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Interiors of continuous images of self-similar sets with overlaps

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Abstract. Let K be the attractor of the following iterated function system

$$\{S_1(x) = \lambda x, S_2(x) = \lambda x + c - \lambda, S_3(x) = \lambda x + 1 - \lambda\},$$

where $S_1(I) \cap S_2(I) \neq \emptyset, (S_1(I) \cup S_2(I)) \cap S_3(I) = \emptyset$, and $I = [0, 1]$ is the convex hull of K. Let $d_1 = \frac{1-c-\lambda}{\lambda} < \frac{1}{1-c-\lambda} = d_2$. Suppose that f is a continuous function defined on an open set $U \subset \mathbb{R}^2$. Denote the image

$$f_U(K, K) = \{f(x, y) : (x, y) \in (K \times K) \cap U\}.$$

If $\partial_x f$, $\partial_y f$ are continuous on U, and there is a point $(x_0, y_0) \in (K \times K) \cap U$ such that

$$\left|\frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}}\right| \in (d_1, d_2) \quad \text{or} \quad \left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right| \in (d_1, d_2),$$

then $f_U(K, K)$ contains an interval. As a result, we let $c = \lambda = \frac{1}{3}$, and if

$$f(x, y) = x^\alpha y^\beta (\alpha\beta \neq 0), \quad x^\alpha \pm y^\alpha (\alpha \neq 0), \quad \sin(x)\cos(y), \quad \text{or } x\sin(xy),$$

then $f_U(C, C)$ contains an interval, where C is the middle-third Cantor set.

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1. Introduction

Representation of real numbers is an important area in number theory. There are many ways of expressing numbers, for instance, the β-expansions ([1], [3], [6], [7], [13], [14], [15], [26]), the continued fractions ([11], [12]), the Lüroth expansions ([5]), and so on. These representations are popular in number theory. There are, however, some other approaches that can represent real numbers ([2], [24], [37], [39]). First, let us introduce some basic definitions. Given two non-empty sets $A, B \subset \mathbb{R}$, we define

$$A * B = \{x * y : x \in A, y \in B\},$$

where $*$ is $+, -, \cdot$ or $\div$ (when $* = \div$, $y \neq 0$). We call $u = x * y$ an arithmetic representation in terms of A and B. Steinhaus proved that

$$C - C = \{x - y : x, y \in C\} = [-1, 1],$$

where C is the middle-third Cantor set. This result also implies that $C + C = [0, 2]$ due to the equation $C = 1 - C$. In [2], Athreya, Reznick and Tyson considered the multiplication on C, and proved that

$$17/21 \leq \mathcal{L}(C \cdot C) \leq 8/9,$$

where $\mathcal{L}$ denotes the Lebesgue measure. Moreover, they also considered the division on C. Due to these two results, it is natural to ask that for the general self-similar sets K_1 and K_2, how large $K_1 * K_2$ is in the sense of Hausdorff dimension or Lebesgue measure, where $* = +$ or $\cdot$. This question was mainly addressed by Peres and Shmerkin [31], Hochman and Shmerkin [19], who proved the following results. Let K_1 and K_2 be two self-similar sets with iterated function systems (IFS's)

$$\{f_i(x) = r_i x + a_i\}_{i=1}^{n} \quad \text{and} \quad \{g_j(x) = r'_j x + b_j\}_{j=1}^{m},$$

respectively, if there are some r_i, r'_j such that $\log|r_i| / \log|r'_j| \notin \mathbb{Q}$, then

$$\dim_H(K_1 + K_2) = \min\{\dim_H(K_1) + \dim_H(K_2), 1\}.$$

The condition in the above result is called the irrational condition. In [34], Shmerkin stated that

$$\dim_H(K_1 \cdot K_2) = \min\{\dim_H(K_1) + \dim_H(K_2), 1\}$$

under the irrational condition. It is natural to consider the Hausdorff dimension of $K_1 * K_2$, where $* = +$ or $\cdot$., without the irrational condition. In [22], Jiang proved that without the irrational condition, K_1+K_2 is either a self-similar set or an attractor of some infinite iterated function system. Moreover, in [23], Jiang considered the sum $K_1 + K_2$ from the projectional perspective, and calculated in certain cases the Hausdorff dimension of the projection of $K_1 \times K_2$ for some special angles. If $\dim_H(K_1 + K_2) = 1$ or $\dim_H(K_1 \cdot K_2) = 1$, then one may ask whether K_1+K_2 or $K_1 \cdot K_2$ contains an interval. For K_1+K_2, there is a celebrated conjecture posed by Palis [30], i.e. whether it is true (at least generically) that the arithmetic sum of dynamically defined Cantor sets either has measure zero or contains an interval. This conjecture was solved in [28]. However, for the general self-similar sets this conjecture is still open. Without the irrational condition, there are few results for $K_1 \cdot K_2$. In [38], Yuki considered the general Cantor sets, and proved under some conditions that $K_1 \cdot K_2$ contains an interval.

Suppose that $K = K_{\lambda,c}$ is the self-similar set generated by the following IFS,

$$\{S_1(x) = \lambda x, S_2(x) = \lambda x + c - \lambda, S_3(x) = \lambda x + 1 - \lambda\},$$

where $S_1(I) \cap S_2(I) \neq \emptyset, (S_1(I) \cup S_2(I)) \cap S_3(I) = \emptyset$, and $I = [0, 1]$ is the convex hull of K. The class $\{K_{\lambda,c}\}_{\lambda,c}$, which is an important class of self-similar sets with overlaps [21], was investigated by many scholars. It was investigated from different aspects, see [9], [10], [16], [20], [25], [29], [32], [39]. For example, Hochman [20] proved the Furstenberg's conjecture which states $\dim_H K_{1/3,c} = 1$ for any $c \notin \mathbb{Q}$, Keyon discussed the necessary and sufficient condition for the open set condition satisfied on $K_{1/3,c}$, Rao and Wen [32] obtained the graph-directed construction when λ^{-1} is a P.V. number and $c \in \mathbb{Q}$.

In [39], Tian *et al.* proved that $K \cdot K = [0, 1]$ if and only if $c \geq (1 - \lambda)^2$. In this paper, we shall analyze the continuous image of K. Let f be a continuous function defined on an open set $U \subset \mathbb{R}^2$. We call

$$f_U(K, K) = \{f(x, y) : (x, y) \in (K \times K) \cap U\}$$

the continuous image of K. For convenience, we write $f(K, K) = f_{\mathbb{R}^2}(K, K)$.

In this paper, we shall give a sufficient condition such that $f_U(K, K)$ contains an interval. In fact, whether a fractal contains an interval is a crucial problem in fractal geometry and dynamical systems. Schief [33], Bandt and Graf [4] showed the relations among the open set condition, positive Hausdorff measure and non-empty interiors. Dajani *et al.* [8], Hare and Sidorov [17], [18] found that the existence of an interior of a class of self-affine sets implies the existence of the simultaneous expansions. In [27], Lagarias and Wang showed that for some self-affine tiles, the existence of interiors is equivalent to the positive Lebesgue measure of the tiles. More results can be found in [27], [29] and references therein.

Now we state the main results of this paper.

Theorem 1. *Let K be the attractor of the following IFS*

$$\{S_1(x) = \lambda x,\ S_2(x) = \lambda x + c - \lambda,\ S_3(x) = \lambda x + 1 - \lambda\},$$

where $S_1(I) \cap S_2(I) \neq \emptyset$, $(S_1(I) \cup S_2(I)) \cap S_3(I) = \emptyset$ and $I = [0,1]$ is the convex hull of K. Let $d_1 = \dfrac{1-c-\lambda}{\lambda} < \dfrac{1}{1-c-\lambda} = d_2$. Suppose that f is a continuous function defined on an open set $U \subset \mathbb{R}^2$. If $\partial_x f$, $\partial_y f$ are continuous on U, and there is a point $(x_0, y_0) \in (K \times K) \cap U$ such that

$$\left|\frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}}\right| \in (d_1, d_2) \quad \text{or} \quad \left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right| \in (d_1, d_2),$$

then $f_U(K, K)$ contains an interval.

Remark 1. We point that it is not easy to prove this result in terms of the Newhouse thickness theorem [36], as the IFS of K consists of very complicated overlaps.

Corollary 1. *Suppose $d_1 < d_2$, and let*

$$f(x,y) = x^{\alpha} \pm y^{\beta} \quad \text{with } \alpha\beta \neq 0 \text{ and } \frac{\alpha-1}{\beta-1} \notin \mathbb{Q}.$$

Then $f_U(K, K)$ contains an interval.

Proof. Note that $\left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right| = \left|\frac{\alpha}{\beta}\right| \cdot \left|\frac{x^{\alpha-1}}{y^{\beta-1}}\right|$. Take $x_0 = \lambda^{k_1}$ and $y_0 = \lambda^{k_2}$, we have $\left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right| = \left|\frac{\alpha}{\beta}\right| \cdot \left|\lambda^{k_1\left(\frac{\alpha-1}{\beta-1}\right)-k_2}\right|^{(\beta-1)}$. Since $\frac{\alpha-1}{\beta-1} \notin \mathbb{Q}$, we can take integers k_1 and k_2 such that

$$k_1\left(\frac{\alpha-1}{\beta-1}\right) - k_2 \text{ is so close to } \frac{1}{\beta-1}\log_\lambda\left|\frac{\beta}{\alpha}\cdot\frac{d_1+d_2}{2}\right|,$$

hence $\left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right|$ is close enough to $\frac{d_1+d_2}{2}$, which implies $\left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right| \in (d_1, d_2)$. Therefore, $f_U(K, K)$ contains an interval. □

Note that if $c = \lambda = 1/3$, then K is the middle-third Cantor set. We have the following results.

Corollary 2. *Let C be the middle-third Cantor sets. If $\partial_x f$, $\partial_y f$ are continuous on U, and there is a point $(x_0, y_0) \in (C \times C) \cap U$ such that*

$$1 < \left| \frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}} \right| < 3 \text{ or } 1 < \left| \frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}} \right| < 3,$$

then $f_U(C, C)$ contains an interval.

Corollary 3. *Let C be the middle-third Cantor set. If $f(x, y)$ is one of the following functions,*

$$x^\alpha y^\beta (\alpha\beta \neq 0), \quad x^\alpha \pm y^\alpha (\alpha \neq 0), \quad x \pm y^2, \quad \sin(x)\cos(y), \quad x\sin(xy),$$

then $f_U(C, C)$ contains an interval.

This paper is organized as follows. In Section 2, we will give proofs of Theorem 1 and Corollary 3.

2. Proofs of the main results

In this section, we assume that $f : U \to \mathbb{R}$ is a function such that $\partial_x f$, $\partial_y f$ are continuous on U.

Fix $\lambda \in (0, 1)$ and $2\lambda \geq c \geq \lambda$ with $c + \lambda < 1$. Given $J = [x_1, x_2]$, we let $g_J(t) = x_1 + (x_2 - x_1)t$ with $g_J([0, 1]) = J$, and

$$\tilde{J} = J^{(1)} \cup J^{(2)} \cup J^{(3)},$$

where

$$J^{(i)} = g_J(S_i([0, 1])).$$

Suppose $G = \cup_{J \in \Lambda} J$ is the union of some closed intervals pairwise disjoint, we let $\tilde{G} = \cup_{J \in \Lambda} \tilde{J}$.

Lemma 1. *Suppose $I = [a, a + t]$ and $J = [b, b + t]$. Then*

$$\bigcap_{1 \leq i,j \leq 2} (I^{(i)} \times J^{(j)}), \quad \bigcap_{1 \leq i \leq 2} (I^{(i)} \times J^{(3)}), \quad \bigcap_{1 \leq j \leq 2} (I^{(3)} \times J^{(j)})$$

are non-empty.

Proof. We note that

$$[a+(c-\lambda)t, a+\lambda t]\times[b+(c-\lambda)t, b+\lambda t]\subset\bigcap_{1\le i,j\le 2}(I^{(i)}\times J^{(j)}),$$

$$[a+(c-\lambda)t, a+\lambda t]\times[b+(1-\lambda)t, b+t]=\bigcap_{1\le i\le 2}(I^{(i)}\times J^{(3)}),$$

$$[a+(1-\lambda)t, a+t]\times[b+(c-\lambda)t, b+\lambda t]=\bigcap_{1\le j\le 2}(I^{(3)}\times J^{(j)}).\qquad\square$$

Let $H=[0,1]$. For any $i_1\cdots i_n\in\{1,2,3\}^n$, we call $f_{i_1\cdots i_n}(H)=(f_{i_1}\circ\cdots\circ f_{i_n})(H)$ a basic interval of rank n, or n-basic interval, which has length λ^n. Denote by H_n the collection of all these basic intervals of rank n. We say that $I\times J$ is a basic square of $K\times K$, if I and J are basic intervals of the same rank. Suppose A and B are the left and right endpoints of some basic intervals in H_k for some $k\ge 1$, respectively. Denote by G_n the union of basic intervals of rank n contained in $[A,B]$.

The following Lemma 2 comes from [2] and [39], here we give its proof just for the self-containedness of the paper.

Lemma 2. *Let $F:U\to\mathbb{R}$ be a continuous function. Suppose A and B (M and N) are the left and right endpoints of some basic intervals in H_{k_0} for some $k_0\ge 1$, respectively, such that $[A,B]\times[M,N]\subset U$. Then $K\cap[A,B]=\bigcap_{n=k_0}^{\infty}G_n$, and $K\cap[M,N]=\bigcap_{n=k_0}^{\infty}G'_n$. Moreover, if for any $n\ge k_0$ and any two n-basic intervals $I\subset G_n$, $J\subset G'_n$ such that*

$$F(I,J)=F(\tilde{I},\tilde{J}),$$

then $F(K\cap[A,B],K\cap[M,N])=F(G_{k_0},G'_{k_0})$.

Proof. By the construction of G_n (G'_n), i.e. $G_{n+1}\subset G_n$ ($G'_{n+1}\subset G'_n$) for any $n\ge k_0$, it follows that

$$K\cap[A,B]=\bigcap\nolimits_{n=k_0}^{\infty}G_n\quad\text{ and }\quad K\cap[M,N]=\bigcap\nolimits_{n=k_0}^{\infty}G'_n.$$

The continuity of F yields that

$$F(K\cap[A,B],K\cap[M,N])=\bigcap\nolimits_{n=k_0}^{\infty}F(G_n,G'_n).$$

In terms of the relation $G_{n+1}=\tilde{G}_n$, $G'_{n+1}=\tilde{G}'_n$ and the condition in the lemma, it follows that

$$\begin{aligned}F(G_n,G'_n)&=\bigcup\nolimits_{1\le i\le t_n}\bigcup\nolimits_{1\le j\le t'_n}F(I_{n,i},J_{n,j})=\bigcup\nolimits_{1\le i\le t_n}\bigcup\nolimits_{1\le j\le t'_n}F(\tilde{I}_{n,i},\tilde{J}_{n,j})\\&=F\left(\bigcup\nolimits_{1\le i\le t_n}\tilde{I}_{n,i},\bigcup\nolimits_{1\le j\le t'_n}\tilde{J}_{n,j}\right)=F(G_{n+1},G'_{n+1}).\end{aligned}$$

Therefore, $F(K\cap[A,B],K\cap[M,N])=F(G_{k_0},G'_{k_0})$. □

Proposition 1. *If there is a point $(x_0,y_0)\in(K\times K)\cap U$ such that*

$$\begin{cases}\partial_x f|_{(x_0,y_0)}>0>\partial_y f|_{(x_0,y_0)},\\ d_1<\left|\frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}}\right|<d_2,\end{cases}$$

then $f_U(K,K)$ contains an interval.

Proof. Let $I\times J(\subset U)$ be a basic square of $K\times K$ close enough to (x_0,y_0), that means $I=[a,a+t]$, $J=[b,b+t]$ are basic intervals such that (a,b) is close enough to (x_0,y_0) and t is small enough. Then

$$\tilde I=[a,a+ct]\cup[a+(1-d)t,a+t],\quad \tilde J=[b,b+ct]\cup[b+(1-d)t,b+t],$$

where $d=\lambda$. In what follows, we let $\delta=1-c-\lambda$. In terms of Lemma 2, it suffices to prove

$$f(I,J)=f(\tilde I,\tilde J)$$

if we want to prove that $f_U(K,K)$ contains an interval. Since

$$\partial_x f|_{(x_0,y_0)}>0>\partial_y f|_{(x_0,y_0)},$$

it follows that

$$f(I,J)=[f(a,b+t),f(a+t,b)].$$

Moreover, in terms of the conditions

$$\frac{\delta}{d}<\left|\frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}}\right|<\frac{1}{\delta}\quad\text{and}\quad c\ge d=\lambda,$$

we conclude that

$$\begin{aligned}h_1(x,y)&=f(x-\delta t,y)-f(x,y+dt)=t(-d\partial_y f-\delta\partial_x f)+o(t)\ge 0,\\ h_2(x,y)&=f(x,y+\delta t)-f(x-t,y)=t(\partial_x f+\delta\partial_y f)+o(t)\ge 0,\\ h_3(x,y)&=f(x-\delta t,y)-f(x,y+ct)=t(-c\partial_y f-\delta\partial_x f)+o(t)\ge 0,\end{aligned}$$

where $o(t)/t\to 0$ uniformly as $t\to 0$, i.e. $o(t)$ is independent of the choice of (x,y) as $\partial_x f$ and $\partial_y f$ are continuous at (x_0,y_0). In fact,

$$f(\tilde I,\tilde J)=\bigcup_{1\le i,j\le 3} f(I^{(i)},J^{(j)}).$$

Using Lemma 1, we obtain that

$$f(\tilde{I},\tilde{J}) = \bigcup_{1\le i,j\le 3} f(I^{(i)},J^{(j)}) = J_1\cup J_2\cup J_3\cup J_4,$$

where

$$J_1=[f(A),f(B)],\qquad J_2=[f(C),f(D)],$$
$$J_3=[f(E),f(F)],\qquad J_4=[f(G),f(H)].$$

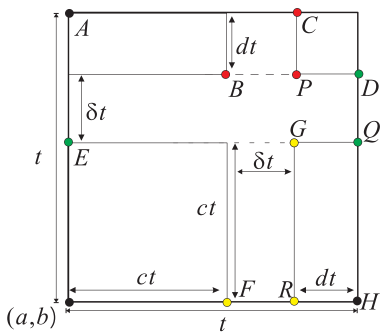

Figure 1. Basic square in Proposition 1

Hence, in order to prove $f(I,J)=f(\tilde{I},\tilde{J})$, we only need to check

$$\begin{cases} f(B)-f(C)\ge 0,\\ f(D)-f(E)\ge 0,\\ f(F)-f(G)\ge 0.\end{cases}$$

However, the above inequalities are the direct consequences of

$$f(B)-f(C)=h_1(P)\ge 0,\quad f(D)-f(E)=h_2(Q)\ge 0,$$
$$f(F)-f(G)=h_3(R)\ge 0. \qquad \square$$

For the function $f(x,y)$ in Proposition 1, when considering the function $-f(x,y)$, we obtain the following Proposition 2.

Proposition 2. *If there is a point $(x_0,y_0)\in(K\times K)\cap U$ such that*

$$\begin{cases} \partial_x f|_{(x_0,y_0)}<0<\partial_y f|_{(x_0,y_0)},\\ d_1<\left|\frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}}\right|<d_2,\end{cases}$$

then $f_U(K,K)$ contains an interval.

By the symmetry of $K \times K$, we obtain the following result.

Proposition 3. *If there is a point $(x_0, y_0) \in (K \times K) \cap U$ such that*

$$\begin{cases} \partial_x f|_{(x_0,y_0)} \cdot \partial_y f|_{(x_0,y_0)} < 0, \\ d_1 < \left| \frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}} \right| < d_2, \end{cases}$$

then $f_U(K, K)$ contains an interval.

With a similar discussion, we are allowed to prove the following result.

Proposition 4. *If there is a point $(x_0, y_0) \in (K \times K) \cap U$ such that*

$$\begin{cases} \partial_x f|_{(x_0,y_0)} > 0, \partial_y f|_{(x_0,y_0)} > 0, \\ d_1 < \left| \frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}} \right| < d_2, \end{cases}$$

then $f_U(K, K)$ contains an interval.

Proof. The proof is similar to Proposition 1. Let $d = \lambda$ and $\delta = 1 - c - \lambda$. Let $I \times J (\subset U)$ be a basic square of $K \times K$ close enough to (x_0, y_0), that means $I = [a, a + t]$, $J = [b, b + t]$ are basic intervals such that (a, b) is close enough to (x_0, y_0) and t is small enough. Then

$$\tilde{I} = [a, a + ct] \cup [a + (1 - d)t, a + t], \quad \tilde{J} = [b, b + ct] \cup [b + (1 - d)t, b + t].$$

In terms of Lemma 2, it suffices to prove

$$f(I, J) = f(\tilde{I}, \tilde{J}).$$

By virtue of

$$\partial_x f|_{(x_0,y_0)} > 0, \quad \partial_y f|_{(x_0,y_0)} > 0,$$

it follows that

$$f(I, J) = [f(a, b), f(a + t, b + t)] = [f(A), f(H)]$$

as in Figure 2.

Moreover, in terms of the conditions

$$\frac{\delta}{d} < \left| \frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}} \right| < \frac{1}{\delta} \quad \text{and} \quad c \geq d = \lambda,$$

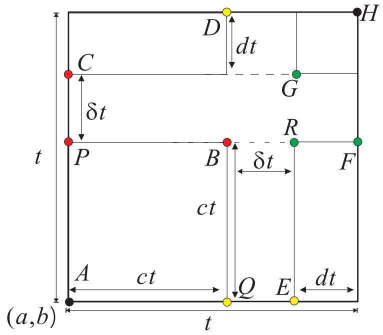

Figure 2. Basic square in Proposition 4

we conclude that

$$\begin{aligned} h_4(x,y) &= f(x+ct,y) - f(x,y+\delta t) = t(c\partial_x f - \delta\partial_y f) + o(t) \geq 0, \\ h_5(x,y) &= f(x,y+t) - f(x+\delta t,y) = t(\partial_y f - \delta\partial_x f) + o(t) \geq 0, \\ h_6(x,y) &= f(x+dt,y) - f(x,y+\delta t) = t(d\partial_x f - \delta\partial_y f) + o(t) \geq 0, \end{aligned}$$

where $o(t)/t \to 0$ uniformly as $t \to 0$, i.e. $o(t)$ is independent of the choice of (x,y), as $\partial_x f$ and $\partial_y f$ are continuous at (x_0,y_0). Using Lemma 1, we obtain that

$$f(\tilde{I},\tilde{J}) = \bigcup_{1\leq i,j\leq 3} f(I^{(i)},J^{(j)}) = J_5 \cup J_6 \cup J_7 \cup J_8,$$

where

$$\begin{aligned} J_5 &= [f(A),f(B)],\ \ J_6 = [f(C),f(D)], \\ J_7 &= [f(E),f(F)],\ \ J_8 = [f(G),f(H)]. \end{aligned}$$

Hence, in order to prove $f(I,J) = f(\tilde{I},\tilde{J})$, we only need to prove

$$\begin{cases} f(B) - f(C) \geq 0, \\ f(D) - f(E) \geq 0, \\ f(F) - f(G) \geq 0. \end{cases}$$

However, the above inequalities follow immediately from

$$\begin{aligned} &f(B) - f(C) = h_4(P) \geq 0, \quad f(D) - f(E) = h_5(Q) \geq 0, \\ &f(F) - f(G) = h_6(R) \geq 0. \end{aligned}$$

□

In the same way, we obtain the following result.

Proposition 5. *If there is a point $(x_0, y_0) \in (K \times K) \cap U$ such that*

$$\begin{cases} \partial_x f|_{(x_0,y_0)} \cdot \partial_y f|_{(x_0,y_0)} > 0, \\ \left|\frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}}\right| \in (d_1, d_2) \quad \text{or} \quad \left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right| \in (d_1, d_2), \end{cases}$$

then $f_U(K, K)$ contains an interval.

PROOF OF THEOREM 1.
It follows from Propositions 1–5. □

PROOF OF COROLLARY 3.
It suffices to check the conditions in Theorem 1.

(1) If $f(x, y) = x^\alpha y^\beta$ with $\alpha\beta \neq 0$, using $(x^\alpha y^\beta) = (xy^{\beta/\alpha})^\alpha$, we only need to deal with $f(x, y) = xy^\gamma$. Using the symmetry, we may assume that $|\gamma| \geq 1$. Now we have

$$\partial_x f = y^\gamma \quad \text{and} \quad \partial_y f = \gamma x y^{\gamma-1}$$

with $\left|\frac{\partial_x f}{\partial_y f}\right| = \frac{1}{|\gamma|}\left|\frac{y}{x}\right|$. If $|\gamma| = 3^k$ for some integer $k \geq 0$, we take $y = 1$ and $x = (2/3) \cdot 3^{-k}$, hence $\left|\frac{\partial_x f}{\partial_y f}\right| = 3/2 \in (1, 3)$ in this case. Otherwise, if $3^k < |\gamma| < 3^{k+1}$ for some integer $k \geq 0$, then we take $y = 1$ and $x = 3^{-(k+1)}$, then $\left|\frac{\partial_x f}{\partial_y f}\right| \in (1, 3)$. Now, $f_U(C, C)$ contains an interval for $f(x, y) = x^\alpha y^\beta$ with $\alpha\beta \neq 0$.

(2) If $f(x, y) = x^\alpha \pm y^\alpha$ with $\alpha \neq 0$, then

$$|\partial_x f| = |\alpha| x^{\alpha-1} \quad \text{and} \quad |\partial_y f| = |\alpha| y^{\alpha-1}$$

with $\left|\frac{\partial_x f}{\partial_y f}\right| = \left|\frac{x^{\alpha-1}}{y^{\alpha-1}}\right|$. When $\alpha \neq 1$, take $x, y \in C$ such that y/x is close enough to 1, then $1 < \left|\frac{\partial_x f}{\partial_y f}\right| < 3$ or $1 < \left|\frac{\partial_y f}{\partial_x f}\right| < 3$. When $\alpha = 1$, the classical result $C + C = [0, 2]$ implies there is an interval in $f(C, C) = C + C$.

(3) If $f(x, y) = x \pm y^2$, then $\partial_x f = 1, |\partial_y f| = 2y$. Take $x_0 = 8/9, y_0 = 1/3$, which implies $1 < |1/(2y_0)| < 3$.

(4) If $f(x, y) = \sin(x)\cos(y)$, then

$$|\partial_x f| = |\cos x \cos y|, \quad |\partial_y f| = |\sin x \sin y|.$$

We take $(x_0, y_0) = (2/3, 2/3)$, and obtain that

$$|\cos(2/3)\cos(2/3)| = 0.6176\ldots, \quad |\sin(2/3)\sin(2/3)| = 0.3823\ldots,$$

and thus $1 < \left|\frac{\cos(2/3)\cos(2/3)}{\sin(2/3)\sin(2/3)}\right| = 1.615\ldots < 3$.

(5) If $f(x,y) = x\sin(xy)$, then

$$|\partial_x f| = |\sin(xy) + xy\cos xy|, \quad |\partial_y f| = |x^2\cos(xy)|.$$

We take $(x_0, y_0) = (2/3, 2/3)$, and obtain that

$$\left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right| = \left|1 + \frac{9\sin(4/9)}{4\cos(4/9)}\right| = 2.071\ldots \in (1,3). \qquad \square$$

Remark 2. Our idea can be implemented for other overlapping self-similar sets. Moreover, in Corollary 3, for some functions, we can obtain that $f_U(C,C)$ contains infinitely many closed intervals.

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