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Estimation on the Walsh–Fejér and Walsh logarithmic kernels

By GYÖRGY GÁT (Debrecen) and GÁBOR LUCSKAI (Debrecen)

Abstract. The main aim of this article is to demonstrate the difference of the trigonometric and the Walsh system with respect to the behaviour of the maximal function of the Fejér kernels. Moreover, properties (positivity among others) of the Walsh logarithmic kernels are also investigated.

1. Introduction and main results

We follow the standard notions of dyadic analysis, see, e.g., [14]. We denote by $\mathbb{N}$ and $\mathbb{P}$ the set of natural numbers and positive integers. Define the set of dyadic intervals as (see, e.g., [1], [12], [14], [15])

$$\mathcal{J} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p, n \in \mathbb{N} \right\}.$$

The dyadic interval $I := [0,1) \subset \mathbb{R}$ is called the unit (dyadic) interval (see [1]). The Lebesgue measure of a set B $(B \subset I)$ is $\lambda(B) = |B|$. Denote by $L^p(I)$ the usual Lebesgue spaces, and by $\|.\|_p$ the corresponding norms $(1 \leq p \leq \infty)$. For a given $x \in I$, let $I_n(x)$ denote the dyadic interval $I_n(x) \in \mathcal{J}$ of length 2^{-n} which contains $x\,(n \in \mathbb{N})$. In particular, we write $I_n := I_n(0)\,(n \in \mathbb{N})$. Denote by

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$\mathbb{Q}_2 := \{\frac{p}{2^n} : p, n \in \mathbb{N}\}$ the set of dyadic rational numbers. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in I$, where either $x_n = 0$ or 1. If $x \in \mathbb{Q}_2$, then we choose the expansion which terminates in 0's.

Set $e_i := 2^{-i-1}$, that is, the i-th coordinate of e_i is 1, and the rest are zeros for all $i \in \mathbb{N}$. The dyadic rationals can be represented as the finite $0, 1$ combinations of the elements of the set $\{e_i : i \in \mathbb{N}\}$.

Set the definition of the n-th ($n \in \mathbb{N}$) Walsh–Paley function at point $x \in I$ as

$$\omega_n(x) := \prod_{j=0}^{\infty} (-1)^{x_j n_j},$$

where $\mathbb{N} \ni n = \sum_{n=0}^{\infty} n_j 2^j$ ($n_j \in \{0, 1\}$ $(j \in \mathbb{N})$).

The so-called dyadic or logical addition is defined for any $x, y \in I$ as

$$x + y := \sum_{n=0}^{\infty} |x_n - y_n| 2^{-(n+1)}.$$

Denote by

$$\hat{f}(n) := \int_I f \omega_n d\lambda, \qquad D_n := \sum_{k=0}^{n-1} \omega_k, \qquad K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k, \qquad R_n := \frac{1}{\log n} \sum_{k=1}^{n-1} \frac{D_k}{k}$$

the Fourier coefficients, the Dirichlet, the Fejér or $(C, 1)$ kernels and Walsh logarithmic means, respectively. Moreover, see ([14]),

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{1.1}$$

$$D_n(x) = \omega_n(x) \sum_{i=0}^{\infty} n_i r_i D_{2^i}(x), \tag{1.2}$$

where r_i is the i-th Rademacher function, that is, $r_i = (-1)^{x_i}$. It is also known that the Fejér or $(C, 1)$ means of the function f and Dirichlet kernels are connected by the equality [14]

$$\begin{aligned} \sigma_n f(y) &:= \frac{1}{n} \sum_{k=0}^{n-1} S_k f(y) = \int_I f(x) K_n(y + x) dx \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \int_I f(x) D_k(y + x) dx \quad (n \in \mathbb{N}, y \in I). \end{aligned}$$

On the other hand, for an introduction to the theory of Walsh functions and Walsh–Fourier series in the context of view of topological groups, see [11] or [14].

The forthcoming lemma will play a prominent role in the proof of some lemmas and it is due to Goginava [10].

Lemma 1.1 ([10]). *For $j, n \in \mathbb{N}$, $j < 2^n$, we have*

$$D_{2^n-j}(x) = D_{2^n}(x) - \omega_{2^n-1}(x) D_j(x).$$

We use the following notations:

$$K_{a,b} := \sum_{l=a}^{b} D_l \quad (a, b \in \mathbb{N}) \quad \text{and} \quad n^{(s)} := \sum_{l=s}^{\infty} n_l 2^l \quad (n, s \in \mathbb{N}).$$

The next two lemmas are proved by Gát [5].

Lemma 1.2.

$$nK_n(x) = \sum_{s=0}^{\infty} n_s K_{n^{(s+1)},2^s}(x).$$

Let $|n| := \lfloor \log_2(n) \rfloor$ $(1 \le n \in \mathbb{N})$. That is, $2^{|n|} \le n < 2^{|n|+1}$.

Lemma 1.3 ([5]). *Let $s, t, n \in \mathbb{N}$, $x \in I_t \setminus I_{t+1}$. If $s \le t \le |n|$, then*

$$|K_{n^{(s+1)},2^s}(x)| \le c 2^{s+t}.$$

If $t < s \le |n|$, then

$$K_{n^{(s+1)},2^s}(x) = \begin{cases} 0, & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x) 2^{s+t-1}, & \text{if } x - x_t e_t \in I_s. \end{cases}$$

In this paper, c denotes an absolute constant which may not be the same at different occurrences.

Gát [5] proved the following inequality for the maximal function of Fejér kernels:

Theorem 1.4 ([5]).

$$\int_{I \setminus I_a} \sup_{|n| \ge A} |K_n(x)| \, dx \le c \sqrt{2^{a-A}}$$

for all $A \ge a \in \mathbb{N}$.

In 2004, Goginava improved the result of Gát. That is, he proved:

Theorem 1.5 ([9])**.**

$$\int_{I\setminus I_a} \sup_{|n|\geq A} |K_n(x)|\, dx \leq c\frac{A-a}{2^{A-a}}$$

for all $a < A \in \mathbb{N}$.

Theorems 1.4 and 1.5 give powerful methods in the investigation of one and two-dimensional problems related to Fejér-type means of Walsh–Fourier series ([8], [10]).

Next, we turn our attention to the trigonometric system in order to show some differences between the behaviour of Fejér kernels with respect to the trigonometric and the Walsh system. The well-known formula for Fejér kernels ([2]) is

$$K_n^{\mathrm{trig}}(x) = \frac{\sin^2\left(\frac{2n+1}{2}x\right)}{2(n+1)\sin^2\frac{x}{2}} \qquad (0 < x < 2\pi),$$

which gives the estimation from below for the trigonometric system

$$\frac{c_1}{N\delta} \leq \int_{\delta}^{1} \sup_{n\geq N} |K_n^{\mathrm{trig}}(x)|\, dx \leq \frac{c_2}{N\delta} \qquad \left(N > \frac{1}{\delta}\right),$$

where c_1 and c_2 are absolute constants.

This naturally raises the question whether the result of Gát and Goginava can be improved by writing a smaller term instead of $c\frac{A-a}{2^{A-a}}$. That is, something smaller instead of $c\frac{\log(N\delta)}{N\delta}$.

The following theorem shows that the result of Goginava cannot be improved. This shows a sharp difference between the theory of the trigonometric and the Walsh system.

Theorem 1.6.

$$\int_{\delta}^{1} \sup_{n\geq N} |K_n(x)|\, dx \geq \frac{c}{N\delta}\log(N\delta) \qquad \left(N > \frac{1}{\delta}\right).$$

After that, we turn our attention to another type of kernel function with respect to the Walsh–Paley system. That is, we investigate the Walsh–Riesz logarithmic kernel.

The next theorem is a trivial and well-known property of the trigonometric Riesz logarithmic kernel and probably its proof can be found somewhere, but we did not find it. However, it is not our result. A proof is given only in order to demonstrate the differences between the two orthonormal systems.

Theorem 1.7. *The trigonometric logarithmic kernels*

$$R_n^{\mathrm{trig}}(x) := \frac{1}{\log n}\sum_{k=1}^{n-1}\frac{D_k^{\mathrm{trig}}(x)}{k}$$

are positive for all $n \geq 2$ and for each $x \in [-\pi,\pi]$.

Proof. We apply the Abel transform:

$$\begin{aligned}\log n \cdot R_n^{\mathrm{trig}}(x) &= \sum_{k=1}^{n-1}\frac{D_k^{\mathrm{trig}}(x)}{k} = \sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\sum_{j=1}^{k}D_j^{\mathrm{trig}}(x) + \frac{1}{n}\sum_{i=1}^{n-1}D_i(x)\\ &= \sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)K_k^{\mathrm{trig}}(x) + \frac{1}{n}K_{n-1}^{\mathrm{trig}}(x) \geq 0,\end{aligned}$$

because $K_n^{\mathrm{trig}}(x) \geq 0$ for all $n \in \mathbb{N}$ and $x \in [-\pi,\pi]$. □

The Walsh–Fejér kernel can take negative values, therefore we cannot use the previous proof. The next theorem shows the behaviour of the logaritmic kernels with respect to the Walsh–Paley system.

Theorem 1.8. *Let $t, n \in \mathbb{N}$ and $x \in I_t \setminus I_{t+1}$. Then we have*

$$R_n(x) \geq \frac{2^t}{16\cdot\log n}, \quad \text{for } n > 2^t \quad \text{and} \quad R_n(x) = \frac{n-1}{\log n}, \quad \text{for } n \leq 2^t.$$

2. Proofs

Proof of Theorem 1.6. Let $a = \left\lfloor \log_2\left(\frac{1}{\delta}\right)\right\rfloor$ and $A = \lceil \log_2 N\rceil$. Then the theorem can be written in the following form:

$$\int_{\overline{I}_a}\sup_{n\geq 2^A}|K_n(x)|\,dx \geq c\frac{A-a}{2^{A-a}}.$$

We prove this lower estimation.

Let $t,l \in \mathbb{P}$ be arbitrary and $n = 2^{2B} + 2^{2B-2} + \cdots + 2^4 + 2^2 + 2^0$, where $|n| = 2B \geq A$ and $B = \lceil \frac{A}{2} \rceil$. If $t,l \in \mathbb{N} \setminus \{0\}$, $t \neq l$, then let J_t^l be the dyadic interval $I_{t+l+1}(x)$, for which coordinate x_i satisfies:

$$x_i = \begin{cases} 1, & \text{if } i \in \{t, t+l\}, \\ 0, & \text{if } i \in \{0,1,\dots t-1\} \cup \{t+1, t+2, \dots, t+l-1\}. \end{cases} \tag{I}$$

We can write interval $\overline{I}_a$ in the following way:

$$\overline{I}_a = \bigcup_{t=0}^{a-1} I_t \setminus I_{t+1} = \bigcup_{t=0}^{a-1} \bigcup_{l=1}^{2B-t} J_t^l \cup \bigcup_{t=0}^{a-1} I_{2B+1}(e_t).$$

Therefore (recall that $n = 2^{2B} + 2^{2B-2} + \cdots + 2^0$),

$$\begin{aligned} \int_{\overline{I}_a} \sup_{n \geq 2^A} |K_n(x)|\, dx &\geq \int_{\bigcup_{t=0}^{a-1} \bigcup_{l=1}^{2B-t} J_t^l} |K_n(x)|\, dx + \int_{\bigcup_{t=0}^{a-1} I_{2B+1}(e_t)} |K_n(x)|\, dx \\ &\geq \sum_{t=0}^{a-1} \sum_{l=1}^{2B-t} \int_{J_t^l} \frac{c}{2^{2B}} \left| \sum_{s=0}^{2B} n_s K_{n^{(s+1)},2^s}(x) \right| dx \\ &\quad + \sum_{t=0}^{a-1} \int_{I_{2B+1}(e_t)} \frac{c}{2^{2B}} \left| \sum_{s=0}^{2B} n_s K_{n^{(s+1)},2^s}(x) \right| dx = \text{(I)} + \text{(II)}. \end{aligned}$$

Estimation of (I). We distinguish three cases depending on the value of s. We can suppose that s is even (because $n_s = 1$ if s is even, and $n_s = 0$ if s is odd).

First case: s is even and $t \leq s \leq t+l$. Let $m := \left\lfloor \frac{t+l}{2} \right\rfloor$ and $2m$ play the role of s. Then by Lemma 1.3 we have

$$\int_{J_t^l} |K_{2^{2B}+2^{2B-2}+\dots+2^{2m+2},2^{2m}}(x)|\, dx = \frac{2^{2m+t-1}}{2^{t+l+1}} = \frac{2^{2\left\lfloor \frac{t+l}{2} \right\rfloor + t-1}}{2^{t+l+1}} \geq \frac{2^{2t+l-2}}{2^{t+l+1}} = \frac{2^t}{8},$$

$$\int_{J_t^l} |K_{2^{2B}+2^{2B-2}+\dots+2^{2m},2^{2m-2}}(x)|\, dx = \frac{2^{2m-2+t-1}}{2^{t+l+1}} \leq \frac{2^{2t+l-3}}{2^{t+l+1}} = \frac{2^t}{4^2},$$

$$\vdots$$

$$\int_{J_t^l} |K_{2^{2B}+2^{2B-2}+\dots+2^{2m-2j+2},2^{2m-2j}}(x)|\, dx = \frac{2^{2m-2j+t-1}}{2^{t+l+1}} \leq \frac{2^{2t-2j-1}}{2^{t+1}} = \frac{2^t}{4^{j+1}}$$

for $j = 1, 2, \dots$.

Therefore,

$$\int\limits_{J_t^l} \sup_{n\geq 2B} \frac{c}{2^{2B}} \left| \sum_{s=0}^{2B} n_s K_{n^{(s+1)},2^s}(x) \right| dx \geq \frac{c}{2^{2B}} \left| \frac{2^t}{8} - \frac{2^t}{4^2} - \frac{2^t}{4^3} - \ldots \right|.$$

Consequently,

$$\begin{aligned}
&\int\limits_{\overline{I_a}} \sup_{n\geq 2B} \frac{c}{2^{2B}} \left| \sum_{s=0}^{2B} n_s K_{n^{(s+1)},2^s}(x) \right| dx \\
&\geq \sum_{t=0}^{a-1} \sum_{l=1}^{2B-t} \frac{c}{2^{2B}} \left| 2^t - \frac{2^{t+1}}{4} - \frac{2^{t+1}}{4^2} - \ldots \right| \geq \sum_{t=0}^{a-1} \sum_{l=1}^{2B-t} \frac{c}{2^{2B}} \left[2^t \cdot \left(1 - \sum_{i=1}^{\infty} \frac{2}{4^i} \right) \right] \\
&\geq \sum_{t=0}^{a-1} \sum_{l=1}^{2B-t} \frac{c}{2^{2B}} \left[2^t \cdot \left(1 - \frac{\frac{1}{2}}{1-\frac{1}{4}} \right) \right] \geq \sum_{t=0}^{a-1} \sum_{l=1}^{2B-t} \frac{c \cdot 2^t}{2^{2B}} \geq c \cdot \frac{2^a}{2^{2B}} (2B-a).
\end{aligned}$$

Second case: $t+l \leq s$. Then $K_{n^{(s)},2^s}(x) = 0$, as it is given by Lemma 1.3.

Third case: $0 \leq s < t$. Then, by Lemma 1.3 again, we have

$$\int\limits_{J_t^l} |K_{2^{2B}+2^{2B-2}+\cdots+2^{s+2},2^s}(x)|\, dx \leq c \frac{2^{s+t-1}}{2^{t+l+1}} \leq c \frac{2^s}{2^l}$$

for the integral on the set J_t^l (recall that s is even and the definition of $J_t^l := I_{t+l+1}$ see in (I)). Then

$$\begin{aligned}
&\sum_{t=0}^{a-1} \sum_{l=1}^{2B-t} \int\limits_{J_t^l} \sup_{|n|\geq 2B} \frac{c}{2^{2B}} \left| \sum_{s=0}^{t-1} n_s K_{n^{(s+1)},2^s}(x) \right| dx \\
&\qquad \leq \frac{c}{2^{2B}} \sum_{t=0}^{a-1} \sum_{l=1}^{2B-t} \sum_{s=0}^{t-1} \frac{2^{s+t}}{2^{t+l}} \leq \frac{c}{2^{2B}} \sum_{t=0}^{a-1} \frac{2^t}{2} \leq c \frac{2^a}{2^{2B}}.
\end{aligned}$$

Estimation of (II) is not important for the lower estimation from below, since (I) will be the major part and since we are to find a lower bound. But anyhow,

$$\begin{aligned}
\text{(II)} &= \sum_{t=0}^{a-1} \int\limits_{I_{2B+1}(e_t)} \sup_{|n|\geq 2B} \frac{c}{2^{2B}} \left| \sum_{s=0}^{2B} 2^{s+t-1} \right| dx \\
&= \sum_{t=0}^{a-1} \frac{c}{2^{2B+1}} \cdot \sum_{s=0}^{2B} \frac{2^{s+t-1}}{2^{2B+1}} \leq \sum_{t=0}^{a-1} \cdot \frac{c}{2^{2B+1}} \frac{2^{2B+1+t}}{2^{2B+1}} \leq \frac{2^a}{2^{2B}}.
\end{aligned}$$

To sum up, we get

$$\int_{\bar{I}_a} \sup_{|n|\geq 2B} |K_n(x)|\,dx \geq c\cdot\frac{2^a}{2^{2B}}(2B-a).$$

This completes the proof. □

Before proving Theorem 1.8, we need two lemmas.

Lemma 2.1. *Let $A \geq 3$. Then*

$$\sum_{k=2^A}^{2^{A+1}-1}\frac{1}{k} < 0.7254. \tag{2.1}$$

Proof. The case of $A \in \{3,4\}$ is an easy calculation. If $A \geq 5$, we get the following estimation:

$$\sum_{k=2^A}^{2^{A+1}-1}\frac{1}{k} < \int_{1-2^{-A}}^{2-2^{-A}}\frac{1}{x}\,dx = \log\left(\frac{2-2^{-A}}{1-2^{-A}}\right) < 0.7254. \qquad \square$$

Lemma 2.2. *If $t \in \{0,1,2\}$ and $x \in I_t \setminus I_{t+1}$, then*

$$\sum_{k=2^A}^{2^{A+1}-1}\frac{|D_k(x)|}{k} \leq 0.3627\cdot 2^t$$

for $A \geq 6$.

Proof. Let $x \in I_t \setminus I_{t+1}$ and apply (1.2):

$$|D_k(x)| = \left|\sum_{i=0}^{t-1} k_i 2^i - k_t 2^t\right|.$$

Use the notation

$$k = \sum_{i=0}^{\infty} k_i 2^i \quad \text{and} \quad k^{(m)} := \sum_{i=m}^{\infty} k_i 2^i,$$

where $k_i \in \{0,1\}$ and $m \in \mathbb{N}$. For $t=0$, we have $|D_k(x)| = k_0$, and by Lemma 2.1,

$$\sum_{k=2^A}^{2^{A+1}-1}\frac{|D_k(x)|}{k} = \sum_{k=2^A}^{2^{A+1}-1}\frac{k_0}{k_0+k^{(1)}} \leq \sum_{j=2^{A-1}}^{2^A-1}\frac{1}{1+2j} \leq \frac{0.7254}{2} = 0.3627\cdot 2^t$$

(recall that $t=0$ and $A \geq 4$ in this case).

For $t=1$, we have $D_k(x)=|k_0-2k_1|$ and

$$\frac{|D_k(x)|}{k}\le\frac{|k_0-2k_1|}{1+k^{(2)}}.$$

Thus,

$$\begin{aligned}\sum_{k=2^A}^{2^{A+1}-1}\frac{|D_k(x)|}{k}&\le\sum_{k_0,k_1\in\{0,1\}}\sum_{j=2^{A-2}}^{2^{A-1}-1}\frac{|k_0-2k_1|}{1+4j}\\&\le\frac14\cdot0.7254\cdot\sum_{k_0,k_1\in\{0,1\}}|k_0-2k_1|=0.7254=0.3627\cdot2^t\end{aligned}$$

(recall that $t=1$ and $A\ge5$).

For $t=2$, we have $D_k(x)=|k_0-2k_1-4k_2|$ and

$$\frac{|D_k(x)|}{k}\le\frac{|k_0-2k_1-4k_2|}{1+k^{(3)}}$$

(recall that $t=2$ now). Thus,

$$\begin{aligned}\sum_{k=2^A}^{2^{A+1}-1}\frac{|D_k(x)|}{k}&\le\sum_{k_0,k_1,k_2\in\{0,1\}}\sum_{j=2^{A-3}}^{2^{A-2}-1}\frac{|k_0-2k_1-4k_2|}{1+8j}\\&\le\frac18\cdot0.7254\cdot\sum_{k_0,k_1,k_2\in\{0,1\}}|k_0-2k_1-4k_2|=2\cdot0.7254=0.3627\cdot2^t\end{aligned}$$

(recall that $t=2$ and $A\ge6$ in this case). □

Proof of Theorem 1.8. The proof is easy in the case $n\le2^t$, because if $x\in I_t\setminus I_{t+1}$, then we have

$$R_n(x)=\frac{1}{\log n}\sum_{k=1}^{n-1}\frac{D_k(x)}{k}=\frac{n-1}{\log n},$$

and consequently, $n\ge2^t$ can be supposed. Suppose that $x\in I$ has at least two coordinates different from 0. Let $s\ge1$ and $x\in I_t\setminus I_{t+1}$ satisfy

$$x_i:=\begin{cases}1, & \text{if } i\in\{t,t+s\},\\0, & \text{if } i\in\{0,1,\dots t-1\}\cup\{t+1,t+2,\dots,t+s-1\},\end{cases}\tag{2.2}$$

and $k = a + k_t 2^t + 2^{t+1} b$, where $k_i \in \{0,1\}$ $(i = 0,1,2,\ldots)$,

$$a = k_{(t-1)} := \sum_{i=0}^{t-1} k_i 2^i, \qquad 2^{t+1} b = k^{(t+1)} := \sum_{i=t+1}^{\infty} k_i 2^i.$$

If there is no such s, then $x = 2^{-t-1}$, which will be discussed later. This case is referred to as $s = \infty$. Use the notation

$$L_a(x) = L_{a,a}(x) := \sum_{k=2^a}^{2^{a+1}-1} \frac{D_k(x)}{k} \quad \text{and} \quad L_{a,b}(x) := \sum_{k=2^a}^{2^{b+1}-1} \frac{D_k(x)}{k}.$$

The following estimations are easy:

$$L_{0,t}(x) = \sum_{k=1}^{2^t-1} \frac{D_k(x)}{k} = 2^t - 1, \tag{2.3}$$

$$\begin{aligned} L_t(x) &= \sum_{k=2^t}^{2^{t+1}-1} \frac{D_k(x)}{k} = \sum_{k=1}^{2^t-1} \frac{D_{2^t+k}(x)}{2^t+k} \\ &= \sum_{k=0}^{2^t-1} \frac{2^t-k}{2^t+k} = \sum_{j=2^t}^{2^{t+1}-1} \frac{2^{t+1}-j}{j} \geq 2^{t+1}\log 2 - 2^t. \end{aligned} \tag{2.4}$$

Now we turn to the values of L_{t+j}, where $j = 1,2,3,\ldots$. Then

$$L_{t+j}(x) = \sum_{k=2^{t+j}}^{2^{t+j+1}-1} \frac{D_k(x)}{k} = \sum_{b=2^{j-1}}^{2^j-1} \sum_{k_t=0}^{1} \sum_{a=0}^{2^t-1} \frac{\omega_{2^{t+1}b}(x)\left[k_t 2^t - \sum_{i=0}^{t-1} k_i 2^i\right]}{a + k_t 2^t + b\cdot 2^{t+1}}.$$

We define the function $L^1_{t+j}(x)$ as the following sum:

$$L^1_{t+j}(x) := \sum_{b=2^{j-1}}^{2^j-1} \sum_{k_t=0}^{1} \sum_{a=0}^{2^t-1} \frac{\omega_b(2^{t+1}x)\,(k_t 2^t - a)}{k_t 2^t + b\cdot 2^{t+1}},$$

where $j = 1,2,3,\ldots$. In the next step, we give an upper estimation of $|L_{t+j}(x) - L^1_{t+j}(x)|$:

$$|L_{t+j}(x) - L^1_{t+j}(x)| \leq \sum_{b=2^{j-1}}^{2^j-1} \sum_{a=0}^{2^t-1} \left(\frac{a2^t - a^2}{(2^t + b\cdot 2^{t+1})^2} + \frac{a^2}{(b\cdot 2^{t+1})^2} \right)$$

$$= \sum_{b=2^{j-1}}^{2^j-1} \left(\frac{2^t(2^t-1)2^t}{2\cdot(2^t+b\cdot 2^{t+1})^2} - \frac{2^t(2^t-1)(2^{t+1}-1)}{6\cdot(2^t+b\cdot 2^{t+1})^2} + \frac{2^t(2^t-1)(2^{t+1}-1)}{6(b\cdot 2^{t+1})^2} \right)$$

$$= \sum_{b=2^{j-1}}^{2^j-1} \left(\frac{2^{3t}}{2\cdot 2^{2t}(1+2b)^2} - \frac{2\cdot 2^{3t}}{6\cdot 2^{2t}(1+2b)^2} + \frac{2\cdot 2^{3t}}{24b^2\cdot 2^{2t}} \right) + \alpha(2^t)$$

$$= 2^t \sum_{b=2^{j-1}}^{2^j-1} \left(\frac{1}{2\cdot(1+2b)^2} - \frac{1}{3\cdot(1+2b)^2} + \frac{1}{12b^2} \right) + \alpha(2^t) \le \frac{2^t}{8} \sum_{b=2^{j-1}}^{2^j-1} \frac{1}{b^2} + \alpha(2^t).$$

For $\alpha(2^t)$, we obtain

$$\alpha(2^t) = \sum_{b=2^{j-1}}^{2^j-1} \left(\frac{-2^{2t}}{2\cdot 2^{2t}(1+2b)^2} + \frac{3\cdot 2^{2t}-2^t}{6\cdot 2^{2t}(1+2b)^2} - \frac{3\cdot 2^{2t}-2^t}{24\cdot 2^{2t}b^2} \right)$$

$$= -\frac{1}{8} \sum_{b=2^{j-1}}^{2^j-1} \frac{1}{b^2} + \frac{1}{24\cdot 2^t} \sum_{b=2^{j-1}}^{2^j-1} \left(\frac{1}{b^2} - \frac{1}{b^2+b+\frac{1}{4}} \right).$$

Therefore,

$$\sum_{j=1}^{\infty} |L_{t+j}(x) - L^1_{t+j}(x)|$$

$$\le \frac{2^t-1}{8} \sum_{j=1}^{\infty} \sum_{b=2^{j-1}}^{2^j-1} \frac{1}{b^2} + \frac{1}{24} \sum_{j=1}^{\infty} \sum_{b=2^{j-1}}^{2^j-1} \frac{b+\frac{1}{4}}{b^2(b^2+b+\frac{1}{4})}$$

$$\le \frac{2^t-1}{8} \sum_{b=1}^{\infty} \frac{1}{b^2} + \frac{1}{24} \sum_{b=1}^{\infty} \frac{1}{b^3} = \frac{2^t-1}{6}\cdot\frac{\pi^2}{6} + \frac{\zeta(3)}{24} < 0.206\cdot 2^t - 0.155, \quad (2.5)$$

where ζ denotes the Riemann zeta-function. Moreover, later we will also need the following estimation:

$$\sum_{j=J+1}^{\infty} |L_{t+j}(x) - L^1_{t+j}(x)|$$

$$\le \frac{2^t-1}{8} \sum_{j=J+1}^{\infty} \sum_{b=2^{j-1}}^{2^j-1} \frac{1}{b^2} + \frac{1}{24} \sum_{j=J+1}^{\infty} \sum_{b=2^{j-1}}^{2^j-1} \frac{1}{b^3}$$

$$= \frac{2^t-1}{8} \sum_{b=2^J}^{\infty} \frac{1}{b^2} + \frac{1}{24} \sum_{b=2^J}^{\infty} \frac{1}{b^3} \le \frac{2^t-1}{8}\cdot\frac{1.645}{2^J} + \frac{1}{24}\cdot\frac{1.202}{2^J}. \quad (2.6)$$

In the proof of this estimation we used the following inequalities, which hold for every $J \geq 0$:

$$2^J \sum_{b=2^J}^{\infty} \frac{1}{b^2} \leq 1.645, \qquad 2^J \sum_{b=2^J}^{\infty} \frac{1}{b^3} \leq 1.202.$$

Now we investigate $L^1_{t+j}(x)$, where $j = 1, 2, \ldots$ and $x \in I_t \setminus I_{t+1}$:

$$\begin{aligned}
L^1_{t+j}(x) &= \sum_{b=2^{j-1}}^{2^j-1} \sum_{k_t=0}^{1} \sum_{a=0}^{2^t-1} \frac{\omega_b(2^{t+1}x) \cdot (k_t 2^t - a)}{k_t 2^t + 2^{t+1} b} \\
&= \sum_{b=2^{j-1}}^{2^j-1} \sum_{k_t=0}^{1} \left[2^t \cdot \frac{\omega_b(2^{t+1}x) \cdot k_t 2^t}{k_t 2^t + 2^{t+1} b} - \frac{2^t(2^t-1)}{2} \cdot \frac{\omega_b(2^{t+1}x)}{k_t 2^t + 2^{t+1} b} \right] \\
&= 2^t \cdot \sum_{b=2^{j-1}}^{2^j-1} \sum_{k_t=0}^{1} \frac{\omega_b(2^{t+1}x)}{k_t + 2b} \cdot \left(k_t - \frac{1}{2} \right) + \frac{1}{2} \sum_{b=2^{j-1}}^{2^j-1} \sum_{k_t=0}^{1} \frac{\omega_b(2^{t+1}x)}{k_t + 2b} \\
&= \frac{2^t}{2} \sum_{b=2^{j-1}}^{2^j-1} \omega_b(2^{t+1}x) \left(\frac{1}{1+2b} - \frac{1}{2b} \right) + \frac{1}{2} \sum_{b=2^{j-1}}^{2^j-1} \omega_b(2^{t+1}x) \left(\frac{1}{1+2b} + \frac{1}{2b} \right) \\
&:= L^{1,1}_{t+j}(x) + L^{1,2}_{t+j}(x).
\end{aligned}$$

Furthermore,

$$|L^{1,1}_{t+j}(x)| \leq \frac{2^t}{2} \sum_{b=2^{j-1}}^{2^j-1} \frac{1}{2b(2b+1)},$$

therefore,

$$\begin{aligned}
\sum_{j=1}^{\infty} |L^{1,1}_{t+j}(x)| &\leq \frac{2^t}{2} \sum_{j=1}^{\infty} \sum_{b=2^{j-1}}^{2^j-1} \frac{1}{2b(2b+1)} \leq \frac{2^t}{2} \sum_{b=1}^{\infty} \frac{1}{2b(2b+1)} \\
&= 2^t \frac{1 - \log(2)}{2} \leq 0.154 \cdot 2^t. \qquad (2.7)
\end{aligned}$$

Moreover,

$$\sum_{j=J+1}^{\infty} |L^{1,1}_{t+j}(x)| \leq \frac{2^t}{2} \sum_{b=J+1}^{\infty} \frac{1}{2b(2b+1)} \leq 0.154 \cdot \frac{2^t}{2^J}, \qquad (2.8)$$

where the role of J will be discussed later.

Now we estimate $L^{1,2}_{t+j}(x)$. Let $\beta_b = \dfrac{1}{1+2b} + \dfrac{1}{2b}$. Then, by an Abel transform, we have:

$$\sum_{j=1}^{J} L_{t+j}^{1,2}(x) = \frac{1}{2}\sum_{j=1}^{J}\sum_{b=2^{j-1}}^{2^j-1}\beta_b\omega_b(2^{t+1}x)$$
$$= \frac{1}{2}\sum_{b=1}^{2^J-1}(\beta_b-\beta_{b+1})\sum_{i=1}^{b}\omega_i(2^{t+1}x) + \frac{1}{2}\beta_{2^J}(D_{2^J}(2^{t+1}x)-1). \quad (2.9)$$

We suppose that $J \le A-t-1$. If $J \le s-1$, then for any $b < 2^J$, we have $\omega_b(2^{t+1}x) = 1$ (recall that $\{i : x_i = 1,\ i \le t+s\} = \{t, t+s\}$ and then $2^{t+1}x \in I_{s-1} \setminus I_s$). If $A \le t+s$, then this is the situation for all $J \le A-t-1$.

If $A \ge t+s$, then in $\sum_{k=1}^{2^A-1}\frac{D_k}{k}$ we have addends $L_{t+j}^{1,2}(x)$ for $j = 1, 2, \dots A-t-1$. Then

$$\sum_{j=1}^{A-t-1} L_{t+j}^{1,2}(x) = \frac{1}{2}\sum_{b=1}^{2^{s-1}-1}(\beta_b-\beta_{b+1})b + \frac{1}{2}\sum_{b=2^{s-1}}^{2^{A-t}-1}(\beta_b-\beta_{b+1})(D_{b+1}(2^{t+1}x)-1)$$
$$+ \frac{1}{2}\beta_{2^{A-t}}(D_{2^{A-t}}(2^{t+1}x)-1)$$
$$\ge \frac{1}{2}\sum_{b=1}^{2^{s-1}-1}(\beta_b-\beta_{b+1})b - \frac{1}{2}\cdot 2^{s-1}\sum_{b=2^{s-1}}^{2^{A-t}-1}(\beta_b-\beta_{b+1}) - \frac{1}{2}\beta_{2^{A-t}}$$
$$\ge \frac{1}{2}\sum_{b=1}^{2^{s-1}-1}(\beta_b-\beta_{b+1})b - 2^{s-2}\left(\frac{1}{1+2^s}+\frac{1}{2^s}\right)$$
$$- \frac{1}{2}\left(\frac{1}{2^{A-t+1}}+\frac{1}{2^{A-t+1}}\right)$$
$$\ge \frac{1}{2}\sum_{b=1}^{2^{s-1}-1}(\beta_b-\beta_{b+1})b - \frac{1}{2} - \frac{2^t}{2^{A+1}}. \quad (2.10)$$

If we use the fact that for any s

$$\frac{1}{2}\sum_{b=1}^{2^{s-1}-1} b(\beta_b-\beta_{b+1}) \ge 0, \quad (2.11)$$

then we have

$$\sum_{j=1}^{A-t-1} L_{t+j}^{1,2}(x) = -\frac{1}{2} - \frac{2^t}{2^{A+1}} \ge -1. \quad (2.12)$$

Now, we give a lower bound for $\sum_{k=1}^{2^A-1} \frac{D_k}{k}$, where $A \geq t$. By inequalities (2.3), (2.4), (2.5), (2.7), (2.12), we have

$$\begin{aligned}\sum_{k=1}^{2^A-1} \frac{D_k}{k} &= L_{0,t} + L_t + \sum_{j=1}^{A-t-1} L_{t+j} \\ &\geq 2^t - 1 + 2^{t+1}\log 2 - 2^t - \sum_{j=1}^{\infty} |L_{t+j} - L^1_{t+j}| + \sum_{j=1}^{A-t-1} L^{1,1}_{t+j} + \sum_{j=1}^{A-t-1} L^{1,2}_{t+j} \\ &\geq 2^{t+1}\log(2) - 1 - 0.206 \cdot 2^t + 0.155 - 0.154 \cdot 2^t - 1 \\ &= 1.026 \cdot 2^t - 1.845. \end{aligned} \tag{2.13}$$

If $A \geq 6$ (that is $n \geq 64$), then we use Lemma 2.1. We get the following inequality:

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{D_k(x)}{k} &= \sum_{k=1}^{2^A-1} \frac{D_k(x)}{k} + \sum_{k=2^A}^{n-1} \frac{D_k(x)}{k} \geq \sum_{k=1}^{2^A-1} \frac{D_k(x)}{k} - 2^t \sum_{k=2^A}^{2^{A+1}-1} \frac{1}{k} \\ &\geq 1.026 \cdot 2^t - 1.845 - 0.7254 \cdot 2^t \geq 0.3006 \cdot 2^t - 1.845.\end{aligned}$$

Let $t \geq 3$. Then the previous inequalities can be written in the following form:

$$\sum_{k=1}^{n-1} \frac{D_k(x)}{k} \geq 2.4 \cdot 2^{t-3} - 1.845 \geq 0.5 \cdot 2^{t-3} \geq \frac{2^t}{16}.$$

To summarize, if $A \geq 6$, $t \geq 3$, $t \leq A$, then we have

$$R_n(x) = \frac{1}{\log n} \sum_{k=1}^{n-1} \frac{D_k(x)}{k} \geq \frac{2^t}{16 \cdot \log n}.$$

The case of $A \leq 6$, $n \in \{1, \ldots, 127\}$ and $t \leq A$ can be investigated by the computer algebra program MATLAB. The computer codes can be found in the Appendix.

We have left the case $t < 3$ and $A \geq 7$ to the end. Suppose that $s \leq 4$. Let the value of J be 7. We apply the inequality (which can also be proved by a computer algebra program)

$$\sum_{k=1}^{2^7-1} \frac{D_k(x)}{k} \geq 0.58 \cdot 2^t, \quad \text{if } t \in \{0, 1, 2\},\ x \in [0, 1].$$

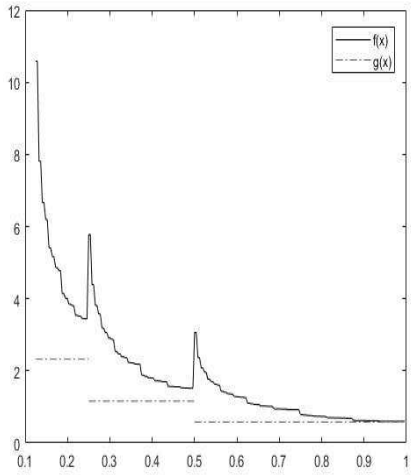

Figure 1. $f(x) = \log 128 \cdot R_{128}(x)$ and $g(x) = 0.58 \cdot 2^t$ plot if $t \in \{0, 1, 2\}$ and $x \in [1/8, 1]$.

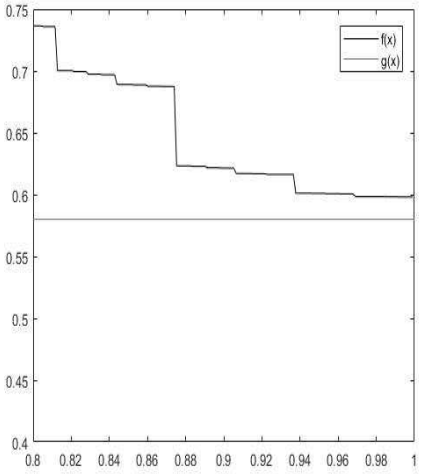

Figure 2. $f(x) = \log 128 \cdot R_{128}$ and $g(x) = 0.58 \cdot 2^t$ plot if $t = 0$ and $x \in [0.8, 1]$.

Then we use (2.6), (2.8), and the following estimation:

$$\sum_{j=J-t-1}^{A-t-1} L_{t+j}^{1,2}(x) = \frac{1}{2}\sum_{j=J-t-1}^{A-t-1}\sum_{b=2^{j-1}}^{2^j-1}\beta_b\omega_b(2^{t+1}x) = \frac{1}{2}\sum_{b=2^{J-t-2}}^{2^{A-t-1}-1}\beta_b\omega_b(2^{t+1}x)$$
$$= \frac{1}{2}\sum_{b=2^{J-t-2}}^{2^{A-t-1}-1}(\beta_b-\beta_{b+1})\sum_{i=2^{J-t-2}}^{b}\omega_i(2^{t+1}x)$$
$$+\frac{1}{2}\beta_{2^{A-t-1}}\left(D_{2^{A-t-1}}(2^{t+1}x)-D_{2^{J-t-1}}(2^{t+1}x)\right).$$

Since $A-t-1 \geq J-t-1 = 6-t \geq 4$ and $2^{t+1}x \in I_{s-1}\setminus I_s$, for any natural number k, we have $|D_k(2^{t+1}x)| \leq 2^{s-1}$, and consequently,

$$\left|\sum_{i=2^{J-t-1}}^{b}\omega_i(2^{t+1}x)\right| \leq 2\cdot 2^{s-1} = 2^s.$$

Besides, $D_{2^{A-t-1}}(2^{t+1}x), D_{2^{J-t-1}}(2^{t+1}x) = 0$. Thus,

$$\sum_{j=J-t-1}^{A-t-1} L_{t+j}^{1,2}(x) \geq -2^{s-1}\sum_{b=2^{J-t-1}}^{2^{A-t-1}-1}(\beta_b-\beta_{b+1}) = -2^{s-1}\left(\beta_{2^{J-t-1}}-\beta_{2^{A-t-1}}\right)$$
$$\geq -2^{s-1}\left(\frac{1}{1+2^{J-t}}+\frac{1}{2^{J-t}}\right) \geq -\frac{2^{s+t}}{2^J}. \tag{2.14}$$

From Lemma 2.2 with $J=7$, $A\geq 7$, $t\leq 2$, $s\leq 4$, it follows that

$$\sum_{k=1}^{n-1}\frac{D_k}{k} = \sum_{k=1}^{2^J-1}\frac{D_k}{k}+\sum_{j=J}^{A-1}\sum_{k=2^j}^{2^{j+1}-1}\frac{D_k}{k}+\sum_{k=2^A}^{n-1}\frac{D_k}{k}$$
$$= \sum_{k=1}^{2^J-1}\frac{D_k}{k}+\sum_{j=J-t}^{A-t-1}L_{t+j}+\sum_{k=2^A}^{n-1}\frac{D_k}{k}$$
$$\geq \sum_{k=1}^{2^7-1}\frac{D_k}{k}-\sum_{j=J-t}^{\infty}|L_{t+j}-L_{t+j}^1|-\sum_{j=J-t}^{\infty}|L_{t+j}^{1,1}|+\sum_{j=J-t}^{A-t-1}L_{t+j}^{1,2}-\sum_{k=2^A}^{2^{A+1}-1}\frac{|D_k|}{k}$$
$$\geq 0.58\cdot 2^t-\frac{2^t-1}{8}\cdot\frac{1.645}{2^{J-t-1}}-\frac{1}{24}\cdot\frac{1.202}{2^{J-t-1}}-0.154\cdot\frac{2^t}{2^{J-t-1}}-\frac{2^{s+t}}{2^J}-0.3627\cdot 2^t.$$

Consequently, for $J=7$, $s\in\{1,2,3,4\}$ and $t\in\{0,1,2\}$, after some numerical calculations we obtain

$$\sum_{k=1}^{n-1}\frac{D_k}{k}(x)-\frac{1}{16}2^t \geq 0.$$

The only task to do is to calculate the term above at 12 points. (The computer algebra code is attached.)

The next task is to discuss the case $A \geq 7$, $s \geq 5$ and $t \in \{0,1,2\}$. By $\beta_b = \dfrac{1}{1+2b} + \dfrac{1}{b}$, we have $\displaystyle\sum_{b=1}^{2^{s-1}-1} b(\beta_b - \beta_{b+1}) \geq 2.1039$. Then we apply (2.5) for the estimation of $\sum |L_{t+j} - L^1_{t+j}|$, and (2.7) for the estimation of $\sum L^{1,1}_{t+j}$. Besides, in the case of $A \geq t+s$, we also apply inequality (2.10) and Lemma 2.2 to estimate $\sum 1/k$.

Taking into account these remarks, we obtain

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{D_k}{k} &= \sum_{k=1}^{2^A-1} \frac{D_k}{k} + \sum_{k=2^A}^{n-1} \frac{D_k}{k} = L_{0,t} + L_t + \sum_{j=1}^{A-t-1} L_{t+j} + \sum_{k=2^A}^{n-1} \frac{D_k}{k} \\
&\geq 2^t - 1 + 2^{t+1}\log 2 - 2^t - \sum_{j=1}^{\infty} |L_{t+j} - L^1_{t+j}| + \sum_{j=1}^{A-t-1} L^{1,1}_{t+j} + \sum_{j=1}^{A-t-1} L^{1,2}_{t+j} - 2^t \sum_{k=2^A}^{n-1} \frac{1}{k} \\
&\geq 2^{t+1}\log(2) - 1 - (0.206 \cdot 2^t - 0.155) - 0.154 \cdot 2^t + (2.1039 - 1) - 0.367 \cdot 2^t \\
&\geq 0.659 \cdot 2^t + 0.258 \geq \frac{1}{16} 2^t.
\end{aligned}$$

In the case of $A \leq t+s-1$, in the calculations above we can apply (2.9) instead of (2.10). In this situation, by $J = A - t - 1$, we have

$$\begin{aligned}
\sum_{j=1}^{A-t-1} L^{1,2}_{t+j}(x) &= \frac{1}{2} \sum_{j=1}^{A-t-1} \sum_{b=2^{j-1}}^{2^j-1} \beta_b \omega_b(2^{t+1}x) = \frac{1}{2} \sum_{j=1}^{A-t-1} \sum_{b=2^{j-1}}^{2^j-1} \beta_b \\
&= \frac{1}{2} \sum_{j=1}^{A-t-1} 2^{j-1} = 2^{A-t-1} - 1 \geq 2^{7-2-1} - 1 = 15.
\end{aligned}$$

Then, in the same way as in the situation of $A \geq t+s$, we have again that $\displaystyle\sum_{k=1}^{n-1} \frac{D_k}{k} \geq \frac{1}{16} 2^t$ for any $s \geq 5$, $t \leq 2$ and $A \geq 7$.

The only situation that remains is $s = \infty$, that is, when $x = \dfrac{1}{2^{t+1}}$. This case can be dealt with the same way as case $A \leq t+s$. Therefore, it is left to the reader. □

Corollary 2.3. *The Walsh logarithmic kernels are positive for each $n \in \mathbb{N}$ and for all $x \in [0,1)$.*

The next corollary shows a sharp contrast between the Walsh–Fejér and the Walsh logarithmic kernels.

Corollary 2.4. *Let $t, a \in \mathbb{N}$ and $t \geq a$. Then we have*

$$\int_{2^{-a}}^{1} \sup_{n \geq 2^t} |R_n(x)| \, dx = \infty.$$

Proof. We apply Theorem 1.8:

$$\begin{aligned}
\int_{2^{-a}}^{1} \sup_{n \geq 2^t} |R_n(x)| \, dx &= \sum_{t=a-1}^{\infty} \int_{2^{-t-1}}^{2^{-t}} \sup_{n \geq 2^t} |R_n(x)| \, dx = \sum_{t=a-1}^{\infty} \int_{2^{-t-1}}^{2^{-t}} \sup_{n \geq 2^t} R_n(x) \, dx \\
&\geq \sum_{t=a-1}^{\infty} \int_{2^{-t-1}}^{2^{-t}} \sup_{n \geq 2^t} \frac{2^t}{16 \cdot \log n} \, dx = c \cdot \sum_{t=a-1}^{\infty} \int_{2^{-t-1}}^{2^{-t}} \frac{2^t}{16 \cdot t} \, dx \\
&= c \cdot \sum_{t=a-1}^{\infty} \frac{1}{t} = \infty.
\end{aligned}$$

□

3. Appendix

Finally, here is the list of MATLAB codes that have been used during our research.

The n-th coordinate of N and x, where N is natural number and $x \in [0, 1)$:

```
> function g = coordN(n,N)
> g = mod(floor(2.^(-n)*N),2);
> end
```

The n-th Rademacher function:

```
> function h = rad(n,x)
> h = (-1).^(mod(floor(2.^(n+1).*x),2));
> end
```

The n-th Walsh function at point x:

```
> function f = walsh(n,x)
> a = floor(log2(1+n));
> for b = 1:1:a+1;
> y(b) = rad(b-1,x)^coordN(b-1,n);
> end
> f = prod(y);
> end
```

The n-th Dirichlet mean at point x:

```
> function f = Dir(n,x)
> for a = 1:1:n;
> y(a) = walsh(a-1,x);
> end
> f = sum(y);
> end
```

The n-th Riesz logarithmic mean at point x:

```
> function f = logkernel(n,x)
> for k = 1:n;
> h(k) = sum(Dir(k,x))/k;
> end
> f = sum(h);
> end
```

The next code counts how many times the code logkernel is less than $2^t/16$:

```
> function f = logkern2t16(n)
> A = floor(log2(n));
> for k = 2:2^A
> x = (k-1)/2^A;t = floor(log2(1/x));
> if logkernel(n,x) < 2^t/16 error(k) = -1;
> else error(k) = 0;
> end
> f = sum(error);
> end
```

The plot of $f(x) = \log(128) \cdot R_{128}$ and $g(x) = 0.58 \cdot 2^t$ for $t \in \{0, 1, 2\}$ and $x \in [1/8, 1]$:

```
> for k = 1:224 x(k) = (k+31)/256; y(k) = logkernel(127,x(k));
> end
> plot(x,y, 'k')
> hold on
> a1=[1/8 1/4];
> a2=[1/4 1/2];
> a3=[1/2 1];
> b3=[0.58 0.58];
> b2=[0.58*2 0.58*2];
> b1=[0.58*4 0.58*4];
> plot(a1,b1, '-.' , a2,b2, '-.', a3,b3, '-.' , 'color' ,
  [0.4 0.4 0.4])
> legend('f(x)' , 'g(x)')
```

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References

[1] S. Baron and F. Schipp, Identical classes of multipliers for Walsh series, *Ann. Univ. Sci. Budapest. Sect. Comput.* **16** (1996), 13–30.

[2] N. K. Bary, A Treatise on Trigonometric Series. Vol. I, *Elsevier Science*, 2014.

[3] N. J. Fine, Cesàro summability of Walsh–Fourier series, *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955), 588–591.

[4] Gy. Gát, On $(C, 1)$P summability for Vilenkin-like systems, *Studia Math.* **144** (2001), 101–120.

[5] Gy. Gát, On the Fejér kernel functions with respect to the Walsh–Paley system, *Acta Acad. Paedagog. Agriensis Sect. Math. (N.S.)* **24** (1997), 105–110.

[6] Gy. Gát, Convergence of sequences of two-dimensional Fejér means of trigonometric Fourier series of integrable functions, *J. Math. Anal. Appl.* **390** (2012), 573–581.

[7] Gy. Gát, On the divergence of the $(C, 1)$ means of double Walsh–Fourier series, *Proc. Amer. Math. Soc.* **128** (2000), 1711–1720.

[8] Gy. Gát, Pointwise convergence of the Cesàro means of double Walsh series, *Ann. Univ. Sci. Budapest. Sect. Comput.* **16** (1996), 173–184.

[9] U. Goginava, Almost everywhere convergence of (C, α)-means of cubical partial sums of d-dimensional Walsh–Fourier series, *J. Approx. Theory* **141** (2006), 8–28.

[10] U. Goginava, Approximation properties of (C, α) means of double Walsh–Fourier series, *Anal. Theory Appl.* **20** (2004), 77–98.

[11] N. Ladhawala, Absolute summability of Walsh–Fourier series, *Pacific J. Math.* **65** (1976), 103–108.

[12] F. Schipp, On the strong summability of Walsh series, *Publ. Math. Debrecen* **52** (1998), 611–633.

[13] F. Schipp, Über gewisse Maximaloperatoren, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **18** (1975), 189–195.

[14] F. Schipp, W. R. Wade, P. Simon and J. Pál, Walsh Series. An Introduction to Dyadic Harmonic Analysis, *Adam Hilger*, *Bristol*, 1990.

[15] F. Weisz, (C, α) summability of Walsh–Fourier series, *Anal. Math.* **27** (2001), 141–155.

[16] A. Zygmund, Trigonometric Series. Second Edition, *Cambridge University Press*, *New York*, 1959.

GYÖRGY GÁT
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4002 DEBRECEN
P. O. BOX 400
HUNGARY

E-mail: gat.gyorgy@science.unideb.hu

GÁBOR LUCSKAI
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4002 DEBRECEN
P. O. BOX 400
HUNGARY

E-mail: gabor.lucskai@science.unideb.hu

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