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The Schur multiplier and stem covers of Leibniz n-algebras

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Abstract. Given a free presentation $0 \to \mathcal{R} \to \mathcal{F} \stackrel{\rho}{\to} \mathcal{G} \to 0$ of a Leibniz n-algebra $\mathcal{G}$, the quotient $\frac{\mathcal{R}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ is known as the Schur multiplier of $\mathcal{G}$. In the article, we construct a four-term exact sequence relating the Schur multiplier of $\mathcal{G}$ and $\mathcal{G}/\mathcal{N}$, from which we derive some formulas concerning dimensions of the underlying vector spaces of the corresponding Schur multipliers. Additionally, this exact sequence is useful to characterize nilpotency of Leibniz n-algebras. Finally, we characterize stem covers of Leibniz n-algebras, showing their existence in case of finite dimension. We also analyze the interaction between stem covers of Leibniz n-algebras and the Schur multiplier.

1. Introduction

A Leibniz n-algebra [14] is a $\mathbb{K}$-vector space $\mathcal{L}$ equipped with a linear map $[-,\ldots,-] : \mathcal{L}^{\otimes n} \to \mathcal{L}$ satisfying the fundamental identity

$$[[x_1,\ldots,x_n],y_1,\ldots,y_{n-1}] = \sum_{i=1}^{n}[x_1,\ldots,x_{i-1},[x_i,y_1,\ldots,y_{n-1}],x_{i+1},\ldots,x_n]$$

for all $x_1,\ldots,x_n,y_1,\ldots,y_{n-1} \in \mathcal{L}$. In case $n = 2$, the fundamental identity becomes the Leibniz identity, so a Leibniz 2-algebra is exactly a Leibniz algebra [21]. The origin of this kind of structures, together with its skew-symmetric version,

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called Lie n-algebras or FILIPPOV algebras [17], [19], was the so called NAMBU mechanics [24], an n-ary generalization of the Hamiltonian mechanics.

This kind of structures have found applications in string theory and M-branes [2], [26], and in a generalization of Nahm's equation proposed by BASU and HARVEY [1]. It also has applications in the construction of solutions of the Yang–Baxter equation [25], and even in the analysis of DNA recombination [27]. On the other hand, the topic of central extensions of an algebraic structure is present in many applications in Physics, for instance, the Witt algebra and its one-dimensional universal central extension, the Virasoro algebra, often appear in problems with conformal symmetry in the setting of string theory [18].

Therefore, our goal in this paper is to continue with the study of central extensions of Leibniz n-algebras started in [9], with special emphasis on the interaction of the Schur multiplier (first Leibniz n-algebra homology with trivial coefficients [5]) and coverings of Leibniz n-algebras.

The outline of the paper is as follows: Section 2 is devoted to recall the background on Leibniz n-algebras, among others, (i-th) nilpotency of an n-sided ideal, exact sequences in (co)-homology and homomorphisms between abelian extensions. In Section 3, we recall that the Schur multiplier of a Leibniz n-algebra $\mathcal{G}$ is the Baer-invariant (see [11], [12], [16]) $\frac{\mathcal{R}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$, where $0\to\mathcal{R}\to\mathcal{F}\overset{\rho}{\to}\mathcal{G}\to 0$ is a free presentation of $\mathcal{G}$. Then we construct a four-term exact sequence that relates the Schur multipliers of $\mathcal{G}$ and $\mathcal{G}/\mathcal{N}$, from which we derive, in case of finite dimension, some formulas concerning dimensions of the underlying vector spaces of the Schur multiplier. In addition, this exact sequence is useful to characterize nilpotency of Leibniz n-algebras. In Section 4, we analyze the interaction between stem covers of Leibniz n-algebras and the Schur multiplier. Specifically, in the case of finite-dimensional Leibniz n-algebras, we show the existence of coverings and we prove that all stem covers, with finite-dimensional Schur multiplier, are isoclinic. In the second part of this section, we characterize stem covers of perfect Leibniz n-algebras, recovering the corresponding results of stem covers in [13] when we restrict to the case $n = 2$.

2. Preliminary results on Leibniz n-algebras

Definition 2.1 ([14]). A Leibniz n-algebra is a $\mathbb{K}$-vector space $\mathcal{L}$ equipped with an n-linear map $[-,\ldots,-] : \mathcal{L}^{\otimes n} \to \mathcal{L}$ satisfying the following fundamental identity:

$$[[x_1,\ldots,x_n],y_1,\ldots,y_{n-1}]=\sum_{i=1}^{n}[x_1,\ldots,x_{i-1},[x_i,y_1,\ldots,y_{n-1}],x_{i+1},\ldots,x_n]. \quad (1)$$

Example 2.2.

(a) If $n = 2$, then Fundamental Identity (1) gives rise to the Leibniz identity, so a Leibniz 2-algebra is simply a Leibniz algebra in the sense of [21].

(b) Lie triple systems [20] are Leibniz 3-algebras satisfying the conditions:

$$[x,y,z]+[y,z,x]+[z,x,y]=0 \quad \text{and} \quad [x,y,y]=0.$$

(c) Leibniz triple systems [3] are vector spaces equipped with a trilinear operation $\langle -,-,-\rangle$ satisfying the following conditions:

- $\langle x,\langle y,z,u\rangle,v\rangle - \langle\langle x,y,z\rangle,u,v\rangle + \langle\langle x,z,y\rangle,u,v\rangle + \langle\langle x,u,y\rangle,z,v\rangle - \langle\langle x,u,z\rangle,y,v\rangle = 0;$ (LTS-A)
- $\langle x,y,\langle z,u,v\rangle\rangle - \langle\langle x,y,z\rangle,u,v\rangle + \langle\langle x,y,u\rangle,z,v\rangle - \langle\langle x,y,v\rangle,u,z\rangle + \langle\langle x,y,v\rangle,z,u\rangle = 0.$ (LTS-B)

If T is a Leibniz triple system, then T is a Leibniz 3-algebra with respect to the operation $[x,y,z] = \langle z,x,y\rangle - \langle z,y,x\rangle$.

(d) Trialgebras are vector spaces equipped with three binary associative operations $\dashv, \perp, \vdash$ satisfying eleven relations (see [23]). The underlying vector space of an associative trialgebra is a non-Lie Leibniz 3-algebra endowed with the 3-bracket $[x,y,z] = x \dashv (y \perp z - z \perp y) - (y \perp z - z \perp y) \vdash x$ [8].

(e) 2-dimensional complex Leibniz 3-algebras with one-dimensional derived algebra are classified in [4].

(f) The algebra of $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$ equipped with the bracket $[f_1,\ldots,f_n] = \mathsf{det}\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\ldots,n}$ is a Leibniz n-algebra.

(g) Any Leibniz algebra $\mathcal{L}$ gives rise to a Leibniz n-algebra under the following n-bracket: $[x_1,x_2,\ldots,x_n] := [x_1,[x_2,\ldots,[x_{n-1},x_n]\ldots]]$.

(h) $\mathbb{R}^{n+1}$ is a Leibniz n-algebra with the bracket given by $[v_1,v_2,\ldots,v_n] := v_1 \times v_2 \times \cdots \times v_n$, where $v_1 \times v_2 \times \cdots \times v_n$ denotes the vector product of the vectors $v_i \in \mathbb{R}^{n+1}$.

A homomorphism of Leibniz n-algebras is a linear map that preserves the bracket. Thus we have defined the category of Leibniz n-algebras, denoted by ${}_n\mathbf{Leib}$.

There is a functor [15] $\mathcal{D}_n : {}_{n+1}\mathbf{Leib} \to {}_2\mathbf{Leib}$ which assigns to a Leibniz $(n+1)$-algebra $\mathcal{L}$ the Leibniz algebra $\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes n}$ with bracket operation

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] := \sum_{i=1}^{n} a_1 \otimes \cdots \otimes [a_i, b_1, \ldots, b_n] \otimes \cdots \otimes a_n.$$

Let $\mathcal{L}$ be a Leibniz n-algebra. A subalgebra $\mathcal{K}$ of $\mathcal{L}$ is called *n-sided ideal* if $[l_1, l_2, \ldots, l_n] \in \mathcal{K}$ as soon as $l_i \in \mathcal{K}$ and $l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n \in \mathcal{L}$, for all $i = 1, 2, \ldots, n$. Let $\mathcal{M}$ and $\mathcal{P}$ be n-sided ideals of a Leibniz n-algebra $\mathcal{L}$. The *commutator ideal* of $\mathcal{M}$ and $\mathcal{P}$, denoted by $[\mathcal{M}, \mathcal{P}, \mathcal{L}, \overset{n-2}{\ldots}, \mathcal{L}]$, is the n-sided ideal of $\mathcal{L}$ spanned by the brackets $[l_1, \ldots, l_i, \ldots, l_j, \ldots, l_n]$ as soon as $l_i \in \mathcal{M}$, $l_j \in \mathcal{P}$ and $l_k \in \mathcal{L}$ for all k different to i, j. Obviously, $[\mathcal{M}, \mathcal{P}, \mathcal{L}, \overset{n-2}{\ldots}, \mathcal{L}] \subset \mathcal{M} \cap \mathcal{P}$. In the particular case $\mathcal{M} = \mathcal{P} = \mathcal{L}$, then we have $[\mathcal{L}, \overset{n}{\ldots}, \mathcal{L}]$, the *derived algebra* of the Leibniz n-algebra $\mathcal{L}$. By $[\overbrace{\mathcal{L}, \ldots, \mathcal{L}}^{i-1}, \mathcal{M}, \overbrace{\mathcal{L}, \ldots, \mathcal{L}}^{n-i}]_i$ we denote the n-sided ideal spanned by the elements $[l_1, \ldots, l_{i-1}, m, l_{i+1}, \ldots, l_n]$, for any $l_j \in \mathcal{L}, j \in \{1, \ldots, i-1, i+1, \ldots, n\}$, and $m \in \mathcal{M}$. Obviously, $[\mathcal{M}, \mathcal{L}, \overset{n-1}{\ldots}, \mathcal{L}] = \sum_{i=1}^{n} [\overbrace{\mathcal{L}, \ldots, \mathcal{L}}^{i-1}, \mathcal{M}, \overbrace{\mathcal{L}, \ldots, \mathcal{L}}^{n-i}]_i$. The *$i$-th center* of a Leibniz n-algebra $\mathcal{L}$ is the *i-th*-sided ideal [4]

$$Z_i(\mathcal{L}) = \{l \in \mathcal{L} \mid [l_1, \ldots, l_{i-1}, l, l_{i+1}, \ldots, l_n]_i = 0, \quad \forall l_j \in \mathcal{L}, j \in \{1, \ldots, \hat{i}, \ldots, n\}\}.$$

The *center* of a Leibniz n-algebra $\mathcal{L}$ is the n-sided ideal $Z(\mathcal{L}) = \bigcap_{i=1}^{n} Z_i(\mathcal{L})$.

An *abelian* Leibniz n-algebra is a Leibniz n-algebra with trivial bracket, that is, the commutator n-sided ideal $[\mathcal{L}, \overset{n}{\ldots}, \mathcal{L}] = 0$. It is clear that a Leibniz n-algebra $\mathcal{L}$ is abelian if and only if $\mathcal{L} = Z(\mathcal{L})$.

Let $\mathcal{M}$ and $\mathcal{P}$ be n-sided ideals of a Leibniz n-algebra $\mathcal{L}$. The *centralizer* of $\mathcal{M}$ and $\mathcal{P}$ on $\mathcal{L}$ [10] is the n-sided ideal

$$C_{\mathcal{L}}(\mathcal{M}, \mathcal{P}) = \{l_i \in \mathcal{L} \mid [l_1, \ldots, l_i, \ldots, l_n] \in \mathcal{P}, i = 1, 2, \ldots, n; l_j \in \mathcal{M},$$
$$j \in \{1, \ldots, \hat{i}, \ldots, n\}; l_k \in \mathcal{L}, k \in \{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\}\}.$$

If $\mathcal{P} = 0$, then $C_{\mathcal{L}}(\mathcal{M}, 0)$ is called the *centralizer* of $\mathcal{M}$ on $\mathcal{L}$. It is denoted briefly by $C_{\mathcal{L}}(\mathcal{M})$. Obviously, $C_{\mathcal{L}}(\mathcal{L}) = Z(\mathcal{L})$.

Definition 2.3 ([10]). We call upper central series of a Leibniz n-algebra $\mathcal{L}$ to the sequence of n-sided ideals defined recursively by

$$\mathcal{Z}_0(\mathcal{L}) = 0; \qquad \mathcal{Z}_k(\mathcal{L}) = C_{\mathcal{L}}(\mathcal{L}, \mathcal{Z}_{k-1}(\mathcal{L})), \quad k \geq 1.$$

Let us observe that $\mathcal{Z}_1(\mathcal{L}) = Z(\mathcal{L})$, and that $\mathcal{Z}_k(\mathcal{L})$ is an n-sided ideal of $\mathcal{L}$ and $\mathcal{Z}_k(\mathcal{L}) \subseteq \mathcal{Z}_{k+1}(\mathcal{L})$.

Definition 2.4 ([10]). For an n-sided ideal $\mathcal{H}$ of a Leibniz n-algebra $\mathcal{L}$, we define recursively the following sequences:

(a) $\mathcal{H}^{<1>_i} = \mathcal{H}, \mathcal{H}^{<k+1>_i} = [\overbrace{\mathcal{L}, \dots, \mathcal{L}}^{i-1}, \mathcal{H}^{<k>_i}, \overbrace{\mathcal{L}, \dots, \mathcal{L}}^{n-i}]_i, \quad k \geq 1, i \in \{1, 2, \dots, n\}.$

(b) $\mathcal{H}^1 = \mathcal{H}, \mathcal{H}^{k+1} = \sum_{i=1}^{n} [\overbrace{\mathcal{L}, \dots, \mathcal{L}}^{i-1}, \mathcal{H}^k, \overbrace{\mathcal{L}, \dots, \mathcal{L}}^{n-i}]_i, \quad k \geq 1.$

The n-sided ideal $\mathcal{H}$ is said to be nilpotent of class k (respectively, i-th nilpotent of class k, $i \in \{1, 2, \dots, n\}$) if there exists $k \in \mathbb{N}$ such that $\mathcal{H}^k \neq 0$ and $\mathcal{H}^{k+1} = 0$, (respectively, $\mathcal{H}^{<k>_i} \neq 0$ and $\mathcal{H}^{<k+1>_i} = 0$).

Remark 2.5. In the case of $\mathcal{H} = \mathcal{L}$, the notion of nilpotent (respectively, i-th nilpotent) Leibniz n-algebra in [4] is recovered.

Proposition 2.6.

(a) *If $\mathcal{L}/Z(\mathcal{L})$ is a nilpotent (respectively, i-th nilpotent) Leibniz n-algebra, then $\mathcal{L}$ is a nilpotent (respectively, i-th nilpotent) Leibniz n-algebra.*

(b) *If $\mathcal{L}$ is a nilpotent and non-trivial Leibniz n-algebra, then $Z(\mathcal{L}) \neq 0$.*

(c) *Let $f : \mathcal{L} \twoheadrightarrow \mathcal{M}$ be a central extension (i.e. $[\mathsf{Ker}(f), \mathcal{L}, \overset{n-1}{\dots}, \mathcal{L}] = 0 \Leftrightarrow \mathsf{Ker}(f) \subseteq Z(\mathcal{L})$) of Leibniz n-algebras. $\mathcal{M}$ is a nilpotent (respectively, i-th nilpotent) Leibniz n-algebra if and only if $\mathcal{L}$ is a nilpotent (respectively, i-th nilpotent) Leibniz n-algebra.*

Proof.

(a) If $\mathcal{L}/Z(\mathcal{L})$ is a nilpotent Leibniz n-algebra (respectively, i-th nilpotent), then there exists a $k \in \mathbb{N}$ such that $(\mathcal{L}/Z(\mathcal{L}))^k = 0$, (respectively, $(\mathcal{L}/Z(\mathcal{L}))^{<k>_i} = 0$), then, $\mathcal{L}^k \subseteq Z(\mathcal{L})$ (respectively, $\mathcal{L}^{<k>_i} \subseteq Z(\mathcal{L})$), hence $\mathcal{L}^{k+1} = \sum_{i=1}^{n} [\overbrace{\mathcal{L}, \dots, \mathcal{L}}^{i-1}, \mathcal{L}^k, \overbrace{\mathcal{L}, \dots, \mathcal{L}}^{n-i}]_i = 0$ (respectively, $\mathcal{L}^{<k+1>_i} = [\overbrace{\mathcal{L}, \dots, \mathcal{L}}^{i-1}, \mathcal{L}^{<k>_i}, \overbrace{\mathcal{L}, \dots, \mathcal{L}}^{n-i}]_i = 0$).

(b) Assume that $\mathcal{L}$ has nilpotency class equal to k, that is $\mathcal{L}^{k+1} = 0$, hence $0 \neq \mathcal{L}^k \subseteq Z(\mathcal{L})$.

(c) There exists a $k \in \mathbb{N}$ such that $\mathcal{M}^k = \sum_{i=1}^{n} [\overbrace{\mathcal{M}, \ldots, \mathcal{M}}^{i-1}, \mathcal{M}^{k-1}, \overbrace{\mathcal{M}, \ldots, \mathcal{M}}^{n-i}]_i = \sum_{i=1}^{n} [\overbrace{f(\mathcal{L}), \ldots, f(\mathcal{L})}^{i-1}, f(\mathcal{L})^{k-1}, \overbrace{f(\mathcal{L}), \ldots, f(\mathcal{L})}^{n-i}]_i = f(\mathcal{L}^k) = 0$. Then $\mathcal{L}^k \subseteq \mathsf{Ker}(f) \subseteq Z(\mathcal{L})$, hence $\mathcal{L}^{k+1} = 0$.
Conversely, the quotient of nilpotent Leibniz n-algebras is nilpotent as well.
Similar argument shows the i-th nilpotency. □

Remark 2.7. Proposition 2.6 (b) is not valid for i-th nilpotency as the following counterexample shows: the two-dimensional Leibniz 3-algebra $\mathcal{L}$ with basis $\{e_1, e_2\}$ and bracket operation given by $[e_2, e_2, e_1] = -e_1, [e_2, e_1, e_2] = e_1, [e_2, e_2, e_2] = e_1$ and zero elsewhere is 1-nilpotent [4] with $Z(\mathcal{L}) = 0$.

2.1. Exact sequences in (co)homology.

A *representation* [14] of a Leibniz n-algebra $\mathcal{L}$ is a $\mathbb{K}$-vector space M equipped with n actions $[-, \ldots, -] : \mathcal{L}^{\otimes i} \otimes \mathsf{M} \otimes \mathcal{L}^{\otimes (n-1-i)} \to \mathsf{M}, 0 \leq i \leq n-1$, satisfying $(2n-1)$ axioms which are obtained from (1) by letting exactly one of the variables $x_1, \ldots, x_n, y_1, \ldots, y_{n-1}$ be in M and all the others in $\mathcal{L}$.

If we define the multilinear applications $\rho_i : \mathcal{L}^{\otimes n-1} \to \mathsf{End}_{\mathbb{K}}(\mathsf{M})$ by

$$\rho_i(l_1, \ldots, l_{n-1})(m) = [l_1, \ldots, l_{i-1}, m, l_{i+1}, \ldots, l_{n-1}], \quad 1 \leq i \leq n,$$

then the axioms of representation can be expressed by means of the following identities [5]:

(1) For $2 \leq k \leq n$,

$$\rho_k([l_1, \ldots, l_n], l_{n+1}, \ldots, l_{2n-2}) = \sum_{i=1}^{n} \rho_i(l_1, \ldots, \hat{l_i}, \ldots, l_n) \cdot \rho_k(l_i, l_{n+1}, \ldots, l_{2n-2}).$$

(2) For $1 \leq k \leq n$,

$$[\rho_1(l_n, \ldots, l_{2n-2}), \rho_k(l_1, \ldots, l_{n-1})] = \sum_{i=1}^{n-1} \rho_k(l_1, \ldots, l_{i-1}, [l_i, l_n, \ldots, l_{2n-2}], l_{i+1}, \ldots, l_{n-1}),$$

where the bracket on $\mathsf{End}_{\mathbb{K}}(\mathsf{M})$ is the usual one for associative algebras.

Let $\mathcal{L}$ be a Leibniz n-algebra, and M be a representation of $\mathcal{L}$. Then $\mathsf{Hom}(\mathcal{L}, \mathsf{M})$ is a $\mathcal{D}_{n-1}(\mathcal{L}) = \mathcal{L}^{\otimes n-1}$-representation as Leibniz algebras [14]. One defines the cochain complex ${}_nCL^*(\mathcal{L}, \mathsf{M})$ to be $CL^*(\mathcal{D}_{n-1}(\mathcal{L}), \mathsf{Hom}(\mathcal{L}, \mathsf{M}))$. We also put ${}_nHL^*(\mathcal{L}, \mathsf{M}) := H^*({}_nCL^*(\mathcal{L}, \mathsf{M}))$. Thus, by definition,

$$ {}_nHL^*(\mathcal{L}, \mathsf{M}) \cong HL^*(\mathcal{D}_{n-1}(\mathcal{L}), \mathsf{Hom}(\mathcal{L}, \mathsf{M})). $$

Here CL^* denotes the Leibniz complex and HL^* its homology, called Leibniz cohomology (see [21], [22] for more information).

Following [14], ${}_nHL^0(\mathcal{L}, \mathsf{M}) \cong \mathsf{Der}(\mathcal{L}, \mathsf{M})$ and ${}_nHL^1(\mathcal{L}, \mathsf{M}) \cong \mathsf{Ext}(\mathcal{L}, \mathsf{M})$, where $\mathsf{Ext}(\mathcal{L}, \mathsf{M})$ denotes the set of isomorphism classes of abelian extensions of $\mathcal{L}$ by M. Exact sequences $E : 0 \to \mathsf{M} \overset{\kappa}{\to} \mathcal{K} \overset{\pi}{\to} \mathcal{L} \to 0$ of Leibniz n-algebras such that $[k_1, \ldots, k_n] = 0, k_1, \ldots, k_n \in \mathcal{K}$, as soon as $k_i, k_j \in \mathsf{M}$ for some $1 \le i, j \le n$ (i.e. $[\mathsf{M}, \mathsf{M}, \mathcal{K}, \overset{n-2}{\ldots}, \mathcal{K}] = 0$), are the objects of the category of abelian extensions of Leibniz n-algebras, whose morphisms are commutative diagrams of the form:

$$
\begin{array}{ccccccccc}
E_1 : 0 & \longrightarrow & \mathsf{M}_1 & \xrightarrow{\kappa_1} & \mathcal{K}_1 & \xrightarrow{\pi_1} & \mathcal{L}_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
E_2 : 0 & \longrightarrow & \mathsf{M}_2 & \xrightarrow{\kappa_2} & \mathcal{K}_2 & \xrightarrow{\pi_2} & \mathcal{L}_2 & \longrightarrow & 0
\end{array}
$$

We denote such morphism as $(\alpha, \beta, \gamma) : E_1 \to E_2$. It is evident that α and γ satisfy the following identities:

$$ \alpha([l_1, \ldots, l_{i-1}, m, l_{i+1}, \ldots, l_n]) = [\gamma(l_1), \ldots, \gamma(l_{i-1}), \alpha(m), \gamma(l_{i+1}), \ldots, \gamma(l_n)], $$

$i = 1, 2, \ldots, n$, provided that M_2 is considered as $\mathcal{L}_1$-representation via γ. That is, α is a morphism of $\mathcal{L}_1$-representations. The equivalence classes in $\mathsf{Ext}(\mathcal{L}, \mathsf{M})$ are provided by the isomorphisms $(1, \beta, 1) : E \to E$.

If E is an abelian extension of Leibniz n-algebras, then M is equipped with an $\mathcal{L}$-representation structure given by $[l_1, \ldots, l_{i-1}, m, l_{i+1}, \ldots, l_n] = [k_1, \ldots, k_{i-1}, \kappa(m), k_{i+1}, \ldots, k_n]$ such that $\pi(k_j) = l_j, j = 1, \ldots, i-1, i+1, \ldots, n, i = 1, 2, \ldots, n$. When the initial $\mathcal{L}$-representation structure of M coincides with the above $\mathcal{L}$-representation structure provided by the extension, then E is said to be an $\mathcal{L}$-extension.

Given an abelian extension E and a homomorphism of Leibniz n-algebras $\gamma : \mathcal{L}_1 \to \mathcal{L}$, we obtain by pulling back along γ an extension E_γ of M by $\mathcal{L}_1$, where $\mathcal{K}_\gamma = \mathcal{K} \times_{\mathcal{L}} \mathcal{L}_1$, together with a morphism of extensions $(1, \gamma', \gamma) : E_\gamma \to E$. The extension E_γ is called the *backward induced extension* of E.

Proposition 2.8 ([9, Proposition 1]). *Every morphism $(\alpha, \beta, \gamma) : E_1 \to E$ of abelian extensions of Leibniz n-algebras admits a unique factorization of the form*

$$ E_1 \overset{(\alpha,\eta,1)}{\to} E_\gamma \overset{(1,\gamma',\gamma)}{\to} E. $$

Given a homomorphism of $\mathcal{L}$-representations $\alpha : \mathsf{M} \to \mathsf{M}_0$, there can be constructed the abelian extension ${}^{\alpha}E : 0 \to \mathsf{M}_0 \overset{\kappa_0}{\to} {}^{\alpha}\mathcal{K} \overset{\pi_0}{\to} \mathcal{L} \to 0$ by putting ${}^{\alpha}\mathcal{K} = (\mathsf{M}_0 \rtimes \mathcal{K})/S$, where $S = \{(\alpha(m), -\kappa(m)) \mid m \in \mathsf{M}\}$. The abelian extension ${}^{\alpha}E$ is called the *forward induced extension* of E.

Proposition 2.9 ([9, Proposition 2]). *Every morphism $(\alpha, \beta, \gamma) : E \to E_0$ of abelian extensions of Leibniz n-algebras admits a unique factorization of the form*

$$E \overset{(\alpha,\alpha',1)}{\to} {}^{\alpha}E \overset{(1,\xi,\gamma)}{\to} E_0$$

through the forward induced extension determined by α.

Homology with trivial coefficients of a Leibniz n-algebra $\mathcal{L}$ is defined in [5] as the homology of the Leibniz complex ${}_nCL_*(\mathcal{L}) := CL_*(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L})$, where the underlying vector space of $\mathcal{L}$ is endowed with a structure of $\mathcal{D}_{n-1}(\mathcal{L})$ symmetric co-representation as Leibniz algebra [22]. We denote the homology groups of this complex by ${}_nHL(\mathcal{L})$, that is

$${}_nHL_*(\mathcal{L}) = H_*(CL_*(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L})) = HL_*(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L}).$$

A direct computation shows that ${}_nHL_0(\mathcal{L}) = \mathcal{L}_{ab} = \frac{\mathcal{L}}{[\mathcal{L}, \overset{n}{\ldots}, \mathcal{L}]}$.

Let be $E : 0 \to \mathsf{M} \overset{\kappa}{\to} \mathcal{K} \overset{\pi}{\to} \mathcal{L} \to 0 \in \mathsf{Ext}(\mathcal{L}, \mathsf{M})$. Then there is an associated natural exact sequence (see [9]):

$$\begin{aligned} 0 \to \mathsf{Der}(\mathcal{L}, \mathsf{M}) \overset{\mathsf{Der}(\pi)}{\to} \mathsf{Der}(\mathcal{K}, \mathsf{M}) \overset{\rho}{\to} \mathsf{Hom}_{\mathcal{L}}(\mathsf{M}, \mathsf{M}) \overset{\theta^*(E)}{\to} \\ {}_nHL^1(\mathcal{L}, \mathsf{M}) \overset{\pi^*}{\to} {}_nHL^1(\mathcal{K}, \mathsf{M}). \end{aligned} \tag{2}$$

From that, we can define $\Delta : \mathsf{Ext}(\mathcal{L}, \mathsf{M}) \to {}_nHL^1(\mathcal{L}, \mathsf{M}), \Delta([E]) = \theta^*(E)(1_{\mathsf{M}})$. The naturality of the sequence (2) implies the well-definition of Δ.

Now for a fixed free presentation $0 \to \mathcal{R} \overset{\chi}{\to} \mathcal{F} \overset{\epsilon}{\to} \mathcal{L} \to 0$, there exists a homomorphism $f : \mathcal{F} \to \mathcal{K}$ such that $\pi \circ f = \epsilon$, which restricts to $\overline{f} : \mathcal{R} \to \mathsf{M}$. Moreover, $\overline{f}$ induces an $\mathcal{L}$-representation homomorphism $\varphi : \mathcal{R}/[\mathcal{R}, \mathcal{R}, \mathcal{F}, \overset{n-2}{\ldots}, \mathcal{F}] \to \mathsf{M}$, where the action of $\mathcal{L}$ on $\mathcal{R}/[\mathcal{R}, \mathcal{R}, \mathcal{F}, \overset{n-2}{\ldots}, \mathcal{F}]$ is given via ϵ, that is,

$$[l_1, \ldots, l_{i-1}, \overline{r}, l_{i+1}, \ldots, l_n] = [x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_n] + [\mathcal{R}, \mathcal{R}, \mathcal{F}, \overset{n-2}{\ldots}, \mathcal{F}],$$

where $\epsilon(x_j) = l_j, j \in \{1, \ldots, i-1, i+1, \ldots, n\}, i \in \{1, \ldots, n\}$. Naturality of sequence (2) induces the following commutative diagram:

$$\begin{array}{ccccccc} \mathsf{Der}(\mathcal{K}, \mathsf{M}) & \longrightarrow & \mathsf{Hom}_{\mathcal{L}}(\mathsf{M}, \mathsf{M}) & \xrightarrow{\theta^*(E)} & {}_nHL^1(\mathcal{L}, \mathsf{M}) & \longrightarrow & {}_nHL^1(\mathcal{K}, \mathsf{M}) \\ \downarrow f^* & & \downarrow \varphi^* & & \downarrow \| & & \downarrow f^* \\ \mathsf{Der}(\mathcal{F}, \mathsf{M}) & \xrightarrow{\tau^*} & \mathsf{Hom}_{\mathcal{L}}(\mathcal{R}/[\mathcal{R}, \mathcal{R}, \mathcal{F}, \overset{n-2}{\ldots}, \mathcal{F}], \mathsf{M}) & \xrightarrow{\sigma^*} & {}_nHL^1(\mathcal{L}, \mathsf{M}) & \longrightarrow & {}_nHL^1(\mathcal{F}, \mathsf{M}) \end{array}$$

Keeping in mind that ${}_nHL^1(\mathcal{F}, \mathsf{M}) = 0$ [14], then

$$\Delta[E] = \theta^*(E)(1_{\mathsf{M}}) = \sigma^*\left(\varphi^*(1_{\mathrm{M}})\right) = \sigma^*(\varphi).$$

Proposition 2.10 ([9, Proposition 3])**.** $\Delta : \mathsf{Ext}(\mathcal{L}, \mathsf{M}) \to {}_nHL^1(\mathcal{L}, \mathsf{M})$ *is an isomorphism.*

Associated to E, there is the isomorphism ${}_nHL^k(\mathcal{L}, \mathsf{M}) \overset{\theta_*}{\cong} \mathsf{Hom}({}_nHL_k(\mathcal{L}), \mathsf{M})$ [5, Theorem 3]. Since $\Delta[E] \in {}_nHL^1(\mathcal{L}, \mathsf{M})$, we have $\theta_*\left(\Delta[E]\right) \in \mathsf{Hom}({}_nHL_1(\mathcal{L}), \mathsf{M})$. Moreover, $\theta_*\left(\Delta[E]\right) = \theta_*(E)$, where $\theta_*(E)$ is the homomorphism given by the following exact sequence (see [5, Theorem 4]):

$$ {}_nHL_1(\mathcal{K}) \to {}_nHL_1(\mathcal{L}) \overset{\theta_*(E)}{\to} \mathsf{M} \to {}_nHL_0(\mathcal{K}) \to {}_nHL_0(\mathcal{L}) \to 0. \tag{3}$$

2.2. Homomorphisms between abelian extensions.

Let $E : 0 \to \mathsf{N} \overset{\chi}{\to} \mathcal{G} \overset{\pi}{\to} \mathcal{Q} \to 0$ be a $\mathcal{Q}$-extension of N, and let $\alpha : \mathsf{N} \to \mathsf{N}'$ be a homomorphism of $\mathcal{Q}$-representations, that is, $\alpha[q_1, \ldots, q_{i-1}, n, q_{i+1}, \ldots, q_n] = [q_1, \ldots, q_{i-1}, \alpha(n), q_{i+1}, \ldots, q_n], n \in \mathsf{N}, q_i \in \mathcal{Q}, 1 \leq i \leq n$, and let $E' : 0 \to \mathsf{N}' \overset{\chi'}{\to} \mathcal{G}' \overset{\pi'}{\to} \mathcal{Q} \to 0$ be a $\mathcal{Q}$-extension with $\Delta[E'] = \xi' \in {}_nHL^1(\mathcal{Q}, \mathsf{N}')$.

Proposition 2.11. *There exists a homomorphism of Leibniz n-algebras $f : \mathcal{G} \to \mathcal{G}'$ such that the diagram*

$$\begin{array}{ccccccccc}
E: 0 & \longrightarrow & \mathsf{N} & \overset{\chi}{\longrightarrow} & \mathcal{G} & \overset{\pi}{\longrightarrow} & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle f} & & \| & & \\
E': 0 & \longrightarrow & \mathsf{N}' & \overset{\chi'}{\longrightarrow} & \mathcal{G}' & \overset{\pi'}{\longrightarrow} & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

is commutative if and only if $\alpha_*(\xi) = \xi' \in {}_nHL^1(\mathcal{Q}, \mathsf{N}')$.

Proof. Naturality of sequence (2) implies $\alpha_*(\xi) = \alpha_*\left(\theta^*(E)(1_{\mathsf{N}})\right) = \theta^*(E) \circ \alpha_* \circ 1_{\mathsf{N}} = \theta^*(E) \circ \alpha = \theta^*(E)\left(\alpha^*(1_{\mathsf{N}'})\right) = \theta^*(E')(1_{\mathsf{N}'}) = \xi'$.

Conversely, for the $\mathcal{Q}$-extension E, we construct the forward induced extension ${}^{\alpha}E$, obtaining the morphism of extensions $(\alpha, f_\alpha, 1) : E \to {}^{\alpha}E$. Thus $\alpha_*(\xi) = \alpha_*\left(\theta^*(E)(1_{\mathsf{N}})\right) = \alpha_*\left(\Delta[E]\right) = \Delta[{}^{\alpha}E]$. Consequently, $\Delta[{}^{\alpha}E] = \alpha_*(\xi) = \xi' = \Delta[E']$, and then ${}^{\alpha}E \equiv E$; thus $E \overset{f_\alpha}{\to} {}^{\alpha}E \overset{\sim}{\to} E'$ concludes the proof. ☐

Let $\gamma : \bar{\mathcal{Q}} \to \mathcal{Q}$ be a homomorphism of Leibniz n-algebras, and let $\bar{E} : 0 \to \mathsf{N} \to \bar{\mathcal{G}} \to \bar{\mathcal{Q}} \to 0$ be a $\bar{\mathcal{Q}}$-extension with $\Delta[\bar{E}] = \bar{\xi} \in {}_nHL^1(\bar{\mathcal{Q}}, \mathsf{N})$.

Proposition 2.12. *There exists a homomorphism of Leibniz n-algebras $\bar{f}$: $\bar{\mathcal{G}} \to \mathcal{G}$ such that the following diagram*

$$\begin{array}{ccccccccc}
\bar{E}: 0 & \longrightarrow & \mathsf{N} & \longrightarrow & \bar{\mathcal{G}} & \longrightarrow & \bar{\mathcal{Q}} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle \gamma} & & \\
E: 0 & \longrightarrow & \mathsf{N} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

is commutative if and only if ${}_nHL^1(\gamma) = \gamma^*(\xi) = \bar{\xi}$.

Proof. If there exists $\bar{f}$, then naturality of sequence (2) implies that $\bar{\xi} = \theta^*(\bar{E})(1_{\mathsf{N}}) = {}_nHL^1(\gamma)\,(\theta^*(E)(1_{\mathsf{N}})) = \gamma^*(\xi)$.

Conversely, from the $\mathcal{Q}$-extension E we construct the backward induced extension E_γ and the morphism of extensions $(1, \gamma^*, \gamma) : E_\gamma \to E$. Hence $\Delta[E_\gamma] = \gamma^*\,(\Delta[E]) = \gamma^*(\xi) = \bar{\xi} = \Delta[\bar{E}]$; consequently, $E_\gamma \equiv \bar{E}$, and then $\bar{f} : \bar{\mathcal{G}} \xrightarrow{\sim} \mathcal{G}_\gamma \to \mathcal{G}$ concludes the proof. □

Proposition 2.13. *Let be given the following diagram:*

$$\begin{array}{ccccccccc}
\bar{E}: 0 & \longrightarrow & \mathsf{N} & \longrightarrow & \bar{\mathcal{G}} & \longrightarrow & \bar{\mathcal{Q}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \gamma} & & \\
E': 0 & \longrightarrow & \mathsf{N}' & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

There exists $f : \bar{\mathcal{G}} \to \mathcal{G}'$ making the diagram commutative if and only if $\alpha_*(\Delta[\bar{E}]) = \gamma^*(\Delta[E'])$.

Proof. If f exists and we consider the decomposition $(\alpha, \sigma, 1) \circ (1, \bar{\gamma}, \gamma)$: $\bar{E} \to E'_\gamma \to E'$ provided by Proposition 2.8, then Propositions 2.11 and 2.12 imply that $\alpha^*(\Delta[\bar{E}]) = \Delta[E'_\gamma] = \gamma^*(\Delta[E'])$.

Conversely, we consider the composition $(\alpha, \sigma, 1) \circ (1, -, 1) \circ (1, \bar{\gamma}, \gamma) : \bar{E} \to {}^{\alpha}\bar{E} \to E'_\gamma \to E'$, and applying Propositions 2.11 and 2.12, we have that $\Delta[{}^{\alpha}\bar{E}] = \alpha_*(\Delta[\bar{E}]) = \gamma^*(\Delta[E']) = \Delta[E'_\gamma]$; consequently, ${}^{\alpha}\bar{E}$ and E'_γ are congruent and $(1, -, 1) = (1, \phi, 1)$ is the wanted morphism. □

3. The Schur multiplier of Leibniz n-algebras

For a free presentation $0 \to \mathcal{R} \to \mathcal{F} \xrightarrow{\rho} \mathcal{G} \to 0$ of a Leibniz n-algebra $\mathcal{G}$, the quotient $\frac{\mathcal{R} \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$ is called the *Schur multiplier* of $\mathcal{G}$, which is denoted by

$\mathsf{M}(\mathcal{G})$. As it is reported in [5], the Schur multiplier is isomorphic to ${}_nHL_1(\mathcal{G})$ and it is a Baer-invariant, which means that it does not depend on the chosen free presentation (see [10], [11], [16]).

Our aim in this section is to show the interaction between the Schur multiplier and nilpotent (respectively, i-th nilpotent) Leibniz n-algebras, as well as the obtention of several formulas concerning dimensions of the underlying vector spaces.

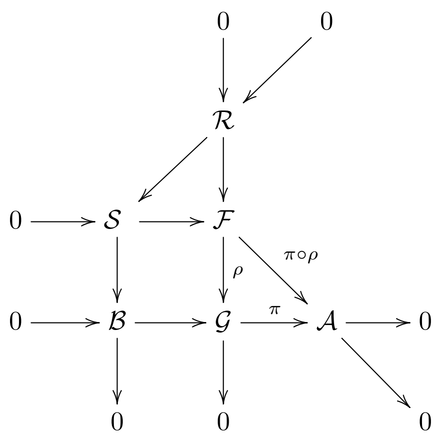
Theorem 3.1. *Let $\mathcal{G}$ be a Leibniz n-algebra with an n-sided ideal $\mathcal{B}$, and set the short exact sequence $0 \to \mathcal{B} \to \mathcal{G} \to \mathcal{A} \to 0$. Then there exists a Leibniz n-algebra $\mathcal{Q}$ with an n-sided ideal $\mathcal{M}$ such that:*

(a) $[\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \cap \mathcal{B} \cong \frac{\mathcal{Q}}{\mathcal{M}}$.

(b) $\mathcal{M} \cong \mathsf{M}(\mathcal{G})$.

(c) $\mathsf{M}(\mathcal{A})$ *is a quotient of* $\mathcal{Q}$.

Proof. Let $0 \to \mathcal{R} \to \mathcal{F} \overset{\rho}{\to} \mathcal{G} \to 0$ be a free presentation of $\mathcal{G}$ and consider the following diagram:



(4)

Then $\mathcal{A} \cong \frac{\mathcal{G}}{\mathcal{B}} \cong \frac{\mathcal{F}/\mathcal{R}}{\mathcal{S}/\mathcal{R}} \cong \frac{\mathcal{F}}{\mathcal{S}}$. Now set $\mathcal{M} \cong \frac{\mathcal{R}\cap[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$ and $\mathcal{Q} \cong \frac{\mathcal{S}\cap[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$. Obviously, $\mathcal{M}$ is an n-sided ideal of $\mathcal{Q}$.

Thus

$$[\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \cap \mathcal{B} \cong \left[\frac{\mathcal{F}}{\mathcal{R}}, \overset{n}{\ldots}, \frac{\mathcal{F}}{\mathcal{R}}\right] \cap \frac{\mathcal{S}}{\mathcal{R}} \cong \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] + \mathcal{R}}{\mathcal{R}} \cap \frac{\mathcal{S}}{\mathcal{R}} \cong \frac{([\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] + \mathcal{R}) \cap \mathcal{S}}{\mathcal{R}}$$

$$\cong \frac{([\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] \cap \mathcal{S}) + \mathcal{R}}{\mathcal{R}} \cong \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] \cap \mathcal{S}}{\mathcal{R} \cap ([\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] \cap \mathcal{S})} \cong \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] \cap \mathcal{S}}{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] \cap \mathcal{R}} \cong$$

$$\cong \frac{([\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] \cap \mathcal{S})/[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{([\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] \cap \mathcal{R})/[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \cong \frac{\mathcal{Q}}{\mathcal{M}}. \tag{5}$$

Now the second statement is obvious. For the third one, since

$$\begin{aligned} \mathsf{M}(\mathcal{A}) &\cong \frac{\mathcal{S} \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \cong \frac{(\mathcal{S} \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}])/[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F},]/[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \\ &\cong \frac{\mathcal{Q}}{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]/[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}, \end{aligned} \tag{6}$$

then $\mathsf{M}(\mathcal{A})$ is the image of $\mathcal{Q}$ under some morphism whose kernel is $\frac{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$. □

Corollary 3.2. *Let $\mathcal{G}$ be a finite-dimensional Leibniz n-algebra, and $\mathcal{B}$ be an n-sided ideal of $\mathcal{G}$ such that $\mathcal{A} \cong \mathcal{G}/\mathcal{B}$. Then*

$$\mathsf{dim}\,(\mathsf{M}(\mathcal{A})) \leq \mathsf{dim}\,(\mathsf{M}(\mathcal{G})) + \mathsf{dim}\,([\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \cap \mathcal{B})\,.$$

Proof. From equation (5) we have the short exact sequence of vector spaces

$$0 \to \mathcal{M} \to \mathcal{Q} \to [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \cap \mathcal{B} \to 0,$$

hence $\mathsf{dim}(\mathcal{Q}) = \mathsf{dim}(\mathcal{M}) + \mathsf{dim}([\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \cap \mathcal{B}) = \mathsf{dim}(\mathsf{M}(\mathcal{G})) + \mathsf{dim}([\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \cap \mathcal{B})$.

On the other hand, equation (6) implies that $\mathsf{dim}(\mathsf{M}(\mathcal{A})) \leq \mathsf{dim}(\mathcal{Q})$, which completes the proof. □

Theorem 3.3. *Let $\mathcal{G}$ be a finite-dimensional Leibniz n-algebra, and $\mathcal{B}$ be a central n-sided ideal of $\mathcal{G}$ (i.e. $\mathcal{B} \subseteq Z(\mathcal{G})$, equivalently, $[\mathcal{B}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}] = 0$) such that $\mathcal{A} \cong \mathcal{G}/\mathcal{B}$. Then*

$$\mathsf{dim}\,(\mathcal{M}(\mathcal{G})) + \mathsf{dim}\,(\mathcal{B} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]) \leq \mathsf{dim}\,(\mathsf{M}(\mathcal{A})) + \sum_{i=1}^{n-1} \mathsf{dim}(J_i)$$

where $J_i = (\mathcal{B} \otimes \overset{n-i}{\ldots} \otimes \mathcal{B} \otimes \mathcal{G}_{\mathrm{ab}} \otimes \overset{i}{\ldots} \otimes \mathcal{G}_{\mathrm{ab}}) \oplus (\mathcal{B} \otimes \overset{n-i-1}{\ldots} \otimes \mathcal{B} \otimes \mathcal{G}_{\mathrm{ab}} \otimes \mathcal{B} \otimes \mathcal{G}_{\mathrm{ab}} \otimes \overset{i-1}{\ldots} \otimes \mathcal{G}_{\mathrm{ab}}) \oplus \cdots \oplus (\mathcal{G}_{\mathrm{ab}} \otimes \overset{i}{\ldots} \otimes \mathcal{G}_{\mathrm{ab}} \otimes \mathcal{B} \otimes \overset{n-i}{\ldots} \otimes \mathcal{B})$.

Proof. From exact sequence (2) in [7], there is the exact sequence

$$\bigoplus_{i=1}^{n-1} J_i \to \mathsf{M}(\mathcal{G}) \to \mathsf{M}(\mathcal{A}) \to \mathcal{B} \to \mathcal{G}_{\mathrm{ab}} \to \mathcal{A}_{\mathrm{ab}} \to 0,$$

and having in mind diagram (4), there is a surjection $C : \bigoplus_{i=1}^{n-1} J_i \to \dfrac{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$.

From the proof of Corollary 3.2 and keeping in mind equation (6), we have:

$$\dim(\mathsf{M}(\mathcal{G})) + \dim([\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \cap \mathcal{B}) = \dim(\mathcal{Q})$$
$$= \dim(\mathsf{M}(\mathcal{A})) + \dim\left(\frac{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}\right) \leq \dim(\mathcal{M}(\mathcal{A})) + \sum_{i=1}^{n-1} \dim(J_i). \qquad \square$$

Lemma 3.4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be n-sided ideals of a Leibniz n-algebra $\mathcal{G}$ such that $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{C}$. Then $\frac{\mathcal{A}+\mathcal{B}}{\mathcal{C}+\mathcal{B}} \cong \frac{\mathcal{A}}{\mathcal{C}}$.*

Proof. It is enough to use the following identification: $a + b + (\mathcal{C} + \mathcal{B}) \equiv a + \mathcal{C}$. $\square$

Lemma 3.5. *Let $0 \to \mathcal{R} \to \mathcal{F} \overset{\rho}{\to} \mathcal{G} \to 0$ be a free presentation of a Leibniz n-algebra $\mathcal{G}$. Let $\mathcal{N}$ be an n-sided ideal of $\mathcal{G}$, and $\mathcal{S}$ be an n-sided ideal of $\mathcal{F}$ such that $\mathcal{N} \cong \frac{\mathcal{S}+\mathcal{R}}{\mathcal{R}}$. Then the quotient $\frac{\mathcal{R} \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$ does not depend on the n-sided ideal $\mathcal{S}$.*

Proof. Assume there is another n-sided ideal $\mathcal{S}'$ such that $\mathcal{N} \cong \frac{\mathcal{S}'+\mathcal{R}}{\mathcal{R}}$. Since $\frac{\mathcal{S}+\mathcal{R}}{\mathcal{R}} \cong \mathcal{N} \cong \frac{\mathcal{S}'+\mathcal{R}}{\mathcal{R}}$, then for any $s \in \mathcal{S}$ there exists a $s' \in \mathcal{S}'$ such that $s - s' \in \mathcal{R}$.

Define $\varphi : \mathcal{R} \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \to \mathcal{R} \cap \big([\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + [\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big)$ by $\varphi(r) = r + 0$. Obviously, φ is a homomorphism satisfying $\varphi\big([\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big) \subseteq [\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap \big([\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + [\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big)$, then φ induces the homomorphism $\overline{\varphi}\left(r + \big([\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big)\right) = r + \big([\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap \big([\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + [\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big)\big)$. It can be easily checked that $\overline{\varphi}$ is a bijection. Now Lemma 3.4 provides the following isomorphisms:

$$\frac{\mathcal{R} \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \cong \frac{\mathcal{R} \cap \big([\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + [\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big)}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap \big([\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + [\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big)}$$
$$\cong \frac{\big(\mathcal{R} \cap [\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big) + [\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{\big([\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]\big) + [\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$$
$$\cong \frac{\mathcal{R} \cap [\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}', \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \qquad \square$$

Theorem 3.6. *Let $0 \to \mathcal{R} \to \mathcal{F} \overset{\rho}{\to} \mathcal{G} \to 0$ be a free presentation of a Leibniz n-algebra $\mathcal{G}$. Let $\mathcal{N}$ be an n-sided ideal of $\mathcal{G}$. Then the following sequence is exact and natural for any n-sided ideal $\mathcal{S}$ of $\mathcal{F}$ such that $\mathcal{N} \cong \frac{\mathcal{S}+\mathcal{R}}{\mathcal{R}}$.*

$$0 \to \frac{\mathcal{R} \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \overset{\Pi}{\to} \mathsf{M}(\mathcal{G}) \overset{\Sigma}{\to} \mathsf{M}\left(\frac{\mathcal{G}}{\mathcal{N}}\right) \overset{\Gamma}{\to} \frac{\mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]}{[\mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}]} \to 0. \quad (7)$$

Proof. Obviously, $\mathsf{M}(\mathcal{G}) \cong \frac{\mathcal{R} \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$. Since $\frac{\mathcal{G}}{\mathcal{N}} \cong \frac{\mathcal{F}/\mathcal{R}}{(\mathcal{S}+\mathcal{R})/\mathcal{R}} \cong \frac{\mathcal{F}}{\mathcal{S}+\mathcal{R}}$, we have that $0 \to \mathcal{S} + \mathcal{R} \to \mathcal{F} \to \mathcal{G}/\mathcal{N} \to 0$ is a free presentation of $\mathcal{G}/\mathcal{N}$, hence $\mathsf{M}\left(\frac{\mathcal{G}}{\mathcal{N}}\right) \cong \frac{(\mathcal{S}+\mathcal{R}) \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{S}+\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$.

On the other hand, we can rewrite

$$\frac{\mathcal{N}\cap[\mathcal{G},\overset{n}{\ldots},\mathcal{G}]}{[\mathcal{N},\mathcal{G},\overset{n-1}{\ldots},\mathcal{G}]}\cong\frac{\frac{\mathcal{S}+\mathcal{R}}{\mathcal{R}}\cap[\frac{\mathcal{F}}{\mathcal{R}},\overset{n}{\ldots},\frac{\mathcal{F}}{\mathcal{R}}]}{[\frac{\mathcal{S}+\mathcal{R}}{\mathcal{R}},\frac{\mathcal{F}}{\mathcal{R}},\overset{n-1}{\ldots},\frac{\mathcal{F}}{\mathcal{R}}]}\cong\frac{\frac{\mathcal{S}+\mathcal{R}}{\mathcal{R}}\cap\frac{[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]+\mathcal{R}}{\mathcal{R}}}{\frac{[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{\mathcal{R}}}$$
$$\cong\frac{(\mathcal{S}+\mathcal{R})\cap([\mathcal{F},\overset{n}{\ldots},\mathcal{F}]+\mathcal{R})}{[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]+\mathcal{R}}.$$

Then it suffices to show that the following sequence is exact:

$$0\to\frac{\mathcal{R}\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}\overset{\Pi}{\to}\frac{\mathcal{R}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}\overset{\Sigma}{\to}$$
$$\frac{(\mathcal{S}+\mathcal{R})\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}\overset{\Gamma}{\to}\frac{(\mathcal{S}+\mathcal{R})\cap([\mathcal{F},\overset{n}{\ldots},\mathcal{F}]+\mathcal{R})}{[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]+\mathcal{R}}\to 0.$$

Define $\Pi:\frac{\mathcal{R}\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}\longrightarrow\frac{\mathcal{R}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ by $\Pi(x+([\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]))=x+[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]$. It is easy to check that Π is an injective well-defined linear map.

Define $\Sigma:\frac{\mathcal{R}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}\longrightarrow\frac{(\mathcal{S}+\mathcal{R})\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ by $\Sigma(x+[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}])=x+[\mathcal{S}+\mathcal{R},\ \mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]$. Obviously, Σ is a well-defined linear map and $\Sigma\circ\Pi=0$, consequently, $\mathsf{Im}(\Pi)\subseteq\mathsf{Ker}(\Sigma)$.

On the other hand, given $x+[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\in\mathsf{Ker}(\Sigma)$, then $x\in[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]$. Hence $x\in\mathcal{R}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]\cap[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]=\mathcal{R}\cap[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]$. Thus $x+[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]=[s+r,f_2,\ldots,f_n]+[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\in\frac{[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$. Summarizing, $x+[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\in\frac{\mathcal{R}\cap[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}=\frac{\mathcal{R}\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$.

Then $x+\left([\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\right)\in\frac{\mathcal{R}\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\cap[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ satisfies that $\Pi\left(x+\left([\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\cap\ [\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\right)\right)=x+[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]$, which implies that $\mathsf{Ker}(\Sigma)\subseteq\mathsf{Im}(\Pi)$.

Define $\Gamma:\frac{(\mathcal{S}+\mathcal{R})\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}\to\frac{(\mathcal{S}+\mathcal{R})\cap([\mathcal{F},\overset{n}{\ldots},\mathcal{F}]+\mathcal{R})}{[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]+\mathcal{R}}$ by $\Gamma\left(x+[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\right)$ $=x+([\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]+\mathcal{R})$. Γ is a well-defined linear map such that $\Gamma\circ\Sigma=0$, then $\mathsf{Im}(\Sigma)\subseteq\mathsf{Ker}(\Gamma)$.

For the converse, let $x+[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\in\frac{(\mathcal{S}+\mathcal{R})\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ such that $\Gamma(x+[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}])=x+([\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]+\mathcal{R})=\overline{0}$.

We need to prove that $x+[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\in\mathsf{Im}(\Sigma)$. This occurs only if $x+[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]\in\frac{\mathcal{R}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$, so it suffices to show that $x+[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]=r+[\mathcal{S}+\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]$ for some $r\in\mathcal{R}$.

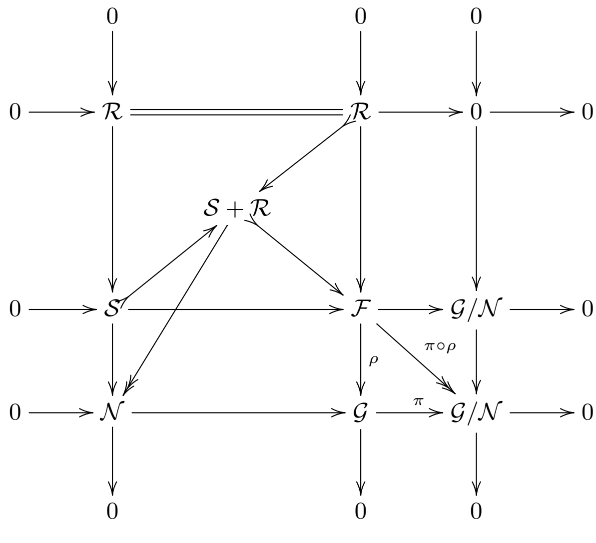
Since $x \in [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + \mathcal{R}$, we get $x - r \in [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]$ for some $r \in \mathcal{R}$. Thus $x - r + [\mathcal{S} + \mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] = \bar{0}$, i.e. $x + [\mathcal{S} + \mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] = r + [\mathcal{S} + \mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]$. Consequently, $\bar{x} \in \frac{\mathcal{R} \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{S}+\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$.

Γ is surjective. Indeed, for $x + ([\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + \mathcal{R}) \in \frac{(\mathcal{S}+\mathcal{R}) \cap ([\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] + \mathcal{R})}{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + \mathcal{R}}$, we have that $x \in \mathcal{S} + \mathcal{R}$ and $\bar{x} \in \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}] + \mathcal{R}}{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + \mathcal{R}} \cong \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + \mathcal{R}}$. Hence $x \in (\mathcal{S} + \mathcal{R}) \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]$ and $\Gamma(x + [\mathcal{S} + \mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]) = x + ([\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] + \mathcal{R})$.

To show that sequence (7) is natural, it is easy to check, keeping in mind Lemma 3.5, that a homomorphism $f : \mathcal{G} \to \mathcal{G}'$ such that $f(\mathcal{N}) \subseteq \mathcal{N}'$ ($\mathcal{N} \trianglelefteq \mathcal{G}, \mathcal{N}' \trianglelefteq \mathcal{G}'$) induces homomorphisms between the Schur multipliers such that the following diagram is commutative:

$$\begin{array}{ccccccc}
\frac{\mathcal{R} \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} & \overset{\pi}{\rightarrowtail} & \mathsf{M}(\mathcal{G}) & \overset{\sigma}{\longrightarrow} & \mathsf{M}\left(\frac{\mathcal{G}}{\mathcal{N}}\right) & \overset{\tau}{\twoheadrightarrow} & \frac{\mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]}{[\mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}]} \\
\downarrow & & \downarrow f* & & \downarrow \overline{f*} & & \downarrow \\
\frac{\mathcal{R}' \cap [\mathcal{S}', \mathcal{F}', \overset{n-1}{\ldots}, \mathcal{F}']}{[\mathcal{R}', \mathcal{F}', \overset{n-1}{\ldots}, \mathcal{F}'] \cap [\mathcal{S}', \mathcal{F}', \overset{n-1}{\ldots}, \mathcal{F}']} & \overset{\pi'}{\rightarrowtail} & \mathsf{M}(\mathcal{G}') & \overset{\sigma'}{\longrightarrow} & \mathsf{M}\left(\frac{\mathcal{G}'}{\mathcal{N}'}\right) & \overset{\tau'}{\twoheadrightarrow} & \frac{\mathcal{N}' \cap [\mathcal{G}', \overset{n}{\ldots}, \mathcal{G}']}{[\mathcal{N}', \mathcal{G}', \overset{n-1}{\ldots}, \mathcal{G}']}
\end{array}$$
□

Remark 3.7. Let us observe that an n-sided ideal $\mathcal{S}$ such that $\mathcal{N} \cong \frac{\mathcal{S}+\mathcal{R}}{\mathcal{R}}$ as in Theorem 3.6 always exists. Indeed, it is enough to consider the following 3×3 diagram:



Corollary 3.8. *Let $\mathcal{G}$ be a nilpotent Leibniz n-algebra of class $k \geq 2$, and $0 \to \mathcal{R} \to \mathcal{F} \overset{\rho}{\to} \mathcal{G} \to 0$ be a free presentation of $\mathcal{G}$, then the following sequence is*

exact and natural:

$$0 \to \frac{\mathcal{F}^{k+1}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap \mathcal{F}^{k+1}} \to \mathsf{M}(\mathcal{G}) \to \mathsf{M}\left(\frac{\mathcal{G}}{\mathcal{G}^{\mathsf{k}}}\right) \to \mathcal{G}^{\mathsf{k}} \to 0.$$

Proof. Take $\mathcal{N} = \mathcal{G}^k$ and $\mathcal{S} = \mathcal{F}^k$ in Theorem 3.6. Then $\mathcal{N} = \mathcal{G}^k \cong \frac{\mathcal{F}^k}{\mathcal{R}} \cong \frac{\mathcal{F}^k + \mathcal{R}}{\mathcal{R}} = \frac{\mathcal{S}+\mathcal{R}}{\mathcal{R}}$ and $[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] = [\mathcal{F}^k, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] = \mathcal{F}^{k+1} \subseteq \mathcal{R}$. Now exact sequence (7) concludes the proof. □

Corollary 3.9. *Let $\mathcal{N}$ be an n-sided ideal of a finite-dimensional Leibniz n-algebra $\mathcal{G}$ satisfying the hypotheses of Theorem 3.6. Then it holds:*

$$\dim\left(\mathsf{M}\left(\frac{\mathcal{G}}{\mathcal{N}}\right)\right) + \dim\left(\frac{\mathcal{R} \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}\right) = \dim\left(\mathsf{M}(\mathcal{G})\right) + \dim\left(\frac{\mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]}{[\mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}]}\right).$$

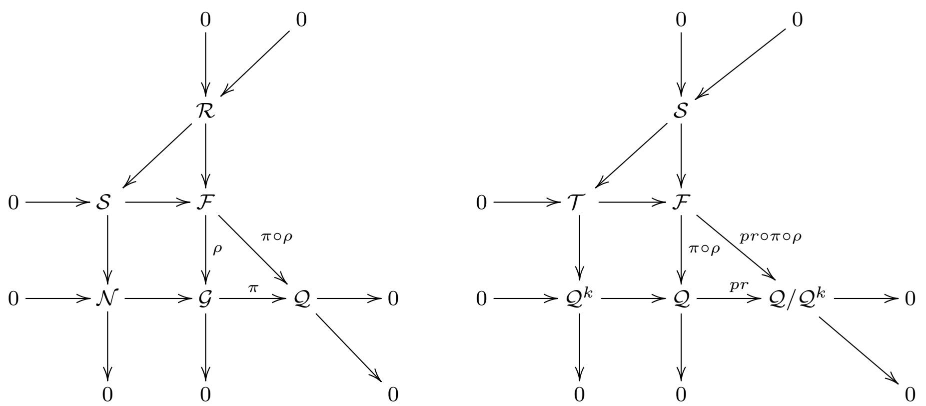
Proof. From exact sequence (7) we have

$$\dim\left(\mathsf{M}\left(\frac{\mathcal{G}}{\mathcal{N}}\right)\right) = \dim\left(\mathsf{Im}(\Sigma)\right) + \dim\left(\frac{\mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]}{[\mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}]}\right) = \dim\left(\mathsf{M}(\mathcal{G})\right) - \dim\left(\frac{\mathcal{R} \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \cap [\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}\right) + \dim\left(\frac{\mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]}{[\mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}]}\right).$$ □

Definition 3.10. Let $\mathcal{Q}$ be a nilpotent Leibniz n-algebra of class k. An extension of Leibniz n-algebras $0 \to \mathcal{N} \to \mathcal{G} \overset{\pi}{\to} \mathcal{Q} \to 0$ is said to be of class k if $\mathcal{G}$ is nilpotent of class k.

Theorem 3.11. *A central extension $0 \to \mathcal{N} \to \mathcal{G} \overset{\pi}{\to} \mathcal{Q} \to 0$ is of class k if and only if $\theta : \mathsf{M}(\mathcal{Q}) \to \mathcal{N}$ vanishes over $\mathsf{Ker}(\tau)$, where $\tau : \mathsf{M}(\mathcal{Q}) \to \mathsf{M}\left(\mathcal{Q}/\mathcal{Q}^{\mathsf{k}}\right)$ is induced by the canonical projection $\mathcal{Q} \twoheadrightarrow \mathcal{Q}^k$.*

Proof. Consider the following diagrams of free presentations:



then $\theta : \mathsf{M}(\mathcal{Q}) = \frac{\mathcal{S}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \to \mathcal{N}$, given by $\theta(x + [\mathcal{S}, \mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]) = \rho(x)$, is well-defined and $\mathsf{Ker}(\tau) \cong \frac{[\mathcal{T},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{[\mathcal{S},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$.

Assume that $\mathcal{G}$ is nilpotent of class k, and consider $x+[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \in \mathsf{Ker}(\tau)$. Therefore $\theta(x+[\mathcal{S}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]) = \rho(x) = 0$, since $\rho(x) \in [\rho(\mathcal{T}), \rho(\mathcal{F}), \overset{n-1}{\ldots}, \rho(\mathcal{F}))] \subseteq [\mathcal{G}^k + \mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}] \subseteq \mathcal{G}^{k+1} = 0$. For the last inclusion, it is necessary to keep in mind that $\pi \circ \rho(\mathcal{T}) \subseteq \mathcal{Q}^k = \pi(\mathcal{G}^k)$, and consequently $\rho(\mathcal{T}) \subseteq \mathcal{G}^k + \mathcal{N}$.

Conversely, $\mathcal{G}^{k+1} = \sum_{i=1}^{n} [\mathcal{G}, \ldots, \mathcal{G}^k, \ldots, \mathcal{G}]_i = \sum_{i=1}^{n} [\rho(\mathcal{F}), \ldots, \rho(\mathcal{F}^k), \ldots, \rho(\mathcal{F})]_i$

$\subseteq \sum_{i=1}^{n} \rho[\mathcal{F}, \ldots, \mathcal{T}, \ldots, \mathcal{F}]_i = \rho[\mathcal{T}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] = 0$, since $[\mathcal{T}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}] \subseteq \mathcal{R}$, because θ vanishes over $\mathsf{Ker}(\tau)$. For the last inclusion, it is necessary to keep in mind that $\pi \circ \rho(\mathcal{F}^k) \subseteq \mathcal{Q}^k$, hence $\mathcal{F}^k \subseteq \mathcal{T}$. □

Proposition 3.12. *Let $\mathcal{G}$ be a nilpotent Leibniz n-algebra, and $f : \mathcal{G} \twoheadrightarrow \mathcal{Q}$ be a surjective homomorphism of Leibniz n-algebras. If $\mathsf{Ker}(f) \subseteq [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]$ and $\mathsf{M}(\mathcal{Q}) = 0$, then f is an isomorphism.*

In particular, if $\mathsf{M}\left(\frac{\mathcal{G}}{[\mathcal{G},\overset{n}{\ldots},\mathcal{G}]}\right) = 0$, then $\mathsf{M}(\mathcal{G}) = 0$.

Proof. Let $\mathcal{N} = \mathsf{Ker}(f)$ be, then $\mathsf{M}(\mathcal{G}/\mathcal{N}) = 0$. From exact sequence (7), we have that $\mathcal{N} = \mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \subseteq [\mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}]$, then $\mathcal{N} \subseteq [\mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}]$. Obviously, $\supseteq$ is true, then $\mathcal{N} = [\mathcal{N}, \mathcal{G}, \overset{n-1}{\ldots}, \mathcal{G}]$.

Obviously, $\mathcal{N} = \mathcal{N}^j \subseteq \mathcal{G}^j$, for all $j \geq 1$. Since $\mathcal{G}$ is nilpotent, there exists $k \in \mathbb{N}$ such that $\mathcal{G}^{k+1} = 0$, which implies that $\mathcal{N} = 0$, and consequently, f is an isomorphism.

The second statement is an obvious consequence of the first one. □

4. Stem covers

The study of different types of central extensions, together with their corresponding characterizations, is the subject of [9, Section 4]. To summarize, let $\mathcal{G}$ and $\mathcal{Q}$ be two Leibniz n-algebras; a central extension $f : \mathcal{G} \twoheadrightarrow \mathcal{Q}$ is said to be a *stem extension* if $\mathsf{Ker}(f) \subseteq [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]$. Additionally, if the induced map $\mathsf{M}(\mathcal{G}) \to \mathsf{M}(\mathcal{Q})$ is the zero map, then the stem extension $f : \mathcal{G} \twoheadrightarrow \mathcal{Q}$ is said to be a *stem cover*. In this last case, $\mathcal{G}$ is said to be a *cover* or a *covering* algebra. Our goal in this section is to analyze the interaction between stem covers of Leibniz n-algebras and the Schur multiplier.

Stem extensions are characterized in [9] as follows:

Proposition 4.1. *The following statements are equivalent for a central extension* $E : 0 \to \mathsf{N} \stackrel{i}{\to} \mathcal{G} \stackrel{f}{\to} \mathcal{Q} \to 0$*:*

(a) E *is a stem extension.*

(b) $\theta_* \left(\Delta[E]\right)$ *is a surjective homomorphism.*

(c) $i_* : \mathsf{N} \to {}_nHL_0(\mathcal{G})$ *is the zero map.*

(d) $f_* : {}_nHL_0(\mathcal{G}) \stackrel{\sim}{\to} {}_nHL_0(\mathcal{Q})$*.*

Lemma 4.2. *Let* $0 \to \mathcal{R} \to \mathcal{F} \stackrel{\rho}{\to} \mathcal{G} \to 0$ *be a free presentation of a Leibniz* n*-algebra* $\mathcal{G}$*, and let* $0 \to \mathsf{M} \to \mathcal{P} \stackrel{\psi}{\to} \mathcal{Q} \to 0$ *be a central extension of another Leibniz* n*-algebra* $\mathcal{Q}$*. Then for each homomorphism* $\alpha : \mathcal{G} \to \mathcal{Q}$*, there exists a homomorphism* $\beta : \frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \to \mathcal{P}$ *such that* $\beta\left(\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}\right) \subseteq \mathsf{M}$ *and the following diagram is commutative:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} & \longrightarrow & \frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} & \stackrel{\overline{\rho}}{\longrightarrow} & \mathcal{G} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_|} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & \mathsf{M} & \longrightarrow & \mathcal{P} & \stackrel{\psi}{\longrightarrow} & \mathcal{Q} & \longrightarrow & 0
\end{array}
$$

where $\overline{\rho}$ *is the natural surjective homomorphism induced by* ρ*.*

Proof. Since $\mathcal{F}$ is a free Leibniz n-algebra, there exists $\omega : \mathcal{F} \to \mathcal{P}$ such that $\psi \circ \omega = \alpha \circ \rho$.

On the other hand, $\psi(\omega(\mathcal{R})) = \alpha(\rho(\mathcal{R})) = 0$, hence $\omega(\mathcal{R}) \subseteq \mathsf{M}$, which implies the vanishing of ω over $[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]$. So ω induces $\beta : \frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \to \mathcal{P}$, and for any $r \in \mathcal{R}$, $\beta(r + [\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]) = \omega(r) \in \mathsf{M}$. □

Theorem 4.3. *Let* $\mathcal{G}$ *be a Leibniz* n*-algebra such that* $\mathsf{M}(\mathcal{G})$ *is finite-dimensional, and let* $0 \to \mathcal{R} \to \mathcal{F} \stackrel{\rho}{\to} \mathcal{G} \to 0$ *be a free presentation of* $\mathcal{G}$*. The extension* $0 \to \mathsf{M} \to \mathcal{P} \stackrel{\psi}{\to} \mathcal{G} \to 0$ *is a stem cover if and only if there exists an* n*-sided ideal* $\mathcal{S}$ *of* $\mathcal{F}$ *such that*

(a) $\mathcal{P} \cong \frac{\mathcal{F}}{\mathcal{S}}$ *and* $\mathsf{M} \cong \frac{\mathcal{R}}{\mathcal{S}}$*.*

(b) $\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \cong \mathsf{M}(\mathcal{G}) \oplus \frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}$*.*

Proof. Let $0 \to \mathsf{M} \to \mathcal{P} \stackrel{\psi}{\to} \mathcal{G} \to 0$ be a stem cover. By Lemma 4.2, there exists a homomorphism $\beta : \frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \to \mathcal{P}$ such that $\psi \circ \beta = \overline{\rho}$ and $\beta\left(\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}\right) \subseteq \mathsf{M}$.

Since $\mathcal{P} = \mathsf{Im}(\beta) + \mathsf{M}$ and $\mathsf{M} \subseteq Z(\mathcal{P})$, we have $\mathsf{M} \subseteq [\mathcal{P}, \overset{n}{\ldots}, \mathcal{P}] = [\mathsf{Im}(\beta) + \mathsf{M}, \overset{n}{\ldots}, \mathsf{Im}(\beta) + \mathsf{M}] \subseteq \mathsf{Im}(\beta)$. Thus β is surjective, and $\beta\left(\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}\right) = \mathsf{M}$.

Now, let $\mathcal{S}$ be an n-sided ideal of $\mathcal{F}$ such that $\mathsf{Ker}(\beta) = \frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$. Then we have the exact sequence $0 \to \frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \to \frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \overset{\beta}{\to} \mathcal{P} \to 0$, which induces the exact sequence $0 \to \frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \to \frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \overset{\beta_|}{\to} \mathsf{M} \to 0$. It follows from these two exact sequences and the third isomorphism theorem that $\mathcal{P} \cong \frac{\mathcal{F}/[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{\mathcal{S}/[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \cong \frac{\mathcal{F}}{\mathcal{S}}$ and $\mathsf{M} \cong \frac{\mathcal{R}/[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{\mathcal{S}/[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \cong \frac{\mathcal{R}}{\mathcal{S}}$. Moreover, $\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \cong \mathsf{M} \oplus \frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ as $\mathbb{K}$-vector spaces, and thus $\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \cong \mathsf{M}(\mathcal{G}) \oplus \frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$, since $\mathsf{M} \cong \mathsf{M}(\mathcal{G})$ by [9, Proposition 9].

Conversely, suppose the existence of an n-sided ideal $\mathcal{S}$ of $\mathcal{F}$ satisfying (a) and (b). Then, $\frac{\mathcal{P}}{\mathsf{M}} \cong \frac{\mathcal{F}/\mathcal{S}}{\mathcal{R}/\mathcal{S}} \cong \frac{\mathcal{F}}{\mathcal{R}} \cong \mathcal{G}$, and $\mathsf{M}(\mathcal{G}) \cong \frac{\mathcal{R}/[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}{\mathcal{S}/[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \cong \frac{\mathcal{R}}{\mathcal{S}} \cong \mathsf{M}$. Moreover, $\mathsf{M} \cong \frac{\mathcal{R}}{\mathcal{S}} \subseteq \frac{\mathcal{S}+[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{\mathcal{S}} \cong \frac{[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]}{\mathcal{S}\cap[\mathcal{F},\overset{n}{\ldots},\mathcal{F}]} \subseteq [\frac{\mathcal{F}}{\mathcal{S}}, \overset{n}{\ldots}, \frac{\mathcal{F}}{\mathcal{S}}] \cong [\mathcal{P}, \overset{n}{\ldots}, \mathcal{P}]$. Therefore, the extension $0 \to \mathsf{M} \to \mathcal{P} \overset{\psi}{\to} \mathcal{G} \to 0$ is a stem cover by [9, Proposition 9]. □

Corollary 4.4. *Any finite-dimensional Leibniz n-algebra has at least one covering.*

Proof. Let $0 \to \mathcal{R} \to \mathcal{F} \overset{\rho}{\to} \mathcal{G} \to 0$ be a free presentation of a finite-dimensional Leibniz n-algebra $\mathcal{G}$. Following the proof of Theorem 4.3, choose an n-sided ideal $\mathcal{S}$ of $\mathcal{F}$ such that $\frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ is the complement of $\mathsf{M}(\mathcal{G})$ in $\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$. Then the extension $0 \to \frac{\mathcal{R}}{\mathcal{S}} \to \frac{\mathcal{F}}{\mathcal{S}} \to \mathcal{G} \to 0$ is a stem cover of $\mathcal{G}$. □

Previously to the following result, we need to introduce some notions and properties concerning isoclinism of Leibniz n-algebras. In the particular case of n-Lie algebras, we recover the corresponding notions and results in [28].

Consider the central extensions $E_i : 0 \to \mathsf{N}_i \overset{\chi_i}{\to} \mathcal{G}_i \overset{\pi_i}{\to} \mathcal{Q}_i \to 0, i = 1, 2$. Let $C_i : \mathcal{Q}_i \times, \overset{n}{\ldots}, \times \mathcal{Q}_i \to [\mathcal{G}_i, \overset{n}{\ldots}, \mathcal{G}_i]$ be given by $C_i(q_{i1}, \ldots, q_{in}) = [g_{i1}, \ldots, g_{in}]$, where $\pi_i(g_{ij}) = q_{ij}, i = 1, 2; j = 1, \ldots, n$, the commutator map associated to the extension E_i.

Definition 4.5. The central extensions E_1 and E_2 are said to be isoclinic when there exist isomorphisms $\eta : \mathcal{Q}_1 \to \mathcal{Q}_2$ and $\xi : [\mathcal{G}_1, \overset{n}{\ldots}, \mathcal{G}_1] \to [\mathcal{G}_2, \overset{n}{\ldots}, \mathcal{G}_2]$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{Q}_1 \times \overset{n}{\ldots} \times \mathcal{Q}_1 & \xrightarrow{C_1} & [\mathcal{G}_1, \overset{n}{\ldots}, \mathcal{G}_1] \\ \downarrow{\scriptstyle \eta\times\overset{n}{\ldots}\times\eta} & & \downarrow{\scriptstyle \xi} \\ \mathcal{Q}_2 \times \overset{n}{\ldots} \times \mathcal{Q}_2 & \xrightarrow{C_2} & [\mathcal{G}_2, \overset{n}{\ldots}, \mathcal{G}_2] \end{array} \tag{8}$$

The pair (η, ξ) is called an isoclinism from E_1 to E_2 and will be denoted by $(\eta, \xi) : E_1 \sim E_2$.

Let $\mathcal{Q}$ be a Leibniz n-algebra, then we can construct the following central extension:

$$E_{\mathcal{Q}} : 0 \to Z(\mathcal{Q}) \to \mathcal{Q} \stackrel{pr_{\mathcal{Q}}}{\to} \mathcal{Q}/Z(\mathcal{Q}) \to 0. \tag{9}$$

Definition 4.6. Let $\mathcal{G}$ and $\mathcal{Q}$ be Leibniz n-algebras. Then $\mathcal{G}$ and $\mathcal{Q}$ are said to be isoclinic when $E_{\mathcal{G}}$ and $E_{\mathcal{Q}}$ are isoclinic central extensions.

An isoclinism $(\eta, \xi) : E_{\mathcal{G}} \sim E_{\mathcal{Q}}$ is also called an isoclinism from $\mathcal{G}$ to $\mathcal{Q}$, denoted by $(\eta, \xi) : \mathcal{G} \sim \mathcal{Q}$.

It is a routine task to show that isoclinism is an equivalence relation.

Definition 4.7. A homomorphism of central extensions $(\alpha, \beta, \gamma) : E_1 \to E_2$ is said to be isoclinic if there exists an isomorphism $\beta' : [\mathcal{G}_1, \stackrel{n}{\ldots}, \mathcal{G}_1] \to [\mathcal{G}_2, \stackrel{n}{\ldots}, \mathcal{G}_2]$ with $(\gamma, \beta') : E_1 \sim E_2$.

If β is in addition a surjective homomorphism (resp., injective homomorphism), then (α, β, γ) is called an isoclinic surjection (resp., isoclinic injection).

Proposition 4.8. *For a homomorphism of central extensions $(\alpha, \beta, \gamma) : E_1 \to E_2$, the following statements hold:*

(a) *(α, β, γ) is isoclinic if and only if γ is an isomorphism and $\mathsf{Ker}(\beta) \cap [\mathcal{G}_1, \stackrel{n}{\ldots}, \mathcal{G}_1] = 0$.*

(b) *If (α, β, γ) is isoclinic and β' is given as in the above Definition, then $\beta' = \beta_{|[\mathcal{G}_1, \stackrel{n}{\ldots}, \mathcal{G}_1]}$.*

Proof. (a) Assume that $(\alpha, \beta, \gamma) : E_1 \to E_2$ is isoclinic, then $(\gamma, \beta') : E_1 \sim E_2$ is an isoclinism for some isomorphism $\beta' : [\mathcal{G}_1, \stackrel{n}{\ldots}, \mathcal{G}_1] \to [\mathcal{G}_2, \stackrel{n}{\ldots}, \mathcal{G}_2]$. This implies by definition that γ is an isomorphism. Now let $m \in \mathsf{Ker}(\beta) \cap [\mathcal{G}_1, \stackrel{n}{\ldots}, \mathcal{G}_1]$. Then $\beta(m) = 0$ and $m = \sum_i \lambda_i [g_{i1}^1, \ldots, g_{in}^1]$ for some $g_{i1}^1, \ldots, g_{in}^1 \in \mathcal{G}_1$. Since $(\gamma, \beta') : E_1 \sim E_2$ is an isoclinism, we have

$$\begin{aligned}
\beta'(m) &= \beta' \left(\sum_i \lambda_i [g_{i1}^1, \ldots, g_{in}^1] \right) = \beta' \left(C_1 \left(\sum_i \lambda_i \left(\pi_1(g_{i1}^1), \ldots, \pi_1(g_{in}^1) \right) \right) \right) \\
&= (\beta' \circ C_1) \left(\sum_i \lambda_i \left(\pi_1(g_{i1}^1), \ldots, \pi_1(g_{in}^1) \right) \right) \\
&= (C_2 \circ (\gamma \times \stackrel{n}{\ldots} \times \gamma)) \left(\sum_i \lambda_i \left(\pi_1(g_{i1}^1), \ldots, \pi_1(g_{in}^1) \right) \right)
\end{aligned}$$

$$= C_2\left(\sum_i \lambda_i\left(\gamma(\pi_1(g_{i1}^1)),\ldots,\gamma(\pi_1(g_{in}^1))\right)\right)$$

$$= C_2\left(\sum_i \lambda_i\left(\pi_2(\beta(g_{i1}^1)),\ldots,\pi_2(\beta(g_{in}^1))\right)\right)$$

$$= \sum_i \lambda_i\left[\beta(g_{i1}^1),\ldots,\beta(g_{in}^1)\right] = \beta\left(\sum_i \lambda_i\left[g_{i1}^1,\ldots,g_{in}^1\right]\right) = \beta(m) = 0.$$

Since β' is one-to-one, it follows that $m = 0$.

Conversely, assume that $\mathsf{Ker}(\beta)\cap[\mathcal{G}_1,\overset{n}{\ldots},\mathcal{G}_1] = 0$. Define $\beta' : [\mathcal{G}_1,\overset{n}{\ldots},\mathcal{G}_1] \to [\mathcal{G}_2,\overset{n}{\ldots},\mathcal{G}_2]$ by $\beta'(g) = \beta(g)$, which is one-to-one. It remains to show that β' is onto. Let $y \in [\mathcal{G}_2,\overset{n}{\ldots},\mathcal{G}_2]$. Then $y = \sum_i \lambda_i[g_{i1}^2,\ldots,g_{in}^2]$ for some $g_{ij}^2 \in \mathcal{G}_2, j = 1,\ldots,n$. Since π_1 and γ are onto, it follows that $\pi_2(g_{ij}^2) = (\gamma\circ\pi_1)(g_{ij}^1)$ for some $g_{ij}^1 \in \mathcal{G}_1, j = 1,\ldots,n$. By the homomorphism (α,β,γ), we have $(\gamma\circ\pi_1)(g_{ij}^1) = (\pi_2\circ\beta)(g_{ij}^1)$, which implies that $g_{ij}^2 - \beta(g_{ij}^1) = \chi_2(n_j)$ for some $n_j \in \mathsf{N}_2, j = 1,\ldots,n$. We now have

$$\beta'\left(\sum_i \lambda_i[g_{i1}^1,\ldots,g_{in}^1]\right) = \beta\left(\sum_i \lambda_i[g_{i1}^1,\ldots,g_{in}^1]\right) = \sum_i \lambda_i[\beta(g_{i1}^1),\ldots,\beta(g_{in}^1)]$$
$$= \sum_i \lambda_i[g_{i1}^2 - \chi_2(n_1),\ldots,g_{in}^2 - \chi_2(n_n)] = y.$$

(b) Follows directly from the proof of (a). □

Proposition 4.9. *Let $\beta : \mathcal{G} \to \mathcal{Q}$ be a homomorphism of Leibniz n-algebras. Then β induces an isoclinic homomorphism from $E_\mathcal{G}$ to $E_\mathcal{Q}$ if and only if $\mathsf{Ker}(\beta)\cap[\mathcal{G},\overset{n}{\ldots},\mathcal{G}] = 0$ and $\mathsf{Im}(\beta) + Z(\mathcal{Q}) = \mathcal{Q}$.*

In this case, we call β an isoclinic homomorphism.

Proof. Assume that $\mathsf{Ker}(\beta)\cap[\mathcal{G},\overset{n}{\ldots},\mathcal{G}] = 0$ and $\mathsf{Im}(\beta) + Z(\mathcal{Q}) = \mathcal{Q}$. First we prove that $\beta(Z(\mathcal{G})) \subseteq Z(\mathcal{Q})$. Indeed, let $q_i \in \mathcal{Q}$, since $\mathsf{Im}(\beta) + Z(\mathcal{Q}) = \mathcal{Q}$, it follows that $q_i = \beta(x_i) + q_{i0}$ for some $x_i \in \mathcal{G}$ and $q_{i0} \in Z(\mathcal{Q}), i = 1,\ldots,n$. Then for any $g \in Z(\mathcal{G})$ we have

$$\Bigg[q_1,\ldots,\underbrace{\beta(g)}_{i},\ldots,q_n\Bigg] = [\beta(x_1)+q_{10},\ldots,\beta(g),\ldots,\beta(x_n)+q_{n0}] = 0.$$

So the maps $\alpha := \beta|_{Z(\mathcal{G})}$ and $\gamma : \mathcal{G}/Z(\mathcal{G}) \to \mathcal{Q}/Z(\mathcal{Q})$ given by $\gamma(\overline{g}) = \overline{\beta(g)}$ are well-defined homomorphisms, and it is readily verified that $(\alpha,\beta,\gamma) : E_\mathcal{G} \to E_\mathcal{Q}$ is a homomorphism of central extensions.

To show that it is isoclinic, it is enough to show by Proposition 4.8 that γ is an isomorphism. To show that γ is one-to-one, let $g \in \mathcal{G}$ such that $\gamma(\overline{g}) = 0$. Then $\beta(g) \in Z(\mathcal{Q})$. We claim that $g \in Z(\mathcal{G})$. Indeed, if $g \notin Z(\mathcal{G})$, then $m := [g_1, \ldots, g_{i-1}, g, g_{i+1}, \ldots, g_n] \neq 0$ for some $g_j \in \mathcal{G}, j = 1, \ldots, i-1, i+1, \ldots, n$. But $\beta(m) = [\beta(g_1), \ldots, \beta(g_{i-1}), \beta(g), \beta(g_{i+1}), \ldots, \beta(g_n)] = 0$, because $\beta(g) \in Z(\mathcal{Q})$. This implies that $m \in \mathsf{Ker}(\beta) \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}]$, and thus $m = 0$, a contradiction. Next we show that γ is onto. Let $q \in \mathcal{Q}$. Since $\mathsf{Im}(\beta) + Z(\mathcal{Q}) = \mathcal{Q}$, it follows that $q = \beta(x) + q_0$ for some $x \in \mathcal{G}$ and $q_0 \in Z(\mathcal{Q})$. Clearly, $\overline{q} = \overline{\beta(x)} = \gamma(\overline{x})$.

Conversely, assume that β induces an isoclinic homomorphism (α, β, γ) from $E_{\mathcal{G}}$ to $E_{\mathcal{Q}}$. Then again by Proposition 4.8, $\mathsf{Ker}(\beta) \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] = 0$. It remains to show that $\mathsf{Im}(\beta) + Z(\mathcal{Q}) = \mathcal{Q}$. Clearly, $\mathsf{Im}(\beta) + Z(\mathcal{Q}) \subseteq \mathcal{Q}$. Now let $q \in \mathcal{Q}$. Following the notation in equation (9), $pr_{\mathcal{Q}}$ and γ are onto, then $pr_{\mathcal{Q}}(q) = \gamma \circ pr_{\mathcal{G}}(g)$ for some $g \in \mathcal{G}$. On the other hand, by the homomorphism (α, β, γ), we have that $(\gamma \circ pr_{\mathcal{G}})(g) = (pr_{\mathcal{Q}} \circ \beta)(g)$, which implies that $q - \beta(g) \in \mathsf{Ker}(pr_{\mathcal{Q}}) = Z(\mathcal{Q})$. Therefore, $q = \beta(g) + n$ for some $n \in Z(\mathcal{Q})$. This completes the proof. □

Proposition 4.10. *Let $\mathcal{N}$ be an n-sided ideal of a Leibniz n-algebra $\mathcal{G}$. The natural homomorphism nat : $\mathcal{G} \twoheadrightarrow \mathcal{G}/\mathcal{N}$ is an isoclinic surjection if and only if $\mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] = 0$.*

Proof. Follows by Proposition 4.9, since $\mathsf{Ker}(\mathsf{nat}) = \mathcal{N}$. In addition, if $\mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \neq 0$, then nat is not an isoclinic homomorphism by Proposition 4.8.

Conversely, if $\mathcal{N} \cap [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] = 0$, then $\xi : [\mathcal{G}, \overset{n}{\ldots}, \mathcal{G}] \to [\mathcal{G}/\mathcal{N}, \overset{n}{\ldots}, \mathcal{G}/\mathcal{N}]$, given by $\xi([g_1, \ldots, g_n]) = [g_1 + \mathcal{N}, \ldots, g_n + \mathcal{N}]$, is an isomorphism and $Z(\mathcal{G}/\mathcal{N}) = \frac{Z(\mathcal{G})}{\mathcal{N}}$, and $\frac{\mathcal{G}}{Z(\mathcal{G})} \overset{\eta}{\cong} \frac{\mathcal{G}/\mathcal{N}}{Z(\mathcal{G})/\mathcal{N}}$ by the third isomorphism theorem. Now the commutativity of diagram (8) immediately follows. □

Corollary 4.11. *If $\mathcal{G}$ is a Leibniz n-algebra such that its Schur multiplier is finite-dimensional, then all stem covers of $\mathcal{G}$ are isoclinic.*

Proof. Let $0 \to \mathcal{R} \to \mathcal{F} \overset{\rho}{\to} \mathcal{G} \to 0$ be a free presentation of $\mathcal{G}$. Let $0 \to \mathsf{M} \to \mathcal{P} \overset{\psi}{\to} \mathcal{G} \to 0$ be a stem cover. By the proof of Theorem 4.3, there exists a surjective homomorphism $\beta : \frac{\mathcal{F}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \to \mathcal{P}$ and an n-sided ideal $\mathcal{S}$ of $\mathcal{F}$ such that $\frac{\mathcal{R}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \cong \mathsf{M}(\mathcal{G}) \oplus \mathsf{Ker}(\beta)$ and $\mathsf{Ker}(\beta) = \frac{\mathcal{S}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$. Moreover, $\mathsf{Ker}(\beta) \cap \left[\frac{\mathcal{F}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}, \overset{n}{\ldots}, \frac{\mathcal{F}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}\right] = \frac{\mathcal{S}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \cap \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} = \frac{\mathcal{S} \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} = \mathsf{M}(\mathcal{G}) \cap \frac{\mathcal{S}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}$, which vanishes thanks to the finite-dimension and the exact sequence $0 \to \mathsf{M}(\mathcal{G}) \cap \frac{\mathcal{S}}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \to \mathsf{M}(\mathcal{G}) \to \mathsf{M} \to 0$.

Now Proposition 4.10 completes the proof. □

Lemma 4.12. *Let $\mathcal{G}$ be a Leibniz n-algebra and*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{P}_1 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \mathsf{M}_2 & \longrightarrow & \mathcal{P}_2 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

be a commutative diagram of short exact sequences of Leibniz n-algebras such that the bottom row is a stem extension. If the homomorphism γ is surjective, then β is a surjective homomorphism as well.

PROOF. Obviously, $\mathcal{P}_2 = \mathsf{Im}(\beta) + \mathsf{M}_2$. Hence $[\mathcal{P}_2, \overset{n}{\ldots}, \mathcal{P}_2] = [\mathsf{Im}(\beta), \overset{n}{\ldots}, \mathsf{Im}(\beta)]$. By [9, Proposition 6], $\mathsf{M}_2 \subseteq [\mathcal{P}_2, \overset{n}{\ldots}, \mathcal{P}_2] = [\mathsf{Im}(\beta), \overset{n}{\ldots}, \mathsf{Im}(\beta)]$. Therefore, $\mathcal{P}_2 \subseteq \mathsf{Im}(\beta) + [\mathsf{Im}(\beta), \overset{n}{\ldots}, \mathsf{Im}(\beta)]$, i.e. β is surjective. □

Theorem 4.13. *Let $\mathcal{G}$ be a Leibniz n-algebra such that $\mathsf{M}(\mathcal{G})$ is finite-dimensional, and let $0 \to \mathsf{M}_i \to \mathcal{P}_i \overset{\psi_i}{\to} \mathcal{G} \to 0, i = 1, 2,$ be two stem covers of $\mathcal{G}$. If $\eta : \mathcal{P}_1 \to \mathcal{P}_2$ is a surjective homomorphism such that $\eta(\mathcal{M}_1) \subseteq \mathcal{M}_2$, then η is an isomorphism.*

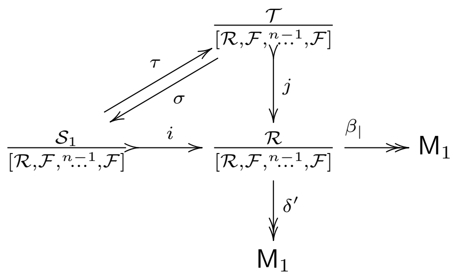
PROOF. Let $0 \to \mathcal{R} \to \mathcal{F} \overset{\rho}{\to} \mathcal{G} \to 0$ be a free presentation of $\mathcal{G}$. By Theorem 4.3, there exist n-sided ideals $\mathcal{S}_i, i = 1, 2,$ of $\mathcal{F}$ such that $\mathcal{P}_i \cong \frac{\mathcal{F}}{\mathcal{S}_i}; \mathsf{M}_i \cong \frac{\mathcal{R}}{\mathcal{S}_i}$ and $\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \cong \mathsf{M}(\mathcal{G}) \oplus \frac{\mathcal{S}_i}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$, $i = 1, 2$.

By Lemmas 4.2 and 4.12 and the proof of Theorem 4.3, there exists a surjective homomorphism $\theta : \frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \to \mathcal{P}_2 \cong \frac{\mathcal{F}}{\mathcal{S}_2}$ such that $\mathsf{Ker}(\theta) = \frac{\mathcal{S}_2}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$.

Since $\mathcal{F}$ is a free Leibniz n-algebra, there exists a homomorphism $\overline{\delta} : \mathcal{F} \to \mathcal{P}_1$ such that $\psi_1 \circ \overline{\delta} = \rho$. Moreover, $\overline{\delta}(\mathcal{R}) \subseteq \mathsf{M}_1$ and $\overline{\delta}$ vanishes on $[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]$, consequently, it induces a homomorphism $\delta' : \frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \to \mathcal{P}_1 \cong \frac{\mathcal{F}}{\mathcal{S}_1}$ such that $\delta' \circ pr = \overline{\delta}$, where $pr : \mathcal{F} \to \frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ is the canonical projection. Since $\psi_1 \circ \delta' = \overline{\rho}$, Lemma 4.12 implies that δ' is a surjective homomorphism. Let $\mathsf{Ker}(\delta') = \frac{\mathcal{T}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$ for some n-sided ideal $\mathcal{T}$ of $\mathcal{R}$.

Since $\theta\left(\frac{\mathcal{T}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}\right) = \eta\left(\delta'\left(\frac{\mathcal{T}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}\right)\right) = 0$, we have $\frac{\mathcal{T}}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]} \subseteq \mathsf{Ker}(\theta) = \frac{\mathcal{S}_2}{[\mathcal{R},\mathcal{F},\overset{n-1}{\ldots},\mathcal{F}]}$, therefore $\mathcal{T} \subseteq \mathcal{S}_2$.

From the diagram



it follows that $\frac{\mathcal{S}_1}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \cong \frac{\mathcal{T}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}$, and by Theorem 4.3, we have $\mathsf{M}(\mathcal{G}) \oplus \frac{\mathcal{S}_2}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \cong \frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \cong \mathsf{M}(\mathcal{G}) \oplus \frac{\mathcal{T}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}$, which implies that $\mathcal{S}_2 \cong \mathcal{T}$. So η is an isomorphism, since $\mathsf{Ker}(\eta) \cong \frac{\mathcal{S}_2}{\mathcal{T}}$. $\square$

Proposition 4.14. *Let* $0 \to \mathsf{M}_i \to \mathcal{P}_i \stackrel{\psi_i}{\to} \mathcal{G} \to 0, i = 1, 2$, *be two stem covers of a finite-dimensional Leibniz* n*-algebra* $\mathcal{G}$ *with finite-dimensional Schur multiplier. Then* $Z(\mathcal{P}_1)/\mathsf{M}_1 \cong Z(\mathcal{P}_2)/\mathsf{M}_2$.

Proof. Let $0 \to \mathcal{R} \to \mathcal{F} \stackrel{\rho}{\to} \mathcal{G} \to 0$ be a free presentation of $\mathcal{G}$. By Corollary 4.4, there exists a covering $\mathcal{G}^*$ of $\mathcal{G}$, i.e. there is an exact sequence $0 \to \mathsf{M} \to \mathcal{G}^* \stackrel{\psi}{\to} \mathcal{G} \to 0$ such that $\mathsf{M} \subseteq Z(\mathcal{G}^*) \cap [\mathcal{G}, \stackrel{n}{\ldots}, \mathcal{G}]$ and $\mathsf{M} \cong \mathsf{M}(\mathcal{G})$ (see [9, Propositions 6 and 9]).

By Theorem 4.3, there exists an n-sided ideal $\mathcal{S}$ such that $\mathcal{G}^* \cong \frac{\mathcal{F}}{\mathcal{S}}$, $\mathsf{M} \cong \frac{\mathcal{R}}{\mathcal{S}}$ and $\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \cong \mathsf{M}(\mathcal{G}) \oplus \frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}$. As $Z\left(\frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}\right) = \frac{\mathcal{T}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}$, $[\mathcal{T}, \mathcal{F}, \stackrel{n-1}{\ldots}, \mathcal{F}] \subseteq [\mathcal{R}, \mathcal{F}, \stackrel{n-1}{\ldots}, \mathcal{F}]$, thus $\frac{\mathcal{T}}{\mathcal{S}} \subseteq Z\left(\frac{\mathcal{F}}{\mathcal{S}}\right)$.

Conversely, for $x + \mathcal{S} \in Z\left(\frac{\mathcal{F}}{\mathcal{S}}\right)$, we must show that $x + \mathcal{S} \in \frac{\mathcal{T}}{\mathcal{S}}$.

Indeed, for any $f + \mathcal{S} \in \frac{\mathcal{F}}{\mathcal{S}}$, $[f_1 + \mathcal{S}, \ldots, f_{i-1} + \mathcal{S}, x + \mathcal{S}, f_{i+1} + \mathcal{S}, \ldots, f_n + \mathcal{S}] = 0$, hence $[f_1, \ldots, f_{i-1}, x, f_{i+1}, \ldots, f_n] \in \mathcal{S} \cap [\mathcal{F}, \stackrel{n}{\ldots}, \mathcal{F}]$, for any $f_i \in \mathcal{F}$, i=1, …, n.

To show that $x \in \mathcal{T}$, we need to prove that $x + [\mathcal{R}, \mathcal{F}, \stackrel{n-1}{\ldots}, \mathcal{F}] \in Z\left(\frac{\mathcal{F}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}\right) = \frac{\mathcal{T}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}$. But this holds, since for any $f_i \in \mathcal{F}$, $[f_1, \ldots, f_{i-1}, x, f_{i+1}, \ldots, f_n] + [\mathcal{R}, \mathcal{F}, \stackrel{n-1}{\ldots}, \mathcal{F}] = \bar{0}$, $i = 1, \ldots, n$, because $[f_1, \ldots, f_{i-1}, x, f_{i+1}, \ldots, f_n] \in \mathcal{S} \cap [\mathcal{F}, \stackrel{n}{\ldots}, \mathcal{F}]$, and by Theorem 4.3, $\frac{\mathcal{R}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \cong \frac{\mathcal{R} \cap [\mathcal{F},\stackrel{n}{\ldots},\mathcal{F}]}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]} \oplus \frac{\mathcal{S}}{[\mathcal{R},\mathcal{F},\stackrel{n-1}{\ldots},\mathcal{F}]}$, hence $\mathcal{R} \cap [\mathcal{F}, \stackrel{n}{\ldots}, \mathcal{F}] \cap \mathcal{S} \subseteq [\mathcal{R}, \mathcal{F}, \stackrel{n-1}{\ldots}, \mathcal{F}]$, but $\mathcal{S} \subseteq \mathcal{R}$, then $\mathcal{S} \cap [\mathcal{F}, \stackrel{n}{\ldots}, \mathcal{F}] \subseteq [\mathcal{R}, \mathcal{F}, \stackrel{n-1}{\ldots}, \mathcal{F}]$.

Consequently, $\frac{\mathcal{T}}{\mathcal{S}} \cong Z\left(\frac{\mathcal{F}}{\mathcal{S}}\right)$. From here, $\frac{Z(\mathcal{G}^*)}{\mathsf{M}} \cong \frac{Z(\mathcal{F}/\mathcal{S})}{\mathcal{R}/\mathcal{S}} \cong \frac{\mathcal{T}/\mathcal{S}}{\mathcal{R}/\mathcal{S}} \cong \frac{\mathcal{T}}{\mathcal{R}}$.

Applying this result to each stem cover, we have $\frac{Z(\mathcal{P}_1)}{\mathsf{M}_1} \cong \frac{\mathcal{T}}{\mathcal{R}} \cong \frac{Z(\mathcal{P}_2)}{\mathsf{M}_2}$. $\square$

The following results are a generalization to Leibniz n-algebras ($n \geq 3$) of the characterizations of stem extensions and stem covers of Leibniz algebras (case $n = 2$) in [13].

Proposition 4.15. *Let $\mathcal{Q}$ be a Leibniz n-algebra, and let $\mathcal{U}$ be a subspace of ${}_nHL_1(\mathcal{Q})$, then there exists a stem extension E with $\mathcal{U} = \mathsf{Ker}\,(\theta_*\,(\Delta[E]))$.*

Proof. Consider the quotient $\mathbb{K}$-vector space $\mathsf{N} = {}_nHL_1(\mathcal{Q})/\mathcal{U}$ as a trivial $\mathcal{Q}$-module. Consider the central extension $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0 \in {}_nHL^1(\mathcal{Q}, \mathsf{N})$. Thus $\theta_*\,(\Delta[E]) = \theta_*(E) \in \mathsf{Hom}({}_nHL_1(\mathcal{Q}), \mathsf{N})$.

If $\theta_*(E) : {}_nHL_1(\mathcal{Q}) \to \mathsf{N} = {}_nHL_1(\mathcal{Q})/\mathcal{U}$ is the canonical projection, then there exists a central extension $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ such that $\theta_*\,(\Delta[E]) = \theta_*(E)$ is the canonical projection.

Associated to E, we have the exact sequence (3), where $\mathcal{U} = \mathsf{Ker}\,(\theta_*(E)) = \mathsf{Ker}\,(\theta_*\,(\Delta[E]))$. Moreover, E is a stem extension by Proposition 4.1. □

Corollary 4.16. *A stem extension is a stem cover if and only if $\mathcal{U} = 0$.*

Remark 4.17. Any stem cover $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ is isomorphic to a stem cover $E' : 0 \to {}_nHL_1(\mathcal{Q}) \to \mathcal{G}' \to \mathcal{Q} \to 0$ with $\theta_*\,(\Delta[E']) = 1_{{}_nHL_1(\mathcal{Q})}$.

Indeed, there always exists E', it suffices to take $\mathcal{U} = 0$ in Proposition 4.15; if $\varphi : \mathsf{N} \to {}_nHL_1(\mathcal{Q})$ is the inverse of $\theta_*\,(\Delta[E])$, which is an isomorphism by [9, Definition 2], then naturality of isomorphism $\theta_* : {}_nHL^1(\mathcal{Q}, \mathsf{N}) \overset{\sim}{\to} \mathsf{Hom}({}_nHL_1(\mathcal{Q}), \mathsf{N})$ implies $\theta_*\,(\varphi_*\,(\Delta[E])) = \varphi_*\,(\theta_*\,(\Delta[E])) = \varphi\,(\theta_*\,(\Delta[E])) = 1_{{}_nHL_1(\mathcal{Q})}$, so we can choose E' such that $\Delta[E'] = \varphi_*\,(\Delta[E])$. By Proposition 2.11, there exists a homomorphism $f : \mathcal{G} \to \mathcal{G}'$ making the following diagram commutative:

$$\begin{array}{ccccccccc}
E : 0 & \longrightarrow & \mathsf{N} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle f} & & \Vert & & \\
E' : 0 & \longrightarrow & {}_nHL_1(\mathcal{Q}) & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

Proposition 4.18. *Every stem extension of $\mathcal{Q}$ is an image by a surjective homomorphism of some stem cover.*

Proof. Let $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ be a stem extension characterized by $\Delta[E] = \xi \in {}_nHL^1(\mathcal{Q}, \mathsf{N})$; then $\varphi = \theta_*(\xi) = \theta_*(\Delta[E]) = \theta_*(E) : {}_nHL_1(\mathcal{Q}) \to \mathsf{N}$ is a surjective homomorphism.

In order to complete the proof, we must find $\eta \in {}_nHL^1(\mathcal{Q}, {}_nHL_1(\mathcal{Q}))$ with $\varphi_*(\eta) = \xi$ and $\theta_*(\eta) = 1_{{}_nHL_1(\mathcal{Q})}$, where φ_* is the morphism induced by naturality of the isomorphism θ_* on φ.

Let $\eta \in {}_nHL^1(\mathcal{Q}, {}_nHL_1(\mathcal{Q}))$ be such that $\theta_*(\eta) = 1_{{}_nHL_1(\mathcal{Q})}$, then $\theta_*(\xi - \varphi_*(\eta)) = \varphi - \varphi_*(\theta_*(\eta)) = 0$; consequently, $\varphi_*(\eta) = \xi$. Obviously, η satisfies the required conditions.

Now, let $E' : 0 \to {}_nHL_1(\mathcal{Q}) \to \mathcal{G}' \to \mathcal{Q} \to 0 \in {}_nHL^1(\mathcal{Q}, {}_nHL_1(\mathcal{Q}))$ be such that $\Delta[E'] = \eta$; by Proposition 2.11, there exists $f : \mathcal{G} \to \mathcal{G}'$ such that $(\varphi, f, 1) : E \to E'$ is a surjective homomorphism. □

Proposition 4.19. *There exists only one isomorphism class of stem covers of $\mathcal{Q}$.*

Proof. By Remark 4.17, stem covers are of the form $E : 0 \to {}_nHL_1(\mathcal{Q}) \to \mathcal{G} \to \mathcal{Q} \to 0$ such that $\theta_*(\Delta[E]) = 1_{{}_nHL_1(\mathcal{Q})}$.

Fix a stem cover E. For any other stem cover E', we have that $\theta_*(\Delta[E]) = \theta_*(\Delta[E']) = 1_{{}_nHL_1(\mathcal{Q})}$, then $[E] = [E']$. □

Proposition 4.20. *Let $\bar{E} : 0 \to {}_nHL_1(\bar{\mathcal{Q}}) \to \bar{\mathcal{G}} \to \bar{\mathcal{Q}} \to 0$ be a stem cover and let $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ be a central extension. Then every homomorphism $f : \bar{\mathcal{Q}} \to \mathcal{Q}$ can be lifted to a map $f' : \bar{\mathcal{G}} \to \mathcal{G}$.*

Proof. Let $\Delta[E] = \xi \in {}_nHL^1(\mathcal{Q}, \mathsf{N})$. We define $\varphi = f^*(\theta_*(\xi)) : {}_nHL_1(\bar{\mathcal{Q}}) \to \mathsf{N}$. Since $\eta = \Delta[\bar{E}] \in {}_nHL^1(\bar{\mathcal{Q}}, {}_nHL_1(\bar{\mathcal{Q}}))$ is a stem cover with $\theta_*(\eta) = \theta_*\left(\Delta[\bar{E}]\right) = \theta_*(\bar{E}) = 1_{{}_nHL_1(\bar{\mathcal{Q}})}$, we have $\theta_*(\varphi_*(\eta)) = \varphi_*(\theta_*(\eta)) = \varphi = f^*(\theta_*(\xi)) = \theta_*(f^*(\xi))$, thus $\varphi_*(\eta) = f^*(\xi)$, i.e. $\varphi_*(\Delta[\overline{E}]) = f^*(\Delta[E])$. Proposition 2.13 concludes the proof. □

Proposition 4.21. *Let $E : 0 \to \mathsf{N} \stackrel{\chi}{\to} \mathcal{G} \stackrel{\pi}{\to} \mathcal{Q} \to 0$ and $E' : 0 \to \mathsf{N}' \stackrel{\chi'}{\to} \mathcal{G}' \stackrel{\pi'}{\to} \mathcal{Q}' \to 0$ be central extensions. Let $\rho : \mathsf{N} \to \mathsf{N}'$ and $\sigma : \mathcal{Q} \to \mathcal{Q}'$ be homomorphisms of Leibniz n-algebras. Then:*

(a) *There exists $\tau : \mathcal{G} \to \mathcal{G}'$ inducing ρ and σ if and only if the following diagram is commutative:*

$$\begin{array}{ccc} {}_nHL_1(\mathcal{Q}) & \xrightarrow{\theta_*(E)} & \mathsf{N} \\ \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \rho} \\ {}_nHL_1(\mathcal{Q}') & \xrightarrow{\theta_*(E')} & \mathsf{N}' \end{array}$$

(b) *If τ exists, it is unique if and only if $\mathsf{Hom}({}_nHL_0(\mathcal{Q}), \mathsf{N}') = 0$.*

Proof. (a) If τ exists, the commutativity of the square follows from the naturality of sequence (3).

Conversely, consider the following diagram provided by the natural isomorphism θ_*:

$$\begin{array}{ccc}
\theta_* : {}_nHL^1(\mathcal{Q}, \mathsf{N}) & \xrightarrow{\sim} & \mathsf{Hom}({}_nHL_1(\mathcal{Q}), \mathsf{N}) \\
\downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\
\theta''_* : {}_nHL^1(\mathcal{Q}, \mathsf{N}') & \xrightarrow{\sim} & \mathsf{Hom}({}_nHL_1(\mathcal{Q}), \mathsf{N}') \\
{\scriptstyle \sigma_*}\uparrow & & {\scriptstyle \sigma_*}\uparrow \\
\theta'_* : {}_nHL^1(\mathcal{Q}', \mathsf{N}') & \xrightarrow{\sim} & \mathsf{Hom}({}_nHL_1(\mathcal{Q}'), \mathsf{N}')
\end{array}$$

Let $\xi = \Delta[E]$ and $\xi' = \Delta[E']$ be, then $\rho_*\left(\theta_*(\xi)\right) = \rho_*\left(\theta_*\left(\Delta[E]\right)\right) = \rho_*\left(\theta_*(E)\right) = \rho\left(\theta_*(E)\right) = \theta_*(E') \circ \sigma_* = \sigma^* \circ \theta_*(E') = \sigma^*\left(\theta'_*\left(\Delta[E']\right)\right) = \sigma^*\left(\theta_*(\xi')\right)$. Thus $\theta''_*\left(\rho_*(\xi)\right) = \rho_*\left(\theta_*(\xi)\right) = \sigma^*\left(\theta'_*(\xi')\right) = \theta''_*\left(\sigma^*(\xi')\right)$, and consequently, $\rho_*(\xi) = \sigma^*(\xi')$. Now Proposition 2.13 provides $\tau : \mathcal{G} \to \mathcal{G}'$ inducing ρ and σ.

(b) Suppose there exists $\tau : \mathcal{G} \to \mathcal{G}'$ that induces ρ and σ, and let $\tau' : \mathcal{G} \to \mathcal{G}'$ be another homomorphism that induces ρ and σ, then there are unique homomorphisms $f : \mathcal{G} \to \mathsf{N}'$ such that $\tau' - \tau = \chi' \circ f$ and $\varphi : \mathcal{Q} \to \mathsf{N}'$ such that $\varphi \circ \pi = f$; consequently, $\tau' = \tau + \chi' \circ \varphi \circ \pi$; that is, for another homomorphism $\tau' : \mathcal{G} \to \mathcal{G}'$, there exists a unique homomorphism $\varphi : \mathcal{Q} \to \mathsf{N}'$ such that $\tau' = \tau + \chi' \circ \varphi \circ \pi$. Conversely, if $\varphi : \mathcal{Q} \to \mathsf{N}'$ is a homomorphism, then $\tau' = \tau + \chi' \circ \varphi \circ \pi$ induces ρ and σ.

τ is unique if and only if $\tau - \tau' = 0$, that is, $\chi' \circ \varphi \circ \pi = 0$, which is equivalent to $\varphi \in \mathsf{Hom}(\mathcal{Q}, \mathsf{N}') = 0$ and then $\mathsf{Hom}({}_nHL_0(\mathcal{Q}), \mathsf{N}') = 0$. □

Corollary 4.22. *Under the hypothesis of Proposition 4.21, the map $\tau : \mathcal{G} \to \mathcal{G}'$ exists and it is unique when $\mathcal{Q}$ is a perfect Leibniz n-algebra ($\mathcal{Q} = [\mathcal{Q}, \overset{n}{\ldots}, \mathcal{Q}]$).*

Proof. If $\mathcal{Q}$ is a perfect Leibniz n-algebra, then $\mathsf{Hom}({}_nHL_0(\mathcal{Q}), \mathsf{N}') = 0$. □

Proposition 4.23. *The isomorphism classes of stem extensions of $\mathcal{Q}$ are in one-to-one correspondence with the subspaces of ${}_nHL_1(\mathcal{Q})$. Moreover, if $\mathcal{U}$ and $\mathcal{V}$ are two subspaces of ${}_nHL_1(\mathcal{Q})$, then $\mathcal{U} \subseteq \mathcal{V}$ if and only if there is a map (necessarily surjective) from the stem extension corresponding to $\mathcal{U}$ to the stem extension corresponding to $\mathcal{V}$.*

Proof. Let $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ be a stem extension. According to Proposition 4.15, let $\mathcal{U} = \mathsf{Ker}\left(\theta_*(E)\right) = \mathsf{Ker}\left(\theta_*(\Delta[E]) : {}_nHL_1(\mathcal{Q}) \to \mathsf{N}\right)$ be the subspace associated to E. It is clear that isomorphic stem extensions yield the same subspace of ${}_nHL_1(\mathcal{Q})$.

Conversely, let $\mathcal{U} \subseteq {}_nHL_1(\mathcal{Q})$ and $\mathsf{N} = {}_nHL_1(\mathcal{Q})/\mathcal{U}$ be; we consider the canonical projection $\tau : {}_nHL_1(\mathcal{Q}) \to \mathsf{N}$, then there exists an element $\Delta[E] \in$

${}_nHL^1(\mathcal{Q}, \mathsf{N})$ such that $\theta_*\left(\Delta[E]\right) = \tau$. Obviously, $[E]$ is unique, $\theta_*\left(\Delta[E]\right)$ is a surjective homomorphism and then E is a stem extension.

Finally, if $E' : 0 \to \mathsf{N}' \to \mathcal{G}' \to \mathcal{Q} \to 0$ is other stem extension associated with $\mathcal{U}$, then there exists an isomorphism $r : \mathsf{N} \to \mathsf{N}'$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{U} & \longrightarrow & {}_nHL_1(\mathcal{Q}) & \xrightarrow{\theta_*(E)} & \mathsf{N} & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow{\scriptstyle r} & & \\
0 & \longrightarrow & \mathcal{U} & \longrightarrow & {}_nHL_1(\mathcal{Q}) & \xrightarrow{\theta_*(E')} & \mathsf{N}' & \longrightarrow & 0
\end{array}$$

By Proposition 4.21 (a), there exists $\sigma : \mathcal{G} \to \mathcal{G}'$ inducing $r : \mathsf{N} \to \mathsf{N}'$ and $1_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ such that $(r, \sigma, 1_{\mathcal{Q}}) : E \to E'$ is a homomorphism of extensions; moreover, σ is an isomorphism, and then E and E' are in the same isomorphism class.

For the second statement, we consider a morphism of stem extensions $(r, t, 1) : E \to E'$. Naturality in sequence (3) implies $\mathcal{U} = \mathsf{Ker}\left(\theta_*(E)\right) \subseteq \mathsf{Ker}\left(\theta_*(E')\right) = \mathcal{V}$.

For the converse, we first recall that every stem extension is isomorphic to an extension E with $\theta_*\left(\Delta[E]\right)$ the canonical projection. It is thus enough to consider those. Let $\mathcal{U} \subseteq \mathcal{V} \subseteq {}_nHL_1(\mathcal{Q})$, $\mathsf{N} = {}_nHL_1(\mathcal{Q})/\mathcal{U}$ and $\mathsf{N}' = {}_nHL_1(\mathcal{Q})/\mathcal{V}$. There exists a surjective homomorphism $r : \mathsf{N} \to \mathsf{N}'$ such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{U} & \longrightarrow & {}_nHL_1(\mathcal{Q}) & \xrightarrow{\tau} & \mathsf{N} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle r} & & \\
0 & \longrightarrow & \mathcal{V} & \longrightarrow & {}_nHL_1(\mathcal{Q}) & \xrightarrow{\sigma} & \mathsf{N}' & \longrightarrow & 0
\end{array}$$

Now, if $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ is an extension with $\theta_*\left(\Delta[E]\right) = \theta_*(E) = \tau$, and $E' : 0 \to \mathsf{N}' \to \mathcal{G}' \to \mathcal{Q} \to 0$ is another extension with $\theta_*\left(\Delta[E']\right) = \theta_*(E') = \sigma$, then by Proposition 4.21 (a), there exists $t : \mathcal{G} \to \mathcal{G}'$ inducing r and 1; moreover, t is surjective. □

Remark 4.24. We recall that when $\mathcal{Q}$ is perfect, then Proposition 4.21 implies that t is uniquely determined by r.

Proposition 4.25. *Let $\mathcal{Q}$ be a perfect Leibniz n-algebra, and let $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ be a stem extension, then the following sequence is exact:*

$$0 \to {}_nHL_1(\mathcal{G}) \to {}_nHL_1(\mathcal{Q}) \stackrel{\theta_*(E)}{\to} \mathsf{N} \to 0.$$

Proof. From exact sequence (3) in [7] and keeping in mind that Proposition 4.1 implies that ${}_nHL_0(\mathcal{G}) = 0$. □

Note that it follows from Propositions 4.23 and 4.25 that when $\mathcal{Q}$ is a perfect Leibniz n-algebra, then the first n-Leibniz homology $\mathbb{K}$-vector spaces with trivial coefficients of a stem extension of $\mathcal{Q}$ are precisely the subspaces of ${}_nHL_1(\mathcal{Q})$.

Corollary 4.26. *Let $\mathcal{Q}$ be a perfect Leibniz n-algebra, and let $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ be a central extension. Then E is a stem cover if and only if ${}_nHL_0(\mathcal{G}) = {}_nHL_1(\mathcal{G}) = 0$.*

Proof. It easily comes from the exact sequence (3) in [7] associated to E, with ${}_nHL_0(\mathcal{G}) = {}_nHL_1(\mathcal{G}) = 0$, it is easily derived that $\theta_*(E)$ is an isomorphism, and conversely. □

Remark 4.27. Let $E : 0 \to \mathsf{N} \to \mathcal{G} \to \mathcal{Q} \to 0$ be a stem extension with ${}_nHL_1(\mathcal{G}) = 0$, then E is a stem cover. Corollary 4.26 shows that the converse is true if, in addition, $\mathcal{Q}$ is a perfect Leibniz n-algebra.

In general, however, there are stem covers with ${}_nHL_1(\mathcal{G}) \neq 0$. For example, let $\mathcal{F}$ be a non-abelian or non-nilpotent free Leibniz n-algebra, and let us consider the sequence $0 \to \mathcal{F}^k/\mathcal{F}^{k+1} \to \mathcal{F}/\mathcal{F}^{k+1} \to \mathcal{F}/\mathcal{F}^k \to 0$, which is central for $k \geq 2$, and moreover, is a stem cover by [9, Proposition 9], since $(\mathcal{F}/\mathcal{F}^{k+1})_{\mathrm{ab}} \cong \mathcal{F}_{\mathrm{ab}} \cong (\mathcal{F}/\mathcal{F}^k)_{\mathrm{ab}}$ and, on the other hand, the map ${}_nHL_1(\mathcal{F}/\mathcal{F}^{k+1}) = \mathcal{F}^{k+1}/\mathcal{F}^{k+2} \to {}_nHL_1(\mathcal{F}/\mathcal{F}^k) = \mathcal{F}^k/\mathcal{F}^{k+1}$ is trivial. Moreover, ${}_nHL_1(\mathcal{F}/\mathcal{F}^{k+1}) = \mathcal{F}^{k+1}/\mathcal{F}^{k+2} \neq 0$.

Proposition 4.28. *Let $\mathcal{Q}$ be a perfect Leibniz n-algebra, and let $0 \to \mathcal{R} \to \mathcal{F} \xrightarrow{f} \mathcal{Q} \to 0$ be a free presentation. Then*

$$0 \to {}_nHL_1(\mathcal{Q}) \to \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \xrightarrow{\varphi} \mathcal{Q} \to 0$$

is a stem cover of $\mathcal{Q}$, where φ is induced by f.

Proof. $0 \to {}_nHL_1(\mathcal{Q}) \to \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \xrightarrow{\varphi} \mathcal{Q} \to 0$ is the universal central extension of $\mathcal{Q}$ [6, Theorem 5]. Moreover, ${}_nHL_1\left(\frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}\right) = {}_nHL_0\left(\frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]}\right)$ $= 0$ by [11, Proposition 5.1], hence it is a stem cover by Corollary 4.26. □

From Proposition 4.19, when $\mathcal{Q}$ is a perfect Leibniz n-algebra, we have that any stem cover is isomorphic to $0 \to {}_nHL_1(\mathcal{Q}) \to \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \to \mathcal{Q} \to 0$.

Proposition 4.29. *Let* $E : 0 \to \mathsf{N} \xrightarrow{\chi} \mathcal{G} \xrightarrow{\pi} \mathcal{Q} \to 0$ *be a central extension, and let* $f : \mathcal{X} \to \mathcal{Q}$ *be a homomorphism of Leibniz* n*-algebras where* $\mathcal{X}$ *is a perfect Leibniz* n*-algebra. Then there exists* $\varphi : \mathcal{X} \to \mathcal{G}$ *such that* $\pi \circ \varphi = f$ *if and only if* $f_*({}_nHL_1(\mathcal{X})) \subseteq \pi_*({}_nHL_1(\mathcal{G}))$.

If φ *exists, then it is uniquely determined.*

Proof. If φ exists, then the functor ${}_nHL_1(-)$ preserves the composition, so

$$f_*({}_nHL_1(\mathcal{X})) = \pi_*\left(\varphi_*({}_nHL_1(\mathcal{X}))\right) \subseteq \pi_*({}_nHL_1(\mathcal{G})).$$

Conversely, let $\mathcal{Q}' = \mathsf{Im}(f) \subseteq \mathcal{Q}$ and $\mathcal{S} = \mathsf{Ker}(f)$, then the exact sequence $0 \to \mathcal{S} \to \mathcal{X} \to \mathcal{Q}' \to 0$ induces the exact sequence $0 \to \mathcal{S}/[\mathcal{S}, \mathcal{X}, \overset{n-1}{\ldots}, \mathcal{X}] = \mathcal{S}' \to \mathcal{X}/[\mathcal{S}, \mathcal{X}, \overset{n-1}{\ldots}, \mathcal{X}] = \mathcal{X}' \xrightarrow{f'} \mathcal{Q}' \to 0$ where $f' : \mathcal{X}' \to \mathcal{Q}'$ is induced by f. Now, sequence (3) implies that $f'_*({}_nHL_1(\mathcal{X}')) = f_*({}_nHL_1(\mathcal{X})) \subseteq \pi_*({}_nHL_1(\mathcal{G})) \subseteq {}_nHL_1(\mathcal{Q})$.

In order to complete the proof, we need to construct $\varphi' : \mathcal{X}' \to \mathcal{G}$ such that the following diagram be commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}' & \longrightarrow & \mathcal{X}' & \xrightarrow{f'} & \mathcal{Q}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi'} & & \downarrow & & \\
0 & \longrightarrow & \mathsf{N} & \longrightarrow & \mathcal{G} & \xrightarrow{\pi} & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

By naturality in sequence (3) and by the fact $f'_*({}_nHL_1(\mathcal{X}')) \subseteq \pi_*({}_nHL_1(\mathcal{G}))$, then there exists an injective map $\beta : \mathsf{Im}(f'_*) \to \mathsf{Im}(\pi_*)$ which induces $\tau' : \mathcal{S}' \to \mathsf{N}$. From Proposition 4.21 it follows the existence of $\varphi' : \mathcal{X}' \to \mathcal{G}$. Moreover, φ' is unique if and only if $\mathsf{Hom}({}_nHL_1(\mathcal{Q}'), \mathsf{N}) = 0$, which is obvious. Now $\varphi : \mathcal{X} \to \mathcal{G}$ is obtained by the composition $\varphi' \circ \mathsf{nat} : \mathcal{X} \to \mathcal{X}' \to \mathcal{G}$. □

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