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# *Tb* criteria for Calderón–Zygmund operators on Lipschitz spaces with para-accretive functions

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**Abstract.** By developing the Littlewood–Paley characterization of Lipschitz spaces $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$ and the new Lipschitz spaces $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ with $b$ a para-accretive function, and establishing a density argument for $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ in the weak sense, the authors prove that the Calderón–Zygmund operators $T$ are bounded from $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ to $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$ if and only if $T(b) = 0$.

## 1. Introduction and main results

The $L^2$-boundedness of convolution singular operators follows from the Plancherel theorem. However, for non-convolution operators, one needs to develop new methods to obtain the $L^2$-boundedness. It is well-known that the $T1$ theorem plays a crucial role in the analysis of $L^2$-boundedness, and furthermore, the $L^p$ boundedness of Calderón-Zygmund singular integral operators, see [3] and [5] among others. For the endpoint boundedness, there are also analogous $T1$ criterions for Calderón-Zygmund operators, see, for example, [8].

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To be more precise, assume $K(x,y)$ is a continuous function with $x \neq y$, satisfying the following estimates for some $\sigma > 0$:

$$|K(x,y)| \leq |x-y|^{-n}; \tag{1.1}$$

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')| \leq \frac{|x-x'|^{\sigma}}{|x-y|^{n+\sigma}}, \quad |x-x'| \leq \frac{1}{2}|x-y|. \tag{1.2}$$

A Calderón–Zygmund singular integral operator $T$ is a continuous linear operator from $C_0^\infty(\mathbb{R}^n)$ into its dual associated to a kernel $K(x,y)$, which can be represented in a bilinear form by

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)g(x)dydx \tag{1.3}$$

whenever $f$ and $g$ are two compactly supported $C^\infty$-functions whose supports are disjoint. The $L^2(\mathbb{R}^n)$-boundedness of such a non-convolution operator $T$ was proved by the remarkable $T1$ theorem of David and Journé [3], which provides a general criterion for the $L^2$-boundedness of Calderón–Zygmund singular integral operators. A Calderón–Zygmund singular integral operator $T$ is said to be a Calderón–Zygmund operator if $T$ is bounded on $L^2(\mathbb{R}^n)$.

However, the $T1$ theorem could not be directly applied to the Cauchy integral on Lipschitz curves

$$\mathcal{C}(f)(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{f(y)}{(x-y)+i(a(x)-a(y))}dy,$$

where the function $a(x)$ satisfies the Lipschitz condition. One does not know how to prove that the Cauchy integral $\mathcal{C}(f)$ on Lipschitz curves maps the function 1 into $BMO$ function without assuming the $L^2$-boundedness of the Cauchy integral $\mathcal{C}(f)$. McIntosh and Meyer in [14] (see also [15]) observed that if function 1 in the $T1$ theorem is allowed to be replaced by a function $b(x) = 1 + ia'(x)$, then $\mathcal{C}(b) = 0$, and it will lead to $L^2$-boundedness of the Cauchy integral on all Lipschitz curves. McIntosh and Meyer [14] proved such a $Tb$ theorem, where $b$ is an accretive function. Here a bounded complex-valued function $b$ is said to be an accretive function if $b$ satisfies $0 < \delta \leq \mathrm{Re}(b(x))$ for almost every $x$. Finally, David, Journé and Semmes [4] gave more general conditions for $L^\infty$ functions $b$ to be para-accretive, namely they proved a $Tb$ theorem in which function 1 in the $T1$ theorem can be replaced by para-accretive functions (see Definition 1.1). See also [1], [2], [7], [16] for more details about $Tb$ theorems.

Assuming $b$ is an accretive function, Meyer and Coifman [15] introduced a new Hardy space $H_b^1(\mathbb{R}^n)$ such that $f \in H_b^1(\mathbb{R}^n)$ if the product $bf$ is in the classical Hardy space $H^1(\mathbb{R}^n)$. These spaces have the advantage of a cancellation adapted to the complex measure $b(x)dx$ and are closely related to the $Tb$ theorem. Han, Lee and Lin [9] developed the Hardy spaces $H_b^p(\mathbb{R}^n)$, $p \leq 1$, in terms of the Littlewood–Paley characterization, where $b$ is a para-accretive function. It seems that $f$ belongs to the Hardy spaces $H_b^p(\mathbb{R}^n)$ if $bf \in H^p(\mathbb{R}^n)$. However, this is not the case, because, for $p < 1$, the elements of both $H^p(\mathbb{R}^n)$ could be distributions, and hence, in general, the product $bf$ does not make sense for $f \in H^p(\mathbb{R}^n)$ and a para-accretive $b$ unless $b$ is a constant.

It was well known that the dual spaces of $H^1(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$ are $BMO$ and the Lipschitz space $\mathrm{Lip}(n(1/p-1))(\mathbb{R}^n)$ for $0 < p < 1$, respectively. Meyer and Coifman [15] proved that for an accretive function $b$, the dual space of $H_b^1(\mathbb{R}^n)$ is $\mathrm{BMO}_b(\mathbb{R}^n)$. Similarly, Han, Lee and Lin [10] proved that the dual space of $H_b^p(\mathbb{R}^n)$ is $\mathrm{Lip}_b(n(1/p-1))(\mathbb{R}^n)$ for $0 < p < 1$. More precisely, denote $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n) = \{f : f = bg, g \in \mathrm{Lip}(\alpha)(\mathbb{R}^n)\}$, where $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$ is the classical Lipschitz space of order $\alpha$. If $f \in \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$, then $f = bg$ where $g \in \mathrm{Lip}(\alpha)(\mathbb{R}^n)$, and the norm of $f$ is defined by $\|f\|_{\mathrm{Lip}_b(\alpha)} = \|g\|_{\mathrm{Lip}(\alpha)}$. Notice that the classical Lipschitz spaces on $\mathbb{R}^n$ play an important role in harmonic analysis and partial differential equations (see [6], [11]–[13], [17]).

The goal of this paper is to give a $Tb$ criteria for the boundedness of Calderón–Zygmund operators on the Lipschitz spaces $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$. Our main machine is to develop the Littlewood–Paley characterization for Lipschitz spaces $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ and $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$, which also has its own value and significance. For this purpose, we first recall some definitions about the para-accretive function, test function spaces and an approximation to the identity.

*Definition 1.1* (Para-accretive function [4]). A bounded complex-valued function $b$ defined on $\mathbb{R}^n$ is said to be para-accretive if there exist constants $C, \eta > 0$ such that, for all cubes $Q \subset \mathbb{R}^n$, there is a $Q' \subset Q$ with $\eta|Q| \leq |Q'|$ satisfying

$$\frac{1}{|Q|}\left|\int_{Q'} b(x)dx\right| \geq C > 0.$$

*Definition 1.2* (Test function [10]). Fix $\beta \in (0,1]$, $\gamma \in (0,\infty)$. Let $b$ be a para-accretive function. A function $f$ defined on $\mathbb{R}^n$ is said to be a test function of type $(\beta,\gamma)$ centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$, if $f$ satisfies the following conditions:

$$|f(x)| \leq \frac{Cd^\gamma}{(d+|x-x_0|)^{n+\gamma}}; \tag{1.4}$$

$$|f(x)-f(x')| \leq \left(\frac{|x-x'|}{d+|x-x_0|}\right)^\beta \frac{Cd^\gamma}{(d+|x-x_0|)^{n+\gamma}}, \tag{1.5}$$

for $|x-x'| \le \frac{d+|x-x_0|}{2}$; and

$$\int_{\mathbb{R}^n} f(x)b(x)dx = 0. \tag{1.6}$$

The collection of all test functions of type $(\beta,\gamma)$ centered at $x_0 \in \mathbb{R}^n$ with width $d>0$ will be denoted by $\mathcal{M}(\beta,\gamma,x_0,d)$. If $f \in \mathcal{M}(\beta,\gamma,x_0,d)$, the norm of $f$ is defined by

$$\|f\|_{\mathcal{M}(\beta,\gamma,x_0,d)} = \inf\{C \ge 0 : (1.4) \text{ and } (1.5) \text{ hold}\}.$$

We denote $\mathcal{M}(\beta,\gamma,0,1)$ simply by $\mathcal{M}(\beta,\gamma)$. Moreover, one can see that $\mathcal{M}(\beta,\gamma)$ is a Banach space under the norm $\|f\|_{\mathcal{M}(\beta,\gamma)} < \infty$. The dual space $(\mathcal{M}(\beta,\gamma))'$ consists of all linear functionals $\mathcal{L}$ from $\mathcal{M}(\beta,\gamma)$ to $\mathbb{C}$ satisfying

$$|\mathcal{L}(f)| \le C\|f\|_{\mathcal{M}(\beta,\gamma)}, \quad \text{for all } f \in \mathcal{M}(\beta,\gamma).$$

It is easy to see that for any fixed $x_0 \in \mathbb{R}^n$ and $d>0$, $\|f\|_{\mathcal{M}(\beta,\gamma,x_0,d)}$ is equivalent to $\|f\|_{\mathcal{M}(\beta,\gamma)}$. We denote by $\langle h,f\rangle$ the natural pairing of elements $h \in (\mathcal{M}(\beta,\gamma))'$ and $f \in \mathcal{M}(\beta,\gamma)$. As usual, we write

$$b\mathcal{M}(\beta,\gamma) = \{f | f = bg \text{ for some } g \in \mathcal{M}(\beta,\gamma)\}.$$

If $f \in b\mathcal{M}(\beta,\gamma)$ and $f = bg$ for $g \in \mathcal{M}(\beta,\gamma)$, then the norm is defined by $\|f\|_{b\mathcal{M}(\beta,\gamma)} = \|g\|_{\mathcal{M}(\beta,\gamma)}$.

To state the Calderón reproducing formula, we also need the definition of an approximation to the identity.

*Definition 1.3* (Approximation to the identity [7]). Let $b$ be a para-accretive function. A sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is called an approximation to the identity associated to $b$ if $S_k(x,y)$, the kernels of $S_k$, are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}$ such that there exist constant $C$ and some $0<\varepsilon\le 1$ for all $k\in\mathbb{Z}$ and all $x, x', y$ and $y' \in \mathbb{R}^n$:

(1) $|S_k(x,y)| \le C\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$;

(2) $|S_k(x,y) - S_k(x',y)| \le C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$, for $|x-x'| \le (2^{-k}+|x-y|)/2$;

(3) $|S_k(x,y) - S_k(x,y')| \le C\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$, for $|y-y'| \le (2^{-k}+|x-y|)/2$;

(4) $|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]|$
$\leq C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$,
for $|x-x'| \leq (2^{-k}+|x-y|)/2$ and $|y-y'| \leq (2^{-k}+|x-y|)/2$;

(5) $\int_{\mathbb{R}^n} S_k(x,y)b(y)dy = 1$ for all $k \in \mathbb{Z}, x \in \mathbb{R}^n$;

(6) $\int_{\mathbb{R}^n} S_k(x,y)b(x)dx = 1$ for all $k \in \mathbb{Z}, y \in \mathbb{R}^n$.

Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity defined above, $D_k = S_k - S_{k-1}$ and $b$ is a para-accretive function. The first result of this paper is the following Littlewood–Paley characterization of the Lipschitz spaces $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$ and $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$.

**Theorem 1.1.** (1) *For $0 < \alpha < \varepsilon$, $f \in \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ if and only if $f \in (\mathcal{M}(\beta,\gamma))'$ with some $\beta \in (0,1]$, $\gamma \in (\alpha,\infty)$, and*

$$\sup_{k\in\mathbb{Z},\ x\in\mathbb{R}^n} 2^{k\alpha}|D_k f(x)| \leq C < \infty.$$

*Moreover,*

$$\|f\|_{\mathrm{Lip}_b(\alpha)} \approx \sup_{k\in\mathbb{Z},\ x\in\mathbb{R}^n} 2^{k\alpha}|D_k f(x)|. \tag{1.7}$$

(2) *For $0 < \alpha < \varepsilon$, $f \in \mathrm{Lip}(\alpha)(\mathbb{R}^n)$ if and only if $f \in (b\mathcal{M}(\beta,\gamma))'$ with some $\beta \in (0,1]$, $\gamma \in (\alpha,\infty)$, and*

$$\sup_{k\in\mathbb{Z},\ x\in\mathbb{R}^n} 2^{k\alpha}|D_k(bf)(x)| \leq C < \infty.$$

*Moreover,*

$$\|f\|_{\mathrm{Lip}(\alpha)} \approx \sup_{k\in\mathbb{Z},\ x\in\mathbb{R}^n} 2^{k\alpha}|D_k(bf)(x)|. \tag{1.8}$$

The main result of this paper is the following $Tb$ criteria for the boundedness of Calderón–Zygmund operators on Lipschitz spaces.

**Theorem 1.2.** *Let $T$ be a Calderón–Zygmund operator, then $T$ is bounded from $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ to $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$ for $0 < \alpha < \varepsilon$ if and only if $T(b) = 0$.*

The organization of this paper is as follows. In Section 2, we will give the proof of Theorem 1.1, and Theorem 1.2 will be proved in Section 3.

Throughout this paper, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. We use the notation $A \approx B$ to denote that there exists a positive constant $C$ such that $C^{-1}B \leq A \leq CB$. Let $j \wedge j'$ be the minimum of $j$ and $j'$.

## 2. Proof of Theorem 1.1

Before the proof of Theorem 1.1, we give two continuous versions of the Calderón type reproducing formula.

**Proposition 2.1** (Continuous Calderón type reproducing formula [9]). *Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity defined above. Set $D_k = S_k - S_{k-1}$. Then there exist two families of operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\widetilde{D}}_k\}_{k\in\mathbb{Z}}$ such that for all $f \in \mathcal{M}(\beta,\gamma)$,*

$$f(x) = \sum_k \widetilde{D}_k b D_k b(f)(x) = \sum_k D_k b \widetilde{\widetilde{D}}_k b(f)(x), \tag{2.1}$$

*the series converges in the $L^p$-norm, $1 < p < \infty$, in the $\mathcal{M}(\beta',\gamma')$-norm for $\beta' < \beta$ and $\gamma' < \gamma$, and in the $(b\mathcal{M}(\beta',\gamma'))'$ for $\beta < \beta'$ and $\gamma < \gamma'$. Moreover, $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$, satisfy the following estimates: for $0 < \varepsilon' < \varepsilon$, where $\varepsilon$ is the regularity exponent of $S_k$, there exists a constant $C > 0$ such that*

$$|\widetilde{D}_k(x,y)| \le C \frac{2^{-k\varepsilon'}}{(2^{-k} + |x-y|)^{n+\varepsilon'}},$$

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le C \left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k} + |x-y|)^{n+\varepsilon'}}$$

*for $|x - x'| \le (2^{-k} + |x-y|)/2$,*

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y) b(x) dx = 0 \text{ for all } k \in \mathbb{Z} \text{ and } y \in \mathbb{R}^n,$$

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y) b(y) dy = 0 \text{ for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R}^n.$$

*$\widetilde{\widetilde{D}}_k(x,y)$, the kernels of $\widetilde{\widetilde{D}}_k$, satisfy the same conditions above but with interchanging the positions of $x$ and $y$.*

*We also have*

$$f(x) = \sum_k b\widetilde{D}_k^\sharp b D_k(f)(x) = \sum_k b D_k b \widetilde{\widetilde{D}}_k^\sharp(f)(x), \tag{2.2}$$

*where the series converges in the $L^p$-norm, $1 < p < \infty$, in the $b\mathcal{M}(\beta',\gamma')$-norm for $\beta' < \beta$ and $\gamma' < \gamma$, and in $(\mathcal{M}(\beta',\gamma'))'$ for $\beta < \beta'$ and $\gamma < \gamma'$. Moreover, $\widetilde{D}_k^\sharp(x,y)$, the kernel of $\widetilde{D}_k^\sharp$, and $\widetilde{\widetilde{D}}_k^\sharp(x,y)$, the kernel of $\widetilde{\widetilde{D}}_k^\sharp$, satisfy the same conditions as $\widetilde{D}_k(x,y)$ and $\widetilde{\widetilde{D}}_k(x,y)$, respectively.*

The following four discrete versions of the Calderón type reproducing formula were given in [9].

**Proposition 2.2** (Discrete Calderón type reproducing formula for the case of $\mathcal{M}(\beta,\gamma)$). *Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity defined above. Set $D_k = S_k - S_{k-1}$. Then there exists a family of operators $\{\tilde{\tilde{D}}_k^\sharp\}_{k\in\mathbb{Z}}$ such that for all $f \in \mathcal{M}(\beta,\gamma)$,*

$$f(x) = \sum_k \sum_Q D_k(x, y_Q) \int_Q \tilde{\tilde{D}}_k^\sharp(bf)(y)b(y)dy, \tag{2.3}$$

*where $Q$ are all dyadic cubes with the side length $2^{-k-j}$ for some fixed positive large integer $j$, $y_Q$ is any fixed point in $Q$, and the series converges in the $\mathcal{M}(\beta',\gamma')$-norm for $\beta' < \beta$ and $\gamma' < \gamma$. Moreover, $\tilde{\tilde{D}}_k^\sharp(x,y)$, the kernel of $\tilde{\tilde{D}}_k^\sharp$, satisfy the same conditions as those in Proposition 2.1.*

**Proposition 2.3** (Discrete Calderón type reproducing formula for the case of $(\mathcal{M}(\beta,\gamma))'$). *Let $S_k$, $D_k$, $\tilde{\tilde{D}}_k^\sharp(x,y)$, $Q$, $y_Q$ be given in Proposition 2.2. Then, for all $f \in (\mathcal{M}(\beta,\gamma))'$,*

$$f(x) = \sum_k \sum_Q D_k(f)(y_Q) \int_Q b(x)\tilde{\tilde{D}}_k^\sharp(y,x)b(y)dy, \tag{2.4}$$

*where the series converges in the sense that for all $g \in \mathcal{M}(\beta',\gamma')$ with $\beta<\beta'$ and $\gamma<\gamma'$,*

$$\lim_{M,N\to\infty} \Big\langle \sum_{|k|<M} \sum_{\mathrm{dist}(0,Q)\le N} D_k(f)(y_Q) \int_Q b(x)\tilde{\tilde{D}}_k^\sharp(y,x)b(y)dy, g \Big\rangle = \langle f, g\rangle.$$

**Proposition 2.4** (Discrete Calderón type reproducing formula for the case of $b\mathcal{M}(\beta,\gamma)$). *Let $S_k$, $D_k$, $\tilde{\tilde{D}}_k^\sharp(x,y)$, $Q$, $y_Q$ be given in Proposition 2.2. Then, for all $f \in b\mathcal{M}(\beta,\gamma)$,*

$$f(x) = \sum_k \sum_Q b(x) D_k(x, y_Q) \int_Q \tilde{\tilde{D}}_k^\sharp(f)(y)b(y)dy, \tag{2.5}$$

*where the series converges in the $b\mathcal{M}(\beta',\gamma')$-norm for the case of $\beta'<\beta$ and $\gamma'<\gamma$.*

**Proposition 2.5** (Discrete Calderón type reproducing formula for the case of $(b\mathcal{M}(\beta,\gamma))'$). *Let $S_k$, $D_k$, $\widetilde{\widetilde{D}}_k^\sharp(x,y)$, $Q$, $y_Q$ be given in Proposition 2.2. Then, for all $f \in (b\mathcal{M}(\beta,\gamma))'$,*

$$f(x) = \sum_k \sum_Q D_k(bf)(y_Q) \int_Q \widetilde{\widetilde{D}}_k^\sharp(y,x) b(y) dy, \tag{2.6}$$

*where the series converges in the sense that for all $g \in b\mathcal{M}(\beta',\gamma')$ with $\beta < \beta'$ and $\gamma < \gamma'$,*

$$\lim_{M,N\to\infty} \left\langle \sum_{|k|<M} \sum_{\operatorname{dist}(0,Q)\le N} D_k(bf)(y_Q) \int_Q \widetilde{\widetilde{D}}_k^\sharp(y,x) b(y) dy, g \right\rangle = \langle f, g \rangle.$$

Now we give the proof of Theorem 1.1. For (1), firstly, it is easy to see that if $f \in \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$, then $f \in (\mathcal{M}(\beta,\gamma))'$ with $\beta > 0$ and $\gamma > \alpha$. To check that $\sup_{k\in\mathbb{Z},\ x\in\mathbb{R}^n} 2^{k\alpha}|D_k f(x)| \le C\|f\|_{\mathrm{Lip}_b(\alpha)}$, notice that $D_k(x,y)$, as the function of $y$ when $x$ fixed, belongs to $\mathcal{M}(\varepsilon,\varepsilon), \alpha < \varepsilon$, and $\int_{\mathbb{R}^n} D_k(x,y)b(y)dy = 0$, so we have

$$\begin{aligned}
|D_k f(x)| &= \left|\int_{\mathbb{R}^n} D_k(x,y) f(y) dy\right| = \left|\int_{\mathbb{R}^n} D_k(x,y)\left(\frac{f(y)}{b(y)} - \frac{f(x)}{b(x)}\right) b(y) dy\right| \\
&\le C\|f\|_{\mathrm{Lip}_b(\alpha)} \int_{\mathbb{R}^n} |D_k(x,y)||x-y|^\alpha |b(x)| dx \\
&\le C 2^{-k\alpha}\|f\|_{\mathrm{Lip}_b(\alpha)} \int_{\mathbb{R}^n} \frac{2^{-k(\varepsilon-\alpha)}}{(2^{-k}+|x-y|)^{n+\varepsilon-\alpha}} dx \\
&\le C 2^{-k\alpha}\|f\|_{\mathrm{Lip}_b(\alpha)}.
\end{aligned} \tag{2.7}$$

We now prove converse implication of Theorem 1.1 (1). Suppose that $f \in (\mathcal{M}(\beta,\gamma))'$ and $\sup_{k\in\mathbb{Z},\ x\in\mathbb{R}^n} 2^{k\alpha}|D_k f(x)| \le C\|f\|_{\mathrm{Lip}_b(\alpha)}$. We first show that $f$ is a continuous function. Recalling the Calderón type reproducing formula (2.4) for $f \in (\mathcal{M}(\beta,\gamma))'$,

$$f(x) = \sum_k \sum_Q D_k f(y_Q) \int_Q b(x) \widetilde{\widetilde{D}}_k^\sharp(y,x) b(y) dy.$$

We split $\sum_{k\in\mathbb{Z}}$ by the sums over $k > 0$ and $k \le 0$, and write $f = f_1 + f_2$ in $(\mathcal{M}(\beta,\gamma))'$ for the corresponding $k$. We will show that $f_1$ and $f_2$ are continuous functions.

For the first case, using the size condition on $\widetilde{\widetilde{D}}_k^\sharp$, we get

$$\begin{aligned}
|f_1(x)| &\le \sum_{k>0}\sum_Q |D_k f(y_Q)| \int_Q |b(x)||\widetilde{\widetilde{D}}_k^\sharp(y,x)||b(y)|dy \\
&\le C\sum_{k>0} 2^{-k\alpha}\sum_Q \int_Q |\widetilde{\widetilde{D}}_k^\sharp(y,x)|dy \\
&\le C\sum_{k>0} 2^{-k\alpha}\sum_Q \int_Q \frac{2^{-k\varepsilon'}}{(2^{-k}+|y-x|)^{n+\varepsilon'}}dy \le C.
\end{aligned}$$

So the series for $f_1$ is converges uniformly in $x$, and it implies that $f_1$ is a continuous function.

For $g \in \mathcal{M}(\beta,\gamma)$, by the cancellation condition $\int_{\mathbb{R}^n} g(x)b(x)dx = 0$, we can write

$$\begin{aligned}
\langle f_2, g\rangle &= \sum_{k\le 0}\sum_Q D_k f(y_Q)\int_Q\int_{\mathbb{R}^n}\left(\widetilde{\widetilde{D}}_k^\sharp(y,x)-\widetilde{\widetilde{D}}_k^\sharp(y,x_0)\right)g(x)b(x)dxb(y)dy \\
&= \left\langle \sum_{k\le 0}\sum_Q D_k f(y_Q)\int_Q b(x)\left(\widetilde{\widetilde{D}}_k^\sharp(y,x)-\widetilde{\widetilde{D}}_k^\sharp(y,x_0)\right)b(y)dy,\ g(x)\right\rangle.
\end{aligned}$$

We focus on the series in the inner product. If $|x-x_0| \le \frac{2^{-k}}{2}$, we can use the smoothness condition on $\widetilde{\widetilde{D}}_k^\sharp$ and get

$$\begin{aligned}
&\left|\sum_{k\le 0}\sum_Q D_k f(y_Q)\int_Q b(x)\left(\widetilde{\widetilde{D}}_k^\sharp(y,x)-\widetilde{\widetilde{D}}_k^\sharp(y,x_0)\right)b(y)dy\right| \\
&\quad\le C\sum_{k\le 0}2^{-k\alpha}\sum_Q\int_Q\left(\frac{|x-x_0|}{2^{-k}+|x-y|}\right)^{\varepsilon'}\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-y|)^{n+\varepsilon'}}|b(x)||b(y)|dy \\
&\quad\le C|x-x_0|^{\varepsilon'}\sum_{k\le 0}2^{k(\varepsilon'-\alpha)}\sum_Q\int_Q\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-y|)^{n+\varepsilon'}}dy \\
&\quad\le C|x-x_0|^{\varepsilon'}\sum_{k\le 0}2^{k(\varepsilon'-\alpha)} \le C|x-x_0|^{\varepsilon'}. \qquad (2.8)
\end{aligned}$$

If $|x-x_0| > \frac{2^{-k}}{2}$, we have

$$\left|\sum_{k\le 0}\sum_Q D_k f(y_Q)\int_Q b(x)\left(\widetilde{\widetilde{D}}_k^\sharp(y,x)-\widetilde{\widetilde{D}}_k^\sharp(y,x_0)\right)b(y)dy\right| \le$$

$$\le C\sum_{k\le 0}2^{-k\alpha}\sum_{Q}\int_Q\left(\frac{|x-x_0|}{2^{-k}}\right)^{\varepsilon'}\left(|\widetilde{\widetilde{D}}_k^{\sharp}(y,x)|+|\widetilde{\widetilde{D}}_k^{\sharp}(y,x_0)|\right)|b(x)||b(y)|dy$$

$$\le C|x-x_0|^{\varepsilon'}\sum_{k\le 0}2^{k(\varepsilon'-\alpha)}\sum_{Q}\int_Q\left(|\widetilde{\widetilde{D}}_k^{\sharp}(y,x)|+|\widetilde{\widetilde{D}}_k^{\sharp}(y,x_0)|\right)dy$$

$$\le C|x-x_0|^{\varepsilon'}\sum_{k\le 0}2^{k(\varepsilon'-\alpha)}\le C|x-x_0|^{\varepsilon'}. \tag{2.9}$$

Thus, we obtain that for any given large $L>0$, the series for $f_2$ converges uniformly for $|x-x_0|\le L$ in the distribution sense. This means that $f_2$ is a continuous function on any compact subset in $\mathbb{R}^n$, and hence, it is continuous on $\mathbb{R}^n$.

Now, we estimate $\|f\|_{\mathrm{Lip}_b(\alpha)}$ as follows. For any $x,y\in\mathbb{R}^n$, we can choose $k_0\in\mathbb{Z}$ such that $2^{-k_0}\le|x-y|\le 2^{-k_0+1}$. The Calderón type reproducing formula (2.4) tells us that

$$\frac{f(x)}{b(x)}-\frac{f(y)}{b(y)}=\left(\sum_{k\ge k_0}+\sum_{k<k_0}\right)\sum_Q D_kf(y_Q)\int_Q\left(\widetilde{\widetilde{D}}_k^{\sharp}(z,x)-\widetilde{\widetilde{D}}_k^{\sharp}(z,y)\right)b(z)dz$$
$$:=I+II.$$

For $I$, the size condition on $\widetilde{\widetilde{D}}_k^{\sharp}$ and the assumption $\sup_{k\in\mathbb{Z},\ y\in\mathbb{R}^n}2^{k\alpha}|D_kf(y)|\le C$ lead to that

$$|I|\le C\sum_{k\ge k_0}2^{-k\alpha}\sum_Q\int_Q\left(\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-z|)^{n+\varepsilon'}}+\frac{2^{-k\varepsilon'}}{(2^{-k}+|y-z|)^{n+\varepsilon'}}\right)|b(z)|dz$$
$$\le C\sum_{k\ge k_0}2^{-k\alpha}\int_{\mathbb{R}^n}\left(\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-z|)^{n+\varepsilon'}}+\frac{2^{-k\varepsilon'}}{(2^{-k}+|y-z|)^{n+\varepsilon'}}\right)dz$$
$$\le C2^{-k_0\alpha}\approx C|x-y|^{\alpha}.$$

For the second term $II$, using the smooth condition on $\widetilde{\widetilde{D}}_k^{\sharp}$, we deal with it similarly as in (2.8) and (2.9),

$$|II|\le C|x-y|^{\varepsilon'}\sum_{k<k_0}2^{k(\varepsilon'-\alpha)}\le C|x-y|^{\varepsilon'}2^{k_0(\varepsilon'-\alpha)}\approx C|x-y|^{\alpha}.$$

Therefore, we have

$$\left|\frac{f(x)}{b(x)}-\frac{f(y)}{b(y)}\right|\le C|x-y|^{\alpha},$$

and we can get the proof of (1.7).

Similarly, using the same methods with the Calderón type reproducing formula (2.5) and (2.6), we can get the conclusion (1.8) in Theorem 1.1 (2).

## 3. Proof of Theorem 1.2

To show Theorem 1.2, we need to prove a density argument for $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$, which will play a crucial role in the proof of Theorem 1.2.

*Definition 3.1* ([9]). Suppose $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity associated to a para-accretive function $b$ with regularity exponent $\varepsilon$. Set $D_k = S_k - S_{k-1}$. The classical Hardy space $H^p(\mathbb{R}^n)$, $0 < p \le 1$, is the collection of $(\mathcal{M}(\beta,\gamma))'$ satisfying

$$\|f\|_{H^p} := \|\mathcal{S}(f)\|_p < \infty,$$

where the square function $\mathcal{S}(f)$ is defined by

$$\mathcal{S}(f) = \left\{ \sum_{k\in\mathbb{Z}} \sum_{Q_k} |D_k(f)(x_{Q_k})|^2 \chi_{Q_k}(x) \right\}^{1/2}.$$

We also have that the Hardy space $H^p_b(\mathbb{R}^n)$, $\frac{n}{n+\varepsilon} < p \le 1$, is the collection of $(b\mathcal{M}(\beta,\gamma))'$ such that

$$\|f\|_{H^p_b} := \|g_b(f)\|_p < \infty,$$

where $g_b(f)$, the discrete Littlewood–Paley $g$-function associated with the para-accretive function $b$, is defined by

$$g_b(f)(x) = \left\{ \sum_{k\in\mathbb{Z}} \sum_{Q_k} |D_k(bf)(x_{Q_k})|^2 \chi_{Q_k}(x) \right\}^{1/2},$$

the dyadic cube $Q_k$ with radius $2^{-k}$ and $x_{Q_k}$ are any fixed points in $Q_k$.

**Lemma 3.1** ([10]). *Suppose that $b$ is a para-accretive function and $\frac{n}{n+\varepsilon} < p \le 1$. The dual space of $H^p_b(\mathbb{R}^n)$ is $\mathrm{Lip}_b(n(1/p-1))(\mathbb{R}^n)$ in the following sense:*

(1) *For each $g \in \mathrm{Lip}_b(n(1/p-1))(\mathbb{R}^n)$, the linear functional $\ell_g : f \mapsto \langle f, g\rangle$ defined initially on $H^p_b(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, has a continuous extension to $H^p_b$ and $\|\ell_g\| \le \|g\|_{\mathrm{Lip}_b(n(1/p-1))}$.*

(2) *Conversely, every continuous linear functional $\ell$ on $H^p_b$ can be realized as $\ell = \ell_g$, defined initially on $H^p_b(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, for some $g \in \mathrm{Lip}_b(n(1/p-1))(\mathbb{R}^n)$ and $\|g\|_{\mathrm{Lip}_b(n(1/p-1))} \le C\|\ell\|$.*

The density argument is the following:

**Lemma 3.2.** *If $f \in \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$, then there exists a sequence $\{f_n\} \in L^2(\mathbb{R}^n) \cap \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ such that*

(1)
$$\|f_n\|_{\mathrm{Lip}_b(\alpha)} \le C\|f\|_{\mathrm{Lip}_b(\alpha)}, \tag{3.1}$$
*where the constant $C$ is independent of $f_n$ and $f$.*

(2) *For any $g \in L^2(\mathbb{R}^n) \cap H_b^p(\mathbb{R}^n)$, $\frac{n}{n+\varepsilon} < p \le 1$,*
$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle. \tag{3.2}$$

Proof. Note that the Calderón type reproducing formula (2.2) holds in the sense of distribution. We construct a sequence $\{f_n\}$ as follows,
$$f_n(y) = \sum_{|k'|<n} b\widetilde{D}_{k'}^{\sharp} b D_{k'}(f)(y). \tag{3.3}$$

Obviously, $f_n \in L^2(\mathbb{R}^n)$ and it converges to $f$ in the sense of distribution. To prove $f_n \in \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$, by the Littlewood–Paley characterization (1.7) of the Lipschitz spaces, we just need to show
$$\sup_{k\in\mathbb{Z},\ x\in\mathbb{R}^n} 2^{k\alpha}|D_k f_n(x)| \le C\|f\|_{\mathrm{Lip}_b(\alpha)}.$$

For the left part of the above inequality, we have
$$\begin{aligned}
|D_k f_n(x)| &= \left|\int_{\mathbb{R}^n} D_k(x,y) \sum_{|k'|<n} b\widetilde{D}_{k'}^{\sharp} b D_{k'}(f)(y)dy\right| \\
&\le \left|\sum_{|k'|<n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_k(x,y)(b\widetilde{D}_{k'}^{\sharp} b)(y,z)dy D_{k'}f(z)dz\right| \\
&\le \sup_{k'\in\mathbb{Z},\ z\in\mathbb{R}^n} 2^{k'\alpha}|D_{k'}f(z)| \sum_{|k'|<n} 2^{-k'\alpha}\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} D_k(x,y)(b\widetilde{D}_{k'}^{\sharp} b)(y,z)dy\right|dz.
\end{aligned}$$

Using the Littlewood–Paley characterization (1.7) and the following almost orthogonal estimate (see [7]),
$$\begin{aligned}
\left|\int_{\mathbb{R}^n} D_k(x,y)(b\widetilde{D}_{k'}^{\sharp} b)(y,z)dy\right| &= \left|D_k b\widetilde{D}_{k'}^{\sharp} b(x,z)\right| \\
&\le C 2^{-|k'-k|\varepsilon'} \frac{2^{-(k'\wedge k)\varepsilon'}}{(2^{-(k'\wedge k)} + |x-z|)^{n+\varepsilon'}},
\end{aligned}$$

we have,

$$2^{k\alpha}|D_k f_n(x)| \le C\|f\|_{\mathrm{Lip}_b(\alpha)} \sum_{|k'|<n} 2^{(k-k')\alpha} 2^{-|k'-k|\varepsilon'} \int_{\mathbb{R}^n} \frac{2^{-(k'\wedge k)\varepsilon'}}{(2^{-(k'\wedge k)}+|x-z|)^{n+\varepsilon'}} dz$$
$$\le C\|f\|_{\mathrm{Lip}_b(\alpha)} \sum_{|k'|<n} 2^{(k-k')\alpha} 2^{-|k'-k|\varepsilon'} \le C\|f\|_{\mathrm{Lip}_b(\alpha)},$$

which implies the estimate in (1).

Next, we show (2). According to the construction of $\widetilde{D}_k^\sharp$ and $\widetilde{\widetilde{D}}_k^\sharp$, we have,

$$\big\langle f - f_n,\ g\big\rangle$$
$$= \Big\langle \sum_{|k|\ge n} b\widetilde{D}_k^\sharp b D_k(f)(y), g(y)\Big\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{|k|\ge n} (b\widetilde{\widetilde{D}}_k^\sharp b)(z,y) g(y) dy\ D_k f(z) dz$$
$$= \int_{\mathbb{R}^n} f(x) \sum_{|k|\ge n} \int_{\mathbb{R}^n} D_k(z,x) b\widetilde{\widetilde{D}}_k^\sharp b(g)(z) dz dx = \Big\langle f(x), \sum_{|k|\ge n} D_k b\widetilde{\widetilde{D}}_k^\sharp b(g)(x)\Big\rangle.$$

Using the Calderón type reproducing formula (2.1), we denote $S_n(g)(x)$ by $\sum_{|k|<n} D_k b\widetilde{\widetilde{D}}_k^\sharp b(g)(x)$, so we have

$$\big\langle f - f_n,\ g\big\rangle = \big\langle f,\ g - S_n(g)\big\rangle.$$

The duality argument (see Lemma 3.1), together with the fact that for each $g \in L^2(\mathbb{R}^n)\cap H_b^p(\mathbb{R}^n)$, $\|g - S_n(g)\|_{H_b^p} \to 0$ as $n\to\infty$, implies that

$$\big|\big\langle f - f_n, g\big\rangle\big| \le C\|f\|_{\mathrm{Lip}_b(n(1/p-1))}\ \|g - S_n(g)\|_{H_b^p} \to 0,\ \text{as } n\to\infty,$$

which shows (2), and hence the proof of Lemma 3.2 is complete. □

**Lemma 3.3.** *Assume* $D_k b(x) = D_k(x, z_k) b(x)$, $z_k$ *is a fixed point, we have* $D_k b \in H^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, $\frac{n}{n+\varepsilon} < p \le 1$ *and*

$$\|D_k b\|_{H^p} \le C 2^{-kn\left(\frac{1}{p}-1\right)}. \tag{3.4}$$

Proof. Firstly, we have

$$\int_{\mathbb{R}^n} |D_k(x,z_k)b(x)|^2 dx \le C\int_{\mathbb{R}^n} \frac{2^{-2k\varepsilon}}{(2^{-k}+|x-z_k|)^{2n+2\varepsilon}} dx \le C 2^{kn}.$$

Next, based on Coifman's construction [15], we may assume that $D_k(x,y)$ has compact support in the sense that $D_k(x,y) = 0$ for $|x-y| \ge 2^{-k}$. For any fixed $k$, we estimate the $H^p$ norm of $D_k b(x) = D_k(x,z_k)b(x)$ as follows. Denote $Q^* = 2Q_k$, the dyadic cube with the center at $z_k$ with radius $2\times 2^{-k}$. We write

$$\int_{\mathbb{R}^n} \mathcal{S}(D_k b)(x)^p dx = \int_{Q^*} \mathcal{S}(D_k b)(x)^p dx + \int_{(Q^*)^c} \mathcal{S}(D_k b)(x)^p dx = III + IV.$$

Applying Hölder's inequality together with the boundedness of the square function on $L^2(\mathbb{R}^n)$ and the size condition of $D_k(x, z_k)$, we have

$$\begin{aligned} III^{\frac{1}{p}} &= \left( \int_{Q^*} |\mathcal{S}(D_k b)(x)|^p dx \right)^{\frac{1}{p}} \le \mu(Q^*)^{\frac{1}{p}-\frac{1}{2}} \left( \int_{Q^*} |\mathcal{S}(D_k b)(x)|^2 dx \right)^{\frac{1}{2}} \\ &\le C\mu(Q^*)^{\frac{1}{p}-\frac{1}{2}} \left( \int_{Q^*} |D_k b(x)|^2 dx \right)^{\frac{1}{2}} \le C2^{kn} \mu(Q^*)^{\frac{1}{p}} \le C2^{-kn\left(\frac{1}{p}-1\right)}. \quad (3.5) \end{aligned}$$

We now estimate term $IV$. Note that if

$$D_j(D_k b)(x) = \int D_j(x,y) D_k(y, z_k) b(y) dy \neq 0,$$

then $|y - z_k| \le 2^{-k}$ and $|x-y| \le 2^{-j}$. If $x \notin Q^*$, then we have $2 \times 2^{-k} \le |x - z_k| \le |x-y| + |y-z_k|$. This implies that $2^{-k} \le |x-y| \le 2^{-j}$, and hence $j \le k$. Using the cancellation condition of $D_k(y, z_k)b(y)$ and the smoothness condition of $D_j$, we have

$$\begin{aligned} |D_j(D_k b)(x)| &= \left| \int_{\mathbb{R}^n} D_j(x,y) D_k(y, z_k) b(y) dy \right| \\ &= \left| \int_{\mathbb{R}^n} \big[ D_j(x,y) - D_j(x, z_k) \big] D_k(y, z_k) b(y) dy \right| \\ &\le C \int_{Q_k} \left( \frac{2^{-k}}{2^{-j}} \right)^{\varepsilon} \frac{2^{-j\varepsilon}}{(2^{-j} + |x - z_k|)^{n+\varepsilon}} \; |D_k(y, z_k)||b(y)| dy \\ &\le C 2^{(j-k)\varepsilon} \frac{2^{-j\varepsilon}}{(2^{-j} + |x - z_k|)^{n+\varepsilon}}, \end{aligned}$$

which implies that for $x \notin Q^*$,

$$\mathcal{S}(D_k b)(x) \le C \left( \sum_{j \le k} 2^{2(j-k)\varepsilon} \frac{2^{-2j\varepsilon}}{(2^{-j} + |x - z_k|)^{2n+2\varepsilon}} \right)^{1/2}.$$

Inserting the above estimate into term $IV$, it follows that

$$\begin{aligned} IV &= \int_{(Q^*)^c} |\mathcal{S}(D_k b)(x)|^p dx \le C \int_{(Q^*)^c} \left( \sum_{j \le k} 2^{2(j-k)\varepsilon} \frac{2^{-2j\varepsilon}}{(2^{-j} + |x - z_k|)^{2n+2\varepsilon}} \right)^{p/2} dx \\ &\le C \sum_{j \le k} 2^{p(j-k)\varepsilon} \; 2^{jn(p-1)} \int_{\mathbb{R}^n} \frac{2^{-j(pn+p\varepsilon-n)}}{(2^{-j} + |x - z_k|)^{n+(pn+p\varepsilon-n)}} dx \le C2^{-kn(1-p)}. \end{aligned}$$

Thus, we completed the proof of Lemma 3.3. □

To show Theorem 1.2, the main result in this paper, we need the boundedness of Calderón–Zygmund operator on Hardy spaces.

**Lemma 3.4** ([10])**.** *Let $T$ be a Calderón–Zygmund operator and $T^*(b) = 0$, then $T$ is bounded from $H^p(\mathbb{R}^n)$ to $H^p_b(\mathbb{R}^n)$ for $n/(n+\varepsilon) < p \leq 1$.*

We are ready to show Theorem 1.2. The necessary conditions of Theorem 1.2 follow directly by

$$\|Tb\|_{\mathrm{Lip}(\alpha)} \leq C\|b\|_{\mathrm{Lip}_b(\alpha)} = \|1\|_{\mathrm{Lip}(\alpha)} = 0.$$

We now prove the sufficiency. The strategy of the proof is applying the duality given in Lemma 3.1. We first show the boundedness of $T$ for $f \in L^2(\mathbb{R}^n) \cap \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ and then extend to the case for $f \in \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$, where $n(1/p-1) = \alpha < \varepsilon$. More precisely, for $f \in L^2(\mathbb{R}^n) \cap \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ and $\psi \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, by Lemma 3.1 and then applying Lemma 3.4 for $T^*$, we have

$$|\langle Tf, \psi\rangle| = |\langle f, T^*\psi\rangle| \leq C\|f\|_{\mathrm{Lip}_b(\alpha)}\|T^*\psi\|_{H^p_b} \leq C\|f\|_{\mathrm{Lip}_b(\alpha)}\|\psi\|_{H^p}.$$

Set $\mathscr{L}_{Tf}(\psi) = \langle Tf, \psi\rangle$, this implies that $\mathscr{L}_{Tf}$ is a bounded linear functional on $H^p(\mathbb{R}^n)$ and

$$\|\mathscr{L}_{Tf}\| \leq C\|f\|_{\mathrm{Lip}_b(\alpha)}.$$

We know that the dual space of $H^p(\mathbb{R}^n)$ is $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$, so there exists $h \in \mathrm{Lip}(\alpha)(\mathbb{R}^n)$ such that $\langle Tf, \psi\rangle = \langle h, \psi\rangle$ and $\|h\|_{\mathrm{Lip}(\alpha)} \leq C\|f\|_{\mathrm{Lip}_b(\alpha)}$. Since $D_k b \in H^p(\mathbb{R}^n)$, we take $\psi$ to be $D_k b$, by Littlewood–Paley characterization (1.8), it follows that

$$\begin{aligned}\|Tf\|_{\mathrm{Lip}(\alpha)} &\approx \sup_{k\in\mathbb{Z},\ y\in\mathbb{R}^n} 2^{k\alpha}|D_k b(Tf)(y)| = \sup_{k\in\mathbb{Z},\ y\in\mathbb{R}^n} 2^{k\alpha}|D_k b(h)(y)| \\ &\approx \|h\|_{\mathrm{Lip}(\alpha)} \leq C\|f\|_{\mathrm{Lip}_b(\alpha)}. \end{aligned} \tag{3.6}$$

We now extend $T$ to the case for $f \in \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ as follows. To do this, for $f \in \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$, by Lemma 3.2, there exists a sequence $f_n \in L^2(\mathbb{R}^n) \cap \mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ with $\|f_n\|_{\mathrm{Lip}_b(\alpha)} \leq C\|f\|_{\mathrm{Lip}_b(\alpha)}$, so that for $g \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we have

$$\langle T(f_n - f_{n'}), g\rangle = \langle f_n - f_{n'}, T^*g\rangle.$$

Using Lemma 3.4 with the fact that $T^*g \in H^p_b(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and then applying Lemma 3.2 (2), we get $\langle f_n - f_{n'}, T^*g\rangle \to 0$ as $n, n' \to \infty$. Thus, we can define

$$\langle Tf, g\rangle = \lim_{n\to\infty} \langle Tf_n, g\rangle,\ g \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Finally, Theorem 1.1 (2) and Lemma 3.2 (1) imply that

$$\begin{aligned}\|Tf\|_{\mathrm{Lip}(\alpha)} &\approx \sup_{k\in\mathbb{Z},\ y\in\mathbb{R}^n} 2^{k\alpha}|D_k b(Tf)(y)| = \sup_{k\in\mathbb{Z},\ y\in\mathbb{R}^n} 2^{k\alpha}|\lim_{n\to\infty} D_k b(Tf_n)(y)| \\ &\le \liminf_{n\to\infty} \sup_{k\in\mathbb{Z},\ y\in\mathbb{R}^n} 2^{k\alpha}|D_k b(Tf_n)(y)| \\ &\approx \liminf_{n\to\infty} \|Tf_n\|_{\mathrm{Lip}(\alpha)} \le C \liminf_{n\to\infty} \|f_n\|_{\mathrm{Lip}_b(\alpha)} \le C\|f\|_{\mathrm{Lip}_b(\alpha)}.\end{aligned}$$

We complete the proof of Theorem 1.2.

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