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The final Moufang variety: FRUTE loops

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Abstract. FRUTE loops are loops that satisfy the identity $(x \cdot xy)z = (y \cdot zx)x$. We show that locally finite FRUTE loops are precisely the products $O \times H$, where O is a commutative Moufang loop in which all elements are of odd order, and H is a 2-group such that the derived subloop H' is of exponent two and $H' \leq Z(H)$.

1. Introduction

A *loop* is a set with a single binary operation such that in $x \cdot y = z$, knowledge of any two of x, y, and z specifies the third uniquely, and with a unique two-sided identity element, denoted by 1. Two of the more actively investigated varieties of loops are the (left) Bol loops and the Moufang loops. Generalizing from the features common to the Moufang and Bol identities leads to the notion of generalized Bol–Moufang identity (definitions of terms in this paragraph are given in Section 2, below). There are 48 varieties of loops of generalized Bol–Moufang type [2]. The classification of varieties of loops of this type was initiated by Fenyves [5], [6]. It is well-known that 3 of these 48 are nonassociative varieties that consist of loops all of which are Moufang – the variety of extra loops, the variety of Moufang loops, and the variety of commutative Moufang loops. An exhaustive search in [2] showed that there exists precisely one more variety of nonassociative loops of generalized Bol–Moufang type all of whose members are Moufang loops: the FRUTE loops. The purpose of this paper is to elucidate the structure of these loops.

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In Section 2, we give a brief overview of loops of (generalized) Bol–Moufang type.

In Section 3, we prove some basic facts about FRUTE loops, e.g., that they are Moufang. We also offer a characterization of FRUTE loops via their commutants and nuclei.

In Section 4, we show that FRUTE loops are automorphic loops. We also derive some basic facts about middle inner mappings and commutator identities, and we show that conjugation is homomorphic.

In Section 5, we offer a simple, elegant decomposition of locally finite FRUTE loops.

In the balance of this section, we fix notation and introduce basic definitions. We have tried to make this paper as self-contained as possible.

Loops admit both a left and a right division, denoted by $\backslash$ and $/$, respectively, satisfying the following four identities [11]: $x \cdot (x \backslash y) = y$, $(y/x) \cdot x = y$, $x \backslash (x \cdot y) = y$ and $(y \cdot x)/x = y$. In the event that $1/x = x \backslash 1$, we use the standard notation denoting two-sided inverse: x^{-1}. Loops are both left and right cancellative.

We usually write xy instead of $x \cdot y$, and reserve $\cdot$ to have lower priority than juxtaposition among factors to be multiplied; for instance, $(x \cdot xy)z = (y \cdot zx)x$ stands for $(x \cdot (x \cdot y)) \cdot z = (y \cdot (z \cdot x)) \cdot x$.

Moufang loops satisfy the *left alternative property* ($x \cdot xy = x^2 \cdot y$) and the *right alternative property* ($y \cdot x^2 = yx \cdot x$) [11]. Each element of a Moufang loop has a (unique) two-sided inverse; and Moufang loops satisfy the *left inverse property* ($x^{-1} \cdot xy = y$), the *right inverse property* ($yx \cdot x^{-1} = y$) and the *antiautomorphic inverse property* ($(xy)^{-1} = y^{-1}x^{-1}$) [11]. Moufang loops are also *diassociative*, i.e., pairs of elements generate associative subloops [11].

Note that the right inverse property can also be given by $u(1/x) = u/x$. Indeed, the choice $u = x$ implies $1/x = x \backslash 1 = x^{-1}$. Setting $u = yx$, thus, gives $yx \cdot x^{-1} = y$.

A triple of permutations, α, β, γ on a loop, Q, is called an *autotopism* if $\forall x, y \in Q$ we have $\alpha(x) \cdot \beta(y) = \gamma(x \cdot y)$; note, we follow the convention of writing functions on the left of their arguments and thus composing from right to left.

We use the standard notation for the right and left translations: $R_y(x) = L_x(y) = xy$. The group of permutations generated by the set of all right and left translations is called the *multiplication group* of Q. The *inner mapping group*, $I(Q)$, is the subgroup of those permutations in the multiplication group that fix 1. Set $T_x = R_x^{-1} L_x$; each T_x is an inner mapping.

A subloop of loop Q is *normal* if it is invariant as a set under the action of $I(Q)$ [11]. *Automorphic loops* are loops all of whose inner mappings are automorphisms. The variety of commutative Moufang loops and the variety of groups are two prominent examples of automorphic loops [9]. As we shall see in Corollary 4.5, FRUTE loops are automorphic Moufang loops. Thus (and obviously), the many structural results about automorphic Moufang loops (see, for example, [9]) apply also to FRUTE loops.

The *left nucleus* of a loop Q is given by $N_\lambda(Q) = \{a : a{\cdot}xy = ax{\cdot}y, \forall x, y \in L\}$. The *middle nucleus*, $N_\mu(Q)$, and the *right nucleus*, $N_\rho(Q)$, are defined analogously. The *nucleus*, then, is given by $N(Q) = N_\lambda(Q) \cap N_\mu(Q) \cap N_\rho(Q)$. Each of the four nuclei is an associative subloop of Q for any loop Q [11]; none of these is necessarily normal. The *commutant* of Q is given by $C(Q) = \{c : \forall x \in Q, cx = xc\}$. It need not be normal, nor even a subloop [10]. If Q is a Moufang loop, then $N(Q)$ is normal [11], while $C(Q)$ is a subloop [11], but it need not be normal [7]. But if Q is a FRUTE loop, then it is straightforward to show that both $C(Q)$ and $N(Q)$ are normal. Finally, the *center* is the subloop given by $Z(Q) = N(Q) \cap C(Q)$; it is a normal subloop for an arbitrary loop, Q [11].

2. Bol–Moufang identities

In the variety of loops, each of the following four identities implies the other three: $z(xy \cdot z) = zx \cdot yz$, $z(x \cdot zy) = (zx \cdot z)y$, $(z \cdot xy)z = zx \cdot yz$, and $(xz \cdot y)z = x(z \cdot yz)$. A loop that satisfies any one (hence, all four) of these identities is called a *Moufang loop*. A *left Bol loop* is a loop satisfying the identity $x(y \cdot xz) = (x{\cdot}yx)z$; *right Bol loops* satisfy the mirror identity. The left Bol identity generates a different variety of loops than does the right Bol identity. The intersection of the two Bol varieties is the variety of Moufang loops [13]. These three varieties are amongst the most prominent and intensively investigated varieties of loops; the six equations axiomatizing them share a number of common features:

(1) they contain only one operation – the loop product;
(2) there are exactly three distinct variables appearing on each side of the equal sign, one appearing twice on each side of the equal sign, the other two appearing once each on each side of the equal sign; and
(3) the order in which the variables appear in is the same on each side of the equal sign.

An identity that satisfies these three conditions is thus called an identity of *Bol–Moufang type*. There are 60 identities of Bol–Moufang type, and there are precisely 14 varietes of loops axiomatized by a single identity of Bol–Moufang type [13]. We will refer to these varieties as *varieties of loops of Bol–Moufang type*. By *nonassociative variety of loops* we mean a variety of loops that is not a subvariety of the variety of groups. Of the 14 varieties of loops of Bol–Moufang type, exactly 2 – the variety of Moufang loops, and the variety of extra loops – are nonassociative varieties that consist of loops all of which are Moufang [13]. There is a large body of literature about Moufang loops; quite a bit is also known about the structure of extra loops [8]. Three different identities of Bol–Moufang type axiomatize the variety of extra loops [5], [13]; here is one of them: $x(y \cdot zx) = (xy \cdot z)x$.

By dropping the third condition in the definition of Bol–Moufang type, we obtain the following definition: an identity is said to be of *generalized Bol–Moufang type* if it satisfies the following two conditions:

(1) it contains only one operation – the loop product; and

(2) there are exactly three distinct variables appearing on each side of the equal sign, one appearing twice on each side of the equal sign, the other two appearing once each on each side of the equal sign.

There are 1215 identities of generalized Bol–Moufang type, and there are precisely 48 varieties of loops axiomatized by a single identity of generalized Bol–Moufang type, including the 14 varieties of Bol–Moufang type [2]. Of the remaining 34 varieties, six are varieties of commutative loops, one of which is the variety of all commutative Moufang loops [2]. Thus, there remains 28 varieties of not necessarily commutative loops of generalized Bol–Moufang type, only three of which – the three Cheban varieties [12] – have been investigated. Of the 25 remaining varieties, one is an associative variety (i.e., it consists of groups), and 6 can each be be described by single, shorter identities; they are: $x \cdot yx = y \cdot xx$, $xy \cdot x = y \cdot xx$, $x \cdot xy = y \cdot xx$, $x \cdot xy = yx \cdot x$, $xx \cdot y = x \cdot yx$ and $xx \cdot y = xy \cdot x$. Thus, there remain 18 varieties of loops of Bol–Moufang type that have not yet been investigated. The variety of FRUTE loops is one of these 18; this variety is axiomatized by the *FRUTE identity*: $(x \cdot xy)z = (y \cdot zx)x$. It should be noted that the name "FRUTE" is a nonsensical pseudo-acronym coined in the service of a prank [2] – Froot Loops® are an iconic American breakfast cereal. But pranks notwithstanding, this variety of loops has interesting structure, as we shall see.

Thus, of the 48 generalized Bol–Moufang varieties, exactly four of them are nonassociative varieties that consist entirely of Moufang loops; these are (1) the variety of Moufang loops, (2) the variety of extra loops, (3) the variety of commutative Moufang loops and (4) the variety of FRUTE loops. As above, there is a rather large body of literature about the first two varieties. There is also, obviously, a large body of literature about the commutative Moufang loops; a good place to start is [1]. Here, we initiate a study of the fourth variety: the FRUTE loops.

It is worth pointing out two things. Firstly, if a variety of loops of generalized Bol–Moufang type is not a subvariety of the variety of Moufang loops, and yet it is a subvariety of the variety of left Bol loops or of the variety of right Bol loops, then it is equal to one of these two varieties. Secondly, of the remaining 17 varieties of loops of generalized Bol–Moufang type that remain to be analyzed, two appear to be especially structurally interesting: (1) the variety of loops axiomatized by $(x \cdot xy)z = (yz \cdot x)x$, and (2) the variety of loops axiomatized by this identity's mirror: $x(x \cdot zy) = z(yx \cdot x)$. These might be worthy candidates for future study.

3. Characterization via commutant and nucleus

Lemma 3.1. *Let Q be a FRUTE loop. Then Q is both left and right alternative. Moreover, Q satisfies the left, right and antiautomorphic inverse properties. Furthermore, $x^2y = yx^2$ for all $x, y \in Q$.*

Proof. Setting $z = 1$ in the FRUTE identity yields $x \cdot xy = yx \cdot x$, and setting $y = 1$ yields $x^2z = zx \cdot x$. Hence, $x^2y = yx \cdot x = x \cdot xy$. Next, in $(x \cdot xy)z = (y \cdot zx)x$ set $z = x$; by right cancellation, $x \cdot xy = yx^2$. Concatenation gives $yx^2 = x^2y$, and so $yx^2 = x \cdot xy = yx \cdot x$.

Setting $z = 1/x$ yields $yx = (x \cdot xy)(1/x)$. Hence $yx = (yx^2)(1/x)$. This implies the right inverse property, since yx can be also expressed as $(yx^2)/x$, by the right alternative property. From

$$x = (x \cdot xy)(xy)^{-1} = (y \cdot (xy)^{-1}x)x$$

it follows that $1 = y \cdot (xy)^{-1}x$. Hence $y^{-1} = y\backslash 1 = (xy)^{-1}x$. Thus, by the right inverse property, $y^{-1}x^{-1} = (xy)^{-1}$. Of course, the right inverse property and the antiautomorphic inverse property together imply the left inverse property. □

Corollary 3.2. *In the variety of loops, the FRUTE identity is equivalent to its mirror: $x(xz \cdot y) = z(yx \cdot x)$.*

Proof. Apply the antiautomorphic inverse property to the FRUTE identity, and note that each element is an inverse. □

Theorem 3.3. *Each FRUTE loop is also a Moufang loop.*

Proof. In the FRUTE identity, $(x \cdot xy)z = (y \cdot zx)x$, set $z = z/x$ to obtain $(x \cdot xy)(z/x) = yz \cdot x$. Next, apply the right inverse property and multiply both sides of this equation on the left by x to obtain $x((x \cdot xy)(zx^{-1})) = x \cdot (yz \cdot x)$. Corollary 3.2, together with the right inverse property, yields $x((x \cdot xy)(zx^{-1})) = xy \cdot zx$; hence $x(yz \cdot x) = xy \cdot zx$. □

Theorem 3.4. *Let Q be a Moufang loop. The following conditions are equivalent:*

(i) *Q is a FRUTE loop;*

(ii) *$x^3 \in N(Q)$ and $x^2 \in C(Q)$ for every $x \in Q$; and*

(iii) *each T_x, $x \in Q$, is either the identity mapping or an involutory automorphism on Q.*

Proof. Note that ${T_x}^{-1} = T_{x^{-1}}$ in each Moufang loop. Now, the identity $(x \cdot xy)(z/x) = yz \cdot x$ is true if and only if each (L_x^2, R_x^{-1}, R_x) is an autotopism. This is equivalent to the condition that $(L_x R_x^{-1}, R_x^{-1} L_x, R_x L_x^{-1}) = (T_x, T_x, T_{x^{-1}})$ is an autotopism, since, in a Moufang loop, $(L_x^{-1} R_x^{-1}, L_x, L_x^{-1})$ is an autotopism for every $x \in Q$ [11]. If α, β and γ are permutations of a loop Q such that $\alpha(1) = \beta(1) = \gamma(1) = 1$ and (α, β, γ) is an autotopism, then $\alpha = \beta = \gamma \in \mathrm{Aut}(Q)$. Hence $(x \cdot xy)z = (y \cdot zx)x$ holds if and only if $T_x^2 = \mathrm{id}_Q$ and $T_x = T_{x^{-1}} \in \mathrm{Aut}(Q)$ for all $x \in Q$. Since $L_x R_x = R_x L_x$, both $T_x^2 = \mathrm{id}_Q$ and $T_x = T_{x^{-1}}$ are equivalent to $L_{x^2} = R_{x^2}$, i.e., $x^2 \in C(Q)$. Finally, we note that in Moufang loops, $T_x \in \mathrm{Aut}(Q)$ if and only if $x^3 \in N(Q)$ [11].

The referee suggests including an alternate, and "autotopism-free", proof. We use the diassociativity of Q freely in this second proof. In the FRUTE identity $(x \cdot xy)z = (y \cdot zx)x$, set $y = x^{-2}y$ to obtain $yz = (x^{-2}y \cdot zx)x$. Next, in the Moufang identity $x(y \cdot xz) = xyx \cdot z$, set $y = x^{-2}y$ and $z = x^{-1}zx$ to obtain $x(x^{-2}y \cdot zx) = x^{-1}yx \cdot x^{-1}zx$. It follows from $yz = (x^{-2}y \cdot zx)x$ that $x \cdot yz \cdot x^{-1} = x(x^{-2}y \cdot zx) = x^{-1}yx \cdot x^{-1}zx$. Hence, the identity $x \cdot yz \cdot x^{-1} = x^{-1}yx \cdot x^{-1}zx$ characterizes the FRUTE loops among Moufang loops. Setting $y = 1$ gives $xzx^{-1} = x^{-1}zx$; that is $T_x^2 = \mathrm{id}_Q$, and hence, T_x is an involutory automorphism or the identity map. □

Let Q be a loop with normal subloops A and B such that $Q = AB$. Put $C = A \cap B$. For each $(a, b) \in A \times B$ send ab to aC; this is a well-defined homomorphism $Q \to A/C$. (Indeed, $a_1 b_1 = a_2 b_2$ implies that $a_1 B = a_2 B$. Of course, $a_1 B = a_2 B \Leftrightarrow a_1/a_2 \in C \Leftrightarrow a_1 C = a_2 C$; i.e., the mapping is well-defined. The rest follows from $a_1 B \cdot a_2 B = (a_1 a_2)B$.)

Thus, $ab \mapsto (aC, bC)$ is a well-defined homomorphism $Q \to A/C \times B/C$, the kernel of which is equal to C. This homomorphism is surjective, since $a = a \cdot 1$ and $b = 1 \cdot b$, for every $(a, b) \in A \times B$.

Proposition 3.5. *Let Q be a FRUTE loop. Then $Q/Z(Q) \cong L \times G$, where L is a commutative Moufang loop of exponent three, and G is a Boolean group.*

Proof. We have $x = x^{-2}x^3$ for each $x \in Q$. Hence $Q = C(Q)N(Q)$. By definition, $Z(Q) = C(Q) \cap N(Q)$. If $x \in C(Q)$, then $x^3 \in Z(Q)$ [11]. Thus, since $C(Q)$ is a commutative Moufang loop, $C(Q)/Z(Q)$ is a commutative Moufang loop of exponent three. If $x \in N(Q)$, then $x^2 \in Z(Q)$. Finally, since $N(Q)$ is a group, $N(Q)/Z(Q)$ is a group of exponent two, i.e., a Boolean group. □

4. Conjugation is homomorphic

Proposition 3.5 suggests a characterization of FRUTE loops via central extensions of loops $L \times G$, where L is a commutative Moufang loop of exponent three, and G is a Boolean group. However, such an approach requires a number of opaque technical considerations. We take a different approach and prove first that $x \mapsto T_x$ is a homomorphism $Q \to \mathrm{Aut}(Q)$. There are other settings in loop theory in which a condition of this kind plays a prominent role [3], [9]; it seems to be indicative of very strong structural assumptions.

Let Q be a FRUTE loop. Since Q is also a Moufang loop, and hence, diassociative [11], $T_x(y) = xyx^{-1}$ for all $x, y \in Q$.

Lemma 4.1. *$T_x = T_{x^3}$, for every $x \in Q$.*

Proof. Diassociativity and the fact that $x^2 \in C(Q)$ together imply that $xyx^{-1} = x^3yx^{-3}$. □

Lemma 4.2. *$T_xT_y = T_{yx}$, for all $x, y \in Q$.*

Proof. If $a, b, c \in N(Q)$, then $T_aT_b = T_c$ is equivalent to $T_{ab} = T_c$, and to the condition that $(ab)^{-1}c \in C(Q)$. Since $T_xT_y = T_{x^3}T_{y^3}$ and $T_{yx} = T_{(yx)^3}$, by Lemma 4.1, it suffices to prove that $y^{-3}x^{-3}(yx)^3 \in C(Q)$. And this follows from applying Theorem 3.4 to:

$$y^{-3}x^{-3}(yx)^2(yx) = (yx)^2y^{-1}x^{-1}y^{-1}x^{-1} = (yx)^2(xy)^{-2}. \qquad \square$$

Theorem 4.3. *Let Q be a FRUTE loop. Then the mapping $T\colon Q \to \mathrm{Aut}(Q)$, $x \mapsto T_x$, is a loop homomorphism with kernel equal to $C(Q)$. Furthermore, $Q/C(Q)$ is a Boolean group, and $C(Q)$ is a commutative Moufang loop.*

Proof. By Theorem 3.4, each T_x is an automorphism of Q that is of order at most two. By Lemma 4.2, all these automorphisms form a subgroup of $\mathrm{Aut}(Q)$. This subgroup is Boolean. Therefore $T_xT_y = T_yT_x = T_{xy}$. The rest is clear. □

We define the *commutator*, $[x, y]$ of x and y, thusly: $xy = yx \cdot [x, y]$. The least normal subloop of Q that contains all commutators is known as the *derived subloop* and is denoted by Q'. By Theorem 4.3, $Q' \le C(Q)$.

Theorem 4.4. *Let Q be a FRUTE loop. Then Q' is a Boolean group and is contained in $Z(Q)$. If $x, y, z \in Q$, then $[x, y] = [y, x] = (xy)^2x^{-2}y^{-2}$ and $[x, yz] = [x, y][x, z]$.*

Proof. First note that $[x, y] \in C(Q)$ by Theorem 4.3. Next, we have $x^3 \in N(Q)$ by Theorem 3.4, and so $[x^3, y] = x^{-3}T_y^{-1}(x^3) \in N(Q)$, since $N(Q)$ is normal in Q. Thus, since $x^2 \in C(Q)$ by 3.4, we have $[x, y] = [x^3, y]$, and hence, $[x, y] \in Z(Q)$.

In any Moufang loop, we have $[a, yz] = [a, z](z^{-1}[a, y]z)$ if a is nuclear. This is because $\langle a, y, z\rangle$ is a group, by Moufang's theorem [4]. Thus, if commutators are in the commutant, as they are in FRUTE loops, we have $[a, yz] = [a, y][a, z]$. Now, recalling that x^3 is nuclear, we obtain $[x, yz] = [x^3, yz] = [x^3, y][x^3, z] = [x, y][x, z]$. And this in turn gives$[x, y]^2 = [x, y^2] = 1$, since $y^2 \in C(Q)$. Hence $[y, x] = [x, y]^{-1} = [x, y]$ for all $x, y \in Q$. Finally, since Q is diassociative, and since $x^2, y^2 \in C(Q)$, we have $(xy)^2x^{-2}y^{-2} = x^{-2}xyy^{-2}xy = x^{-1}y^{-1}xy = [x, y]$, again, for all $x, y \in Q$. □

Corollary 4.5. *Let Q be a FRUTE loop. Then Q is an automorphic Moufang loop.*

Proof. The standard inner mappings are pseudoautomorphisms, the companions of which are nuclear.Thus, the inner mappings are automorphisms [1]. □

5. Locally finiteness takes a coproduct face

By a *p-group* we mean a group in which each element is of an order p^k for some $k \ge 0$.

Each locally finite commutative Moufang loop is a coproduct (i.e., a direct sum) of abelian p-groups, $p \ne 3$, and of a commutative Moufang 3-loop, i.e., a commutative Moufang loop in which all elements are of order a power of 3 [1].

Lemma 5.1. *Let G be a locally finite group such that $x^2 \in Z(G)$ for each $x \in G$. Then $G = O \times H$, where O is an abelian group in which all elements are* *of odd order, and H is a 2-group such that H' is of exponent two and $H' \leq Z(H)$. Conversely, every such group $G = O \times H$ satisfies the condition that $x^2 \in Z(G)$ for each $x \in G$.*

Proof. Indeed, every element of odd order is central. Hence, all elements of odd order form a central normal subgroup. Now, $[x, y] = (xy)^2 x^{-2} y^{-2}$ implies that $[x, y] = [y, x]$ is central. Hence $[x, yz] = [x, y][x, z]$ for all $x, y, z \in G$, and, in particular, $1 = [x, y^2] = [x, y]^2$. Hence G' is a central Boolean group, and H may be defined as a preimage of the 2-group in the decomposition of G/G'. For the converse direction, first note that $O \leq Z(G)$ and that $G' = H' \leq Z(G)$. Since $[x, yz] = [x, yz] = [x, y][x, z]$ for all $x, y, z \in G$, and since $[x, y]$ is of order at most two, $[x, y^2] = 1$ for all $x, y \in G$. Thus $y^2 \in Z(G)$ for every $y \in G$. □

Theorem 5.2. *Let Q be a locally finite FRUTE loop. Then $Q = O \times H$, where O is a commutative Moufang loop in which all elements are of odd order, and H is a 2-group such that H' is of exponent two and $H' \leq Z(H)$. On the other hand, every such loop $Q = O \times H$ is a FRUTE loop.*

Proof. Put $\bar{Q} = Q/Q'$ and denote by π the projection $Q \to \bar{Q}$. By Theorem 3.4 and the remark before Lemma 5.1, $\bar{Q} = \bar{O} \times \bar{H}$, where $\bar{H}$ is a 2-group, and $\bar{O}$ consists of all odd order elements of $\bar{Q}$. Put $H = \pi^{-1}(\bar{H})$. All elements of H are of order that is a power of two. By Theorem 3.4, H is a group, or – more precisely – a 2-group. The group satisfies the law $x^2 yz = yzx^2$. Hence $H' \leq Z(H)$, by Lemma 5.1. By Theorem 4.4, $Q' \leq Z(Q)$. All elements of $\pi^{-1}(\bar{O})$ thus belong to $C(Q)$. Indeed, if $x \in Q$ is expressed as yz, where $y, z \in <x>$, $|y|$ is odd and $|z|$ is a power of two, then $\pi(x) \in \bar{O}$ if and only if $z \in Q' \leq Z(Q)$. Therefore $Q' = H'$, again by Theorem 4.4. Now, $\pi^{-1}(\bar{O})$ is a commutative Moufang loop. Hence it can be expressed as $O \times H'$, where O consists of all odd order elements that are contained in Q [1]. On the other hand, H consists of all elements in Q the order of which is a power of two. Clearly, $O \cap H = O \cap \pi^{-1}(\bar{O}) \cap H = O \cap H' = 1$. Both O and H are normal since each inner mapping is an automorphism, by Corollary 4.5.

The converse is trivial, since both commutative Moufang loops and groups with central squares satisfy the FRUTE identity. □

Thus, by Theorem 5.2, we see that the minimal order for a noncommutative FRUTE loop is $8 = 2^3$, while the minimal order for a nonassociative FRUTE loop is $81 = 3^4$.

References

[1] R. H. Bruck, A Survey of Binary Systems, Third Printing, Corrected, Ergebnisse der Mathematik und Ihrer Grenzgebiete, New Series, Vol. **20**, *Springer-Verlag*, *Berlin – Heidelberg*, 1971.

[2] B. Coté, B. Harvill, M. Huhn and A. Kirchman, Classification of loops of generalized Bol–Moufang type, *Quasigroups Related Systems* **19** (2011), 193–206.

[3] A. Drápal, Conjugacy closed loops and their multiplication groups, *J. Algebra* **272** (2004), 838–850.

[4] A. Drápal, A simplified proof of Moufang's theorem, *Proc. Amer. Math. Soc.* **139** (2011), 93–98.

[5] F. Fenyves, Extra loops. I., *Publ. Math. Debrecen* **15** (1968), 235–238.

[6] F. Fenyves, Extra loops. II. On loops with identities of Bol–Moufang type, *Publ. Math. Debrecen* **16** (1969), 187–192.

[7] A. N. Grishkov and A. V. Zavarnitsine, On the commutative center of Moufang loops, arXiv:1711.07001.

[8] M. K. Kinyon and K. Kunen, The structure of extra loops, *Quasigroups Related Systems* **12** (2004), 39–60.

[9] M. K. Kinyon, K. Kunen and J. D. Phillips, Every diassociative A-loop is Moufang, *Proc. Amer. Math. Soc.* **130** (2002), 619–624.

[10] M. K. Kinyon and J. D. Phillips, Commutants of Bol loops of odd order, *Proc. Amer. Math. Soc.* **132** (2004), 617–619.

[11] H. O. Pflugfelder, Quasigroups and Loops: Introduction, *Heldermann Verlag*, *Berlin*, 1990.

[12] J. D. Phillips and V. A. Shcherbacov, Cheban loops, *J. Gen. Lie Theory Appl.* **4** (2010), Art. ID G100501, 5 pp.

[13] J. D. Phillips and P. Vojtěchovský, The varieties of loops of Bol–Moufang type, *Algebra Universalis* **54** (2005), 259–271.

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