E(r) = \frac{k}{\omega}(4\sin kr + kr(k^2-4)\cos kr),$$

where $\omega = k^3 r \sin kr - 4kr\sin kr - 8\cos kr + 8$;

Case 3. $c + 3 < 0$

$$A(r) = -\frac{k}{2}\coth\frac{kr}{2}, \qquad B(r) = \frac{k}{2}\tanh\frac{kr}{2},$$
$$C(r) = -\frac{k^3}{\theta}\sinh kr, \qquad D(r) = \frac{k^2+4}{\theta}(kr\sinh kr - 2\cosh kr + 2),$$
$$E(r) = \frac{k}{\theta}(4\sin kr - kr(k^2+4)\cos kr),$$

where $\theta = 8\cosh kr - k^3 r\sinh kr - 4kr\sinh kr - 8$.

Finally, it follows easily from the expressions for the curvature tensor of the Sasakian space form (M,φ,ξ,η,g) and from the Gauss equation that the Ricci operator $Q^\sigma(p)$ of $\mathcal{P}_\sigma(r)$ has a similar expression as that of $S^\sigma(p)$ in (31) but with other functions as entries.

4. Extrinsic geometry of tubes about φ-geodesics on Sasakian manifolds

In this section we derive some new characterizations of Sasakian space forms and locally φ-symmetric spaces related to the shape operator. We show that its action has some very particular features when restricted to the parallel plane $\{\xi,\varphi v\}$ along a φ-geodesic γ through $\sigma(t)$, tangent to v and meeting $\dot\sigma(t)$ and $\varphi\dot\sigma(t)$ orthogonally. From now on, we call these tubes *φ-geodesic tubes* and we use the notations as at the end of Section 3.

Here we also note that along γ we have $v = \gamma'(s) = \frac{\partial}{\partial s}$ and at $p = \exp_m(rv)$ this vector is a unit normal vector of $\mathcal{P}_\sigma(r)$.

Theorem 1. *Let $(M^{2n+1},\varphi,\eta,\xi,g)$ be a Sasakian manifold of dimension ≥ 5. Then, with the conventions made above, M is a Sasakian space form if and only if, for every sufficiently small φ-geodesic tube $\mathcal{P}_\sigma(r)$, $S^\sigma(p)\varphi v$ (or $S^\sigma(p)\xi$, respectively) belongs to the plane $\{\xi,\varphi v\}(p)$ for all $m\in M$ and all v, where $p=\exp_m(rv)$.*

Proof. First, if M is a Sasakian space form, using the explicit expressions for the shape operator S^σ of the tube $\mathcal{P}_\sigma(r)$ at p and derived in Section 3, we see immediately that in this case S^σ preserves the plane $\{\xi,\varphi v\}(p)$.

To prove the converse we first write down the formulas for $S^\sigma(p)\varphi v$ and $S^\sigma(p)\xi$ using (21) and (25):

$$(32)\quad \begin{aligned} S^\sigma(p)\varphi v &= \frac{1}{r}\varphi v + \xi - \frac{r}{6}\left[3\varphi v + 2R_{v\varphi v}v + R_{uv\varphi vv}u\right] \\ &\quad - \frac{r^2}{12}\left[6\xi + (\nabla_v R)_{uv\varphi vv}u + 3(\nabla_v R)_{v\varphi v}v\right] + O(r^3), \end{aligned}$$

$$(33)\quad \begin{aligned} S^\sigma(p)\xi &= \frac{1}{r}\xi - \varphi v - \frac{5}{6}r\xi \\ &\quad + \frac{r^2}{12}\left[2\varphi v + R_{v\varphi vvu}u + R_{v\varphi v}v\right] + O(r^3). \end{aligned}$$

Now, $S^\sigma(p)\varphi v$ belongs to the parallel plane field $\{\xi(p),\varphi v(p)\}$ if and only if

$$g(S^\sigma(p)\varphi v, x(r)) = 0$$

for all parallel vectors x which are orthogonal to that plane along the φ-geodesic γ. Using (32), this gives

$$R_{v\varphi vvx} + R_{v\varphi vvu}\, g(u,x) = 0 \tag{34}$$

at all $m \in M$, for all horizontal u and all horizontal v orthogonal to u and φu at m. Further, since u is orthogonal to ξ and φv at m, after replacing u by x in relation (34), we get $R_{v\varphi vvu} = 0$. Therefore, (34) implies $R_{v\varphi vvx} = 0$ and then, since $R_{v\varphi vv\xi} = 0$, we obtain that $R_{v\varphi v}v$ is proportional to φv. Finally, using Theorem D, we conclude that M is a Sasakian space form.

To complete the proof, it is enough to use next (33) and to repeat a similar procedure.

Theorem 2. *Let $(M^{2n+1}, \varphi, \eta, \xi, g)$ be a Sasakian manifold of dimension ≥ 5. Then with the conventions made above, M is a Sasakian space form if and only if, for all v and for every sufficiently small φ-geodesic tube $\mathcal{P}_\sigma(r)$, the integral curves of $\varphi v = \varphi\frac{\partial}{\partial s}$ on $\mathcal{P}_\sigma(r)$ are geodesics on these tubes.*

Proof. Let M be a Sasakian manifold and denote by $\tilde{\nabla}$ the induced Riemannian connection on $\mathcal{P}_\sigma(r)$. Then the integral curves of φv are geodesics if and only if

$$\tilde{\nabla}_{\varphi v}(\varphi v) = 0,$$

that is, if and only if

$$g(\nabla_{\varphi v}(\varphi v), X) = 0$$

for all vectors X tangent to $\mathcal{P}_\sigma(r)$. Using (1) and (2), we see that this is equivalent to the condition

$$g(S^\sigma(p)\varphi v, \varphi X) = 0,$$

which means that $S^\sigma(p)\varphi v$ must belong to the plane $\{\xi, \varphi v\}(p)$ and conversely. Hence, the result follows from Theorem 1.

Remark. We also note that it is easy to prove when $\dim M \geq 5$ that a Sasakian manifold has constant φ-sectional curvature if and only if $\varphi v = \varphi\frac{\partial}{\partial r}$ on each sufficiently small $\mathcal{P}_\sigma(r)$ satisfies the Killing equation at the points $p = \exp_m(rv)$, for all $m \in M$ and all v with the conventions as above.

To finish this section we give one characterization of locally φ-symmetric spaces. Therefore we consider the geodesic of $\mathcal{P}_\sigma(r)$ tangent to φv at $p = \exp_m(rv)$. Its curvature $\kappa^\sigma(p)$ in M is given by

$$\kappa^\sigma(p) = g(S^\sigma(p)\varphi v, \varphi v). \tag{35}$$

Using this real valued function and the φ-geodesic symmetry s_m centered at m, we obtain

Theorem 3. *Let $(M^{2n+1}, \varphi, \eta, \xi, g)$ be a Sasakian manifold. Then, with the conventions made above, M is locally φ-symmetric space if and only if*

$$\kappa^\sigma(p) = \kappa^\sigma(s_m(p)), \quad p = \exp_m(rv), \tag{36}$$

for all $m \in M$, all v and all sufficiently small r.

Proof. First, if M is a locally φ-symmetric Sasakian manifold, owing to Theorem A (i), s_m is a local automorphism of the Sasakian structure (φ, η, ξ, g) and hence we obtain the result at once.

We now prove the converse. Using (25) and (32), we get for (35)

$$\kappa^\sigma(p) = \frac{1}{r} - 3r + \frac{8r}{3} R_{v\varphi v\, v\varphi v} - \frac{r^2}{4} (\nabla_v R)_{v\varphi v\, v\varphi v} + O(r^3). \tag{37}$$

Then (36) and (37) imply

$$(\nabla_v R)_{v\varphi v\, v\varphi v} = 0$$

for all horizontal v orthogonal to u and φu at m and all $m \in M$. Now, the required result follows by using Theorem C, since the horizontal vector u may be chosen arbitrarily.

5. Intrinsic geometry of tubes about φ-geodesics on Sasakian manifolds

This section is devoted to research about the intrinsic geometry of φ-geodesic tubes by treating their Ricci operator Q^σ and its action restricted to the parallel plane $\{\xi, \varphi v\}$ along a φ-geodesic γ as before. Analogous theorems as for the shape operator S^σ will be proved here, although in several cases the proofs require a more elaborated treatment.

Theorem 4. *Let $(M^{2n+1}, \varphi, \eta, \xi, g)$ be a Sasakian manifold of dimension ≥ 5. Then with the conventions made before, M is a Sasakian space form if and only if, for every sufficiently small φ-geodesic tube $\mathcal{P}_\sigma(r)$, $Q^\sigma(p)\varphi v$ (or $Q^\sigma(p)\xi$, respectively) belongs to the plane $\{\xi, \varphi v\}(p)$ for all $m \in M$ and all v, where $p = \exp_m(rv)$.*

Proof. We start with the "if" part of the theorem. Therefore, let $\{E_1, \ldots, E_n\}$ be an orthonormal basis parallel along γ defined as in Section 3. In that way, using (22) and (25), we get

$$\begin{aligned} Q^\sigma(p)\varphi v = {} & \frac{2n-2}{r^2}\varphi v + \frac{2n-2}{r}\xi + Q\varphi v - \frac{2n}{3} R_{v\varphi v} v \\ & - \frac{n}{3} R_{v\varphi v v u} u - \left[n - 1 + \frac{1}{3}(\rho_{vv} + 2R_{vuvu}) \right] \varphi v \end{aligned}$$

$$
\begin{aligned}
&+ r\Big\{\frac{1}{3}(3n+1-\rho_{vv}-2R_{vuvu})\xi+(\nabla_v Q)\varphi v-\frac{2n+1}{4}(\nabla_v R)_{v\varphi v}v\\
&-\frac{2n+1}{12}(\nabla_v R)_{v\varphi vvu}u-\frac{1}{4}\left[(\nabla_v\rho)_{vv}+(\nabla_v R)_{vuvu}\right]\varphi v\Big\}\\
&+ r^2\Big\{\frac{1}{2}Q\varphi v-\frac{2n+3}{12}R_{v\varphi v}v+\frac{1}{2}(\nabla^2_{vv}Q)\varphi v-\frac{n+1}{5}(\nabla^2_{vv}R)_{v\varphi v}v\\
&+\frac{1}{9}\rho_{vv}R_{v\varphi v}v+\frac{2}{9}R_{vuvu}R_{v\varphi v}v-\frac{n+1}{18}R_{v\varphi vvu}R_{vu}v\\
&-\frac{2n+2}{45}R_{vR_{v\varphi v}v}v+\Big[\frac{2-17n}{12}+\frac{1}{6}(\rho_{vv}+2R_{vuvu})-\frac{1}{10}(\nabla^2_{vv}\rho)_{vv}\\
&-\frac{1}{15}(\nabla^2_{vv}R)_{vuvu}-\frac{1}{3}R^2_{vuvu}-\frac{2}{15}\sum_{\lambda=3}^{2n+1}R^2_{v\lambda vu}-\frac{1}{45}\sum_{\lambda,\mu=3}^{2n+1}R^2_{v\lambda v\mu}\Big]\varphi v\\
&-\frac{1}{4}\left[(\nabla_v\rho)_{vv}+(\nabla_v R)_{vuvu}\right]\xi+\Big[-\frac{1}{12}R_{v\varphi vvu}\\
&-\frac{n+1}{20}(\nabla^2_{vv}R)_{v\varphi vvu}+\frac{1}{18}\rho_{vv}R_{v\varphi vvu}-\frac{27n+7}{180}R_{vuvu}R_{v\varphi vvu}\\
&-\frac{7n+7}{180}\sum_{\lambda=3}^{2n+1}R_{v\varphi vv\lambda}R_{v\lambda vu}\Big]u\Big\}+O(r^3),
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
Q^\sigma(p)\xi &= \frac{2n-2}{r^2}\xi-\frac{2n-2}{r}\varphi v+\frac{1}{3}\left(n+3-\rho_{vv}-2R_{vuvu}\right)\xi\\
&+ r\Big\{\frac{2n-3}{12}R_{v\varphi v}v+\frac{2n-1}{12}R_{v\varphi vvu}u+\Big[-\frac{14n+1}{12}+\frac{1}{3}(\rho_{vv}\\
&+2R_{vuvu})\Big]\varphi v-\frac{1}{4}\left[(\nabla_v\rho)_{vv}+(\nabla_v R)_{vuvu}\right]\xi\Big\}\\
&+ r^2\Big\{(\nabla_v Q)\varphi v-\frac{2n+2}{5}(\nabla_v R)_{v\varphi v}v-\frac{n+1}{10}(\nabla_v R)_{v\varphi vvu}u\\
&+\Big[\frac{5}{18}\rho_{vv}-\frac{113n+23}{180}+\frac{5}{9}R_{vuvu}-\frac{1}{10}(\nabla^2_{vv}\rho)_{vv}-\frac{1}{15}(\nabla^2_{vv}R)_{vuvu}\\
&-\frac{1}{3}R^2_{vuvu}-\frac{2}{15}\sum_{\lambda=3}^{2n+1}R^2_{v\lambda vu}-\frac{1}{45}\sum_{\lambda,\mu=3}^{2n+1}R^2_{v\lambda v\mu}\Big]\xi\Big\}+O(r^3).
\end{aligned}
\tag{39}
$$

First we consider the action of Q^σ on φv and for this purpose let X be a parallel vector field along γ with $X(0)=x$ and orthogonal to the parallel plane spanned by $\xi(p)$ and $(\varphi v)(p)$. Then it follows that $Q^\sigma(p)\varphi v$ belongs to that plane if and only if

$$
g(Q^\sigma(p)\varphi v, X(p)) = 0
$$

for all such X. Using (38), this implies the following necessary conditions:

(40) $3g(Q\varphi v,x) = 2nR_{v\varphi vvx} + nR_{v\varphi vvu}g(u,x)$,

(41) $12g((\nabla_v Q)\varphi v,x) = (2n+1)\Big(3(\nabla_v R)_{v\varphi vvx} + (\nabla_v R)_{v\varphi vvu}g(u,x)\Big)$,

(42)
$$\frac{1}{2}g(Q\varphi v,x) + \frac{1}{2}g\Big((\nabla^2_{vv}Q)\varphi v,x\Big) - \frac{2n+3}{12}R_{v\varphi vvx} - \frac{n+1}{5}(\nabla^2_{vv}R)_{v\varphi vvx} + \frac{1}{9}\rho_{vv}R_{v\varphi vvx} + \frac{2}{9}R_{uvuv}R_{v\varphi vvx} - \frac{n+1}{18}R_{v\varphi vvu}R_{vuvx} - \frac{2n+2}{45}R_{vR_{v\varphi v}vvx}$$
$$= \Big(\frac{1}{6}R_{v\varphi vvu} + \frac{n+1}{20}(\nabla^2_{vv}R)_{v\varphi vvu} - \frac{1}{18}\rho_{vv}R_{v\varphi vvu} + \frac{27n+7}{180}R_{uvuv}R_{v\varphi vvu} + \frac{7n+7}{180}\sum_{\lambda=3}^{2n+1}R_{v\varphi vv\lambda}R_{v\lambda vu}\Big)g(u,x).$$

Differentiation of (40) leads to

(43) $$3g\Big((\nabla_v Q)\varphi v,x\Big) = n\Big(2(\nabla_v R)_{v\varphi vvx} + (\nabla_v R)_{v\varphi vvu}g(u,x)\Big)$$

and then (41) implies

(44) $$(3-2n)(\nabla_v R)_{v\varphi vvx} = (2n-1)(\nabla_v R)_{v\varphi vvu}g(u,x).$$

Since u is orthogonal to ξ and φv at m, replacing x by u in the last relation gives

(45) $$(\nabla_v R)_{v\varphi vvu} = 0.$$

The last relation yields together with (44)

(46) $$(\nabla_v R)_{v\varphi vvx} = 0$$

and hence

(47) $$(\nabla^2_{vv} R)_{v\varphi vvx} = -R_{v\varphi vvx}.$$

The local fibration formulas (5) and (10) show that (46) is equivalent to

(48) $$(\overline{\nabla}_w \overline{R})_{wJwwy} = 0,$$

at $\bar{m} = \pi(m)$, where $v = w^*,\ x = y^*$, for all (unit) w orthogonal to Jy. To handle this condition we use the method of integration, instead of the usual linearization and contraction technique. (See for example [8], [14],[10], [12] for more details.) Using now the well-known integration formulas, the last relation implies

$$2\overline{\nabla}_y\overline{\tau} - 7(\overline{\nabla}\overline{\rho})(y,y) + 3(\overline{\nabla}_y\overline{R})_{yJyyJy} = 0 \tag{49}$$

for all unit vectors y of $T_{\bar{m}}\mathcal{B}$.

Further, using the Kähler and the second Bianchi identity in relation (48) when $w \in \{y, Jy\}^{\perp}$, and integrating the obtained relation over the unit sphere $S^{2n-3}(1)$ in $\{y, Jy\}^{\perp} \subset T_{\bar{m}}\mathcal{B}$, we get

$$\overline{\nabla}_y\overline{\tau} - 4(\overline{\nabla}_y\overline{\rho})(y,y) + 2(\overline{\nabla}_y\overline{R})_{yJyyJy} = 0. \tag{50}$$

Then, (49) and (50) yield

$$\overline{\nabla}_y\overline{\tau} - 2(\overline{\nabla}_y\overline{\rho})(y,y) = 0. \tag{51}$$

Linearization and contraction of this relation yields

$$\overline{\nabla}_z\overline{\tau} = 0.$$

Finally, using (50) and (51), the last relation implies

$$(\overline{\nabla}_y\overline{R})_{yJyyJy} = 0, \tag{52}$$

for all unit vectors y of $T_{\bar{m}}\mathcal{B}$. Theorem A (ii) then implies that $\mathcal{B}$ is locally Hermitian symmetric and using Theorem B we may conclude that M is locally φ-symmetric.

Further, since u is orthogonal to ξ and φv at m, replacing x by u in (40) gives

$$g(Q\varphi v, u) = nR_{v\varphi vvu} \tag{53}$$

and hence, relation (40) implies

$$3g(Q\varphi v, x) = 2nR_{v\varphi vvx} + g(Q\varphi v, u)g(u, x). \tag{54}$$

On the other hand relation (43), by means of (45) and (46), reduces to

$$g((\nabla_v Q)\varphi v, x) = 0. \tag{55}$$

Next, since $Q\xi = 2n\xi$, (54) and (55) give

$$\begin{cases} 3g((\nabla_v Q)\xi, x) = 2nR_{v\varphi vvx} + g(Q\varphi v, u)g(u, x), \\ 3g((\nabla^2_{vv} Q)\varphi v, x) = -2nR_{v\varphi vvx} - g(Q\varphi v, u)g(u, x). \end{cases} \tag{56}$$

Now, put $x = u$ in (42) and (56). Then we obtain

$$\begin{aligned} R_{v\varphi vvu}\left(\frac{n-2}{12} + \frac{1}{6}\rho_{vv} + \frac{23-37n}{180}R_{vuvu}\right)& \\ - \frac{7n+7}{180}\sum_{\lambda=3}^{2n+1} R_{v\varphi vv\lambda}R_{v\lambda vu} - \frac{2n+2}{45}R_{vR_{v\varphi v}vvu} &= 0. \end{aligned} \tag{57}$$

Further, from (42), (47), (53), (56) and (57) we get

$$
\begin{aligned}
&\frac{2n-3}{60}R_{v\varphi vvx}+\frac{1}{9}\rho_{vv}R_{v\varphi vvx}+\frac{2}{9}R_{vuvu}R_{v\varphi vvx}\\
(58)\qquad &-\frac{n+1}{18}R_{v\varphi vvu}R_{vuvx}-\frac{2n+2}{45}R_{vR_{v\varphi v}vvx}+\Big(\frac{3-2n}{60}R_{v\varphi vvu}\\
&-\frac{1}{9}\rho_{vv}R_{v\varphi vvu}+\frac{n-3}{18}R_{vuvu}R_{v\varphi vvu}+\frac{2n+2}{45}R_{vR_{v\varphi v}vvu}\Big)g(u,x)=0.
\end{aligned}
$$

On the other hand, we get on $(\mathcal{B},G,J)$, using (54) and taking into account the local fibration formulas (5), (7) and (8),

$$
3G(\overline{Q}Jw,y)=2n\overline{R}_{wJwwy}+G(\overline{Q}Jw,z)G(z,y), \tag{59}
$$

where $v=w^*$, $u=z^*$ and $x=y^*$. First, we consider the case when $\mathcal{B}$ is locally irreducible. Then it is an Einstein space and because y and z are orthogonal to Jw at $\bar{m}=\pi(m)\in\mathcal{B}$, (59) then yields

$$
\overline{R}_{wJw}w=\zeta Jw,
$$

which implies, using Theorem E, that $\mathcal{B}$ has constant holomorphic sectional curvature since $\dim\mathcal{B}\geq 4$. Secondly, if $\mathcal{B}$ is locally reducible, it is locally a product $\mathcal{B}_1\times\cdots\times\mathcal{B}_k$ of Kählerian Einstein spaces. For each factor $\mathcal{B}_i$ with $\dim\mathcal{B}_i\geq 4$, (59) is satisfied and hence, these $\mathcal{B}_i$ are complex space forms. If $\dim\mathcal{B}_i=2$, the same result follows from (51). To handle this case we first note that (59) may be written in the form

$$
3\overline{Q}Jw-2n\overline{R}_{wJw}w-\bar{\rho}(Jw,z)z=\eta Jw. \tag{60}
$$

Now we project this onto the tangent space of the factor $\mathcal{B}_1$. Then we get

$$
3(\overline{Q}Jw)_1-2n(\overline{R}_{wJw}w)_1-\nu z_1=\eta(Jw)_1, \tag{61}
$$

where $\nu=\bar{\rho}(Jw,z)$. Since this factor has constant holomorphic sectional curvature, say c_1, we have

$$
(\overline{Q}Jw)_1=\frac{\bar{\tau}_1}{n_1}(Jw)_1,\quad (\overline{R}_{wJw}w)_1=(c_1\cos^2\alpha_1)(Jw)_1 \tag{62}
$$

where $G(w_1,w_1)=\cos^2\alpha_1$. Here $n_1=\dim\mathcal{B}_1$ and $\bar{\tau}_1$ denotes the scalar curvature of $\mathcal{B}_1$, that is

$$
4\bar{\tau}_1=n_1(n_1+2)c_1. \tag{63}
$$

Using (62) and (63), (61) yields

$$
\frac{3}{4}(n_1+2)c_1(Jw)_1-2nc_1\cos^2\alpha_1(Jw)_1-\nu z_1=\eta(Jw)_1. \tag{64}
$$

Similarly, by taking the projection on the tangent space of the second factor $\mathcal{B}_2$ in (60), we get

$$\frac{3}{4}(n_2+2)c_2(Jw)_2 - 2nc_2\cos^2\alpha_2(Jw)_2 - \nu z_2 = \eta(Jw)_2. \tag{65}$$

Here $n_2 = \dim \mathcal{B}_2$, $\bar{\tau}_2$ denotes the scalar curvature of $\mathcal{B}_2$ and $G(w_2, w_2) = \cos^2\alpha_2$. Since the horizontal vectors u and v (orthogonal to φu at m) may be chosen arbitrarily we obtain from (64) and (65)

$$c_1 + c_2 = 0. \tag{66}$$

Finally, we shall prove that $c_1 = c_2 = 0$. For that purpose we first note that from (58) we obtain on $(\mathcal{B}, G, J)$

$$\begin{aligned}&\Big(\frac{6n-5}{36} + \frac{1}{9}\bar{\rho}_{ww} + \frac{2}{9}\overline{R}_{zwzw}\Big)\overline{R}_{wJw}w - \frac{n+1}{18}\overline{R}_{wJwwz}\overline{R}_{wz}w \\ &- \frac{2n+2}{45}\overline{R}_{w\overline{R}_{wJw}w}w + \Big(\frac{5-6n}{36}\overline{R}_{wJwwz} - \frac{1}{9}\bar{\rho}_{ww}\overline{R}_{wJwwz} \\ &+ \frac{n-3}{18}\overline{R}_{zwzw}\overline{R}_{wJwwz} + \frac{2n+2}{45}\overline{R}_{w\overline{R}_{wJw}wwz}\Big)z = \zeta Jw,\end{aligned} \tag{67}$$

using the local fibration formulas (5), (7) and (8), where $v = w^*$ and $u = z^*$.

Moreover, multiplying (67) by Jz we get

$$\begin{aligned}&\Big(\frac{6n-5}{36} + \frac{1}{9}\bar{\rho}_{ww} + \frac{2}{9}\overline{R}_{zwzw}\Big)\overline{R}_{wJwwJz} \\ &\qquad - \frac{n+1}{18}\overline{R}_{wJwwz}\overline{R}_{wzwJz} - \frac{2n+2}{45}\overline{R}_{w\overline{R}_{wJw}wwJz} = 0.\end{aligned} \tag{68}$$

Further, let $w \in \{y, Jy\}^{\perp}$. Then (68) also implies

$$\begin{aligned}&\Big(\frac{6n-5}{36} + \frac{1}{9}\bar{\rho}_{ww} + \frac{2}{9}\overline{R}_{zJwzJw}\Big)\overline{R}_{JwwJwJz} \\ &\qquad - \frac{n+1}{18}\overline{R}_{JwwJwz}\overline{R}_{JwzJwJz} - \frac{2n+2}{45}\overline{R}_{Jw\overline{R}_{Jww}JwJwJz} = 0,\end{aligned} \tag{69}$$

or, equivalently, taking into account the Kähler identity,

$$\begin{aligned}&\Big(\frac{6n-5}{36} + \frac{1}{9}\bar{\rho}_{ww} + \frac{2}{9}\overline{R}_{JzwJzw}\Big)\overline{R}_{wJwwz} \\ &\qquad - \frac{n+1}{18}\overline{R}_{JwwJwz}\overline{R}_{wJzwz} + \frac{2n+2}{45}\overline{R}_{Jw\overline{R}_{Jww}JwJwJz} = 0.\end{aligned} \tag{70}$$

Now, using (67), (70) and the Kähler identity, we obtain

$$\begin{aligned}(71)\qquad &\left(\frac{6n-5}{36}+\frac{1}{9}\bar{\rho}_{ww}+\frac{2}{9}\overline{R}_{zwzw}\right)\overline{R}_{wJw}w-\frac{n+1}{18}\overline{R}_{wJwwz}\overline{R}_{wz}w\\&-\frac{2n+2}{45}\overline{R}_{w\overline{R}_{wJw}w}w+\left(\frac{2}{9}\overline{R}_{wJzwJz}\overline{R}_{wJwwz}\right.\\&\left.-\frac{n+1}{18}\overline{R}_{wzwJz}\overline{R}_{wJwwJz}+\frac{n-3}{18}\overline{R}_{zwzw}\overline{R}_{wJwwz}\right)z=\zeta Jw.\end{aligned}$$

Finally, projecting this onto the tangent spaces of the factors $\mathcal{B}_1$ and $\mathcal{B}_2$ and using (66) we get

$$c_1 = c_2 = 0.$$

The same procedure for the other factors leads to the conclusion that $\mathcal{B}$ is flat.

We conclude that $\mathcal{B}$ is a complex space form and hence, using Theorem F, M is a Sasakian space form. This completes the proof of the first part of the theorem.

To prove the second part, let $Q^\sigma(p)\xi$ belong to the plane spanned by $\xi(p)$ and $(\varphi v)(p)$. Then (39) yields

$$(72)\qquad (2n-3)R_{v\varphi vvx} = (1-2n)R_{v\varphi vvu}\, g(u,x)$$

for all horizontal u, v such that $u \perp \varphi v,\ u \perp v$ at $m \in M$ and all horizontal x orthogonal to φv and ξ. Putting $x = u$ in the last relation yields

$$R_{v\varphi vvx} = 0$$

for all horizontal v and all horizontal x orthogonal to φv and ξ (since u can be chosen arbitrarily). Finally, we conclude that $R_{v\varphi v}v$ is proportional to φv, and the result follows using Theorem D.

The converse follows directly from the formulas for the Ricci operator of these φ-geodesic tubes in Sasakian space forms mentioned in Section 3.

This concludes the proof of the theorem.

Now we turn to the properties which are similar to the ones derived by using the function κ^σ. Therefore, let $\mathcal{P}_\sigma(r)$ be the φ-geodesic tube defined earlier. The associated *Ricci curvature* of $\mathcal{P}_\sigma(r)$ with respect to φv at p is defined by

$$(73)\qquad \rho^\sigma(\varphi v, \varphi v)(p) = g(Q^\sigma(p)\varphi v, \varphi v).$$

Using (25) and (38) we get the following expansion for this curvature:

$$
\begin{aligned}
(74) \qquad \rho^\sigma(\varphi v, \varphi v)(p) &= \frac{2n-2}{r^2} + \frac{2}{3}\Big(\rho_{vv} - 3R_{vuvu} - nR_{v\varphi v\, v\varphi v}\Big) \\
&- \frac{r}{4}\Big[5(\nabla_v \rho)_{vv} + (2n+1)(\nabla_v R)_{v\varphi v\, v\varphi v} + (\nabla_v R)_{vuvu}\Big] + O(r^2).
\end{aligned}
$$

Then we have

Theorem 5. *Let $(M^{2n+1}, \varphi, \eta, \xi, g)$ be a Sasakian manifold of dimension ≥ 5. Then, with the conventions as in Section 4, M is locally φ-symmetric if and only if, for every sufficiently small φ-geodesic tube $\mathcal{P}_\sigma(r)$*

$$
(75) \qquad \rho^\sigma(\varphi v, \varphi v)(\exp_m(rv)) = \rho^\sigma(\varphi v, \varphi v)(\exp_m(-rv))
$$

for all $m \in M$ and all v.

Proof. First suppose (75) holds. Then using again the local fibration technique and with the same convention as above we get

$$
(76) \qquad (2n+1)(\overline{\nabla}_w \overline{R})_{w\varphi w\, w\varphi w} + (\overline{\nabla}_w \overline{R})_{wzwz} + 5(\overline{\nabla}_w \bar{\rho})_{ww} = 0.
$$

After linearizing and contracting the last relation and using the Kähler and the Bianchi identities, we obtain

$$
\begin{aligned}
&2(n+3)(\overline{\nabla}_x \overline{R})_{zxzx} + 2(11n+18)(\overline{\nabla}_x \bar{\rho})_{xx} \\
(77) \qquad &+3G(x,x)(\overline{\nabla}_x \bar{\rho})_{zz} - 2G(x,x)(\overline{\nabla}_z \bar{\rho})_{zx} + 10G(x,x)\overline{\nabla}_x \bar{\tau} = 0.
\end{aligned}
$$

Repeating the same procedure in (77) yields

$$
(78) \qquad (7n+15)(\overline{\nabla}_w \bar{\rho})_{zz} - 4(n+2)(\overline{\nabla}_z \bar{\rho})_{zw} + 2(16n+23)\overline{\nabla}_w \bar{\tau} = 0
$$

and using the same procedure once again in (78), gives

$$
(79) \qquad \overline{\nabla}_w \bar{\tau} = 0.
$$

Finally, doing the same for z in (77) implies

$$
(80) \qquad (\overline{\nabla}_y \bar{\rho})_{yy} = 0.
$$

This yields $(\overline{\nabla}_y \bar{\rho})_{zz} + 2(\overline{\nabla}_z \bar{\rho})_{yz} = 0$ and so, (78) leads to $(\overline{\nabla}_w \bar{\rho})_{zz} = (\overline{\nabla}_z \bar{\rho})_{wz} = 0$. Then (76) and (77) yield

$$
(\overline{\nabla}_w \overline{R})_{wJw\, wJw} = 0
$$

and hence, from Theorem B, we obtain that $(\mathcal{B}, G, J)$ is locally Hermitian symmetric. Then, using Theorem A (ii) we conclude that $(M^{2n+1}, \varphi, \eta, \xi, g)$ is a locally φ-symmetric Sasakian manifold.

The converse follows easily using Theorem A (i).

References

[1] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics 509, *Springer-Verlag*, *Berlin-Heidelberg-New York*, 1976.

[2] D. E. Blair and B. J. Papantoniou, A characterization of Sasakian space forms by geodesic tubes, *Math. J. Toyama Univ.* **16** (1993), 65–90.

[3] D. E. Blair and L. Vanhecke, Geodesic spheres and Jacobi vector fields in Sasakian space forms, *Proc. Roy. Soc. Edinburgh Sect.* **A105** (1987), 17–22.

[4] D. E. Blair and L. Vanhecke, Symmetries and φ-symmetric spaces, *Tôhoku Math. J.* **39** (1987), 373–383.

[5] D. E. Blair and L. Vanhecke, New characterizations of φ-symmetric spaces, *Kodai Math. J.* **10** (1987), 102–107.

[6] P. Bueken, Reflections and rotations in contact geometry, doctoral dissertation, *Katholieke Universiteit Leuven*, 1992.

[7] P. Bueken and L. Vanhecke, Geometry and symmetry on Sasakian manifolds, *Tsukuba Math. J.* **12** (1988), 403–422.

[8] B. Y. Chen and L. Vanhecke, Differential geometry of geodesic spheres, *J. Reine Angew. Math.* **325** (1981), 28–67.

[9] M. Djorić and L. Vanhecke, Almost Hermitian geometry, geodesic spheres and symmetries, *Math. Okayama Univ.* **32** (1990), 187–206.

[10] M. Djorić and L. Vanhecke, Geometry of geodesic spheres on Sasakian manifolds, *Rend. Sem. Mat. Univ. Pol. Torino* **49** (1991), 329–357.

[11] M. Djorić and L. Vanhecke, Geometry of tubes about characteristic curves on Sasakian manifolds, *Rend. Circ. Mat. Palermo* **XLI** (1992), 111–122.

[12] M. Djorić, Geometry of tubes about φ-geodesics on Sasakian manifolds, (*submitted*).

[13] L. Gheysens and L. Vanhecke, Total scalar curvature of tubes about curves, *Math. Nachr.* **103** (1981), 177–197.

[14] A. Gray and L. Vanhecke, Riemannian geometry as determined by the volumes of small geodesic balls, *Acta Math.* **142** (1979), 157–198.

[15] A. Gray and L. Vanhecke, The volume of tubes about curves in a Riemannian manifold, *Proc. London Math. Soc.* **44** (1982), 215–243.

[16] A Gray, Tubes, *Addison-Wesley Publ. Co.*, *Reading*, 1990.

[17] S. Kobayashi and K. Nomizu, Foundations of Differential geometry, I-II, *J. Wiley*, 1963.

[18] K. Ogiue, On fiberings of almost contact manifolds, *Kōdai Math. Sem. Rep.* **17** (1965), 53–62.

[19] M. Okumura, Some remarks on spaces with a certain contact structure, *Tôhoku Math. J.* **14** (1962), 135–145.

[20] S. Sekigawa and L. Vanhecke, Symplectic geodesic symmetries on Kähler manifolds, *Quart. J. Math. Oxford* **37** (1986), 95–103.

[21] T. Takahashi, Sasakian φ-symmetric spaces, *Tôhoku Math. J.* **29** (1977), 91–113.

[22] S. Tanno, Constancy of holomorphic sectional curvature in almost Hermitian manifolds, *Kōdai Math. Sem. Rep.* **25** (1973), 190–201.

[23] L. Vanhecke, Geometry in normal and tubular neighborhoods, *Rend. Sem. Fac. Sci. Univ. Cagliari, Supplemento al Vol. 58*, 1988, pp. 73–176.

[24] K. Yano and M. Kon, Structures on manifolds, *Series in Pure Mathematics, 3, World Scientific Publ. Co.*, *Singapore*, 1984.
endRefs

M. Djorić
Faculty of Mathematics
University of Belgrade
Studentski trg 16, P.B. 550
11000 Belgrade
Yugoslavia

(Received August 10, 1994)