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Isosceles orthogonally additive mappings and inner product spaces

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Dedicated to Professor János Aczél on his 70th birthday

Abstract. In a normed vector space $(X, \|\cdot\|)$, consider James' *isosceles orthogonality*, i.e., $x \perp y \iff \|x+y\| = \|x-y\|$. It is known that any odd, orthogonally additive mapping from X into an Abelian group is unconditionally additive whenever $\dim X \geq 3$. In this paper a complementary result is presented: the existence of a nontrivial even orthogonally additive mapping characterizes inner product spaces for $\dim X \geq 2$. The proof uses some interesting connectivity theorems.

1. Introduction

Mappings from a vector space into an Abelian group that are additive on orthogonal pairs of vectors have been studied for long by several authors. For an early contribution, see [3]. Besides the usual inner product orthogonality, some other relation were considered such as orthogonalities defined by a bilinear/sesquilinear form (see e.g. [13, 10]), by a norm of the space (see e.g. [7, 8]) or just by some abstract properties (see e.g. [1, 4]).

Based on some weak assumptions and the homogeneity, the most crucial property of such a relation, a general abstract theory was developed in [5], resulting in an additive/quadratic representation of orthogonally additive mappings: odd solutions are additive while the even ones are quadratic. Moreover, the existence of a nontrivial, even solution is characteristic for (generalized) inner product orthogonalities (see e.g. [9, 11]).

In normed vector spaces, however, there are some natural generalizations of the usual inner product orthogonality, homogeneity of which characterizes inner product spaces. Such relations are, for instance, James'

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isosceles orthogonality or the Pythagorean orthogonality (see [2]). Due to the lack of homogeneity, these relations are not covered by the general theory mentioned above.

Using connectivity theorems for intersection of spheres in normed spaces of dimension ≥ 3, we have succeded recently in proving the additivity of an odd, isosceles orthogonally additive mapping (see [12]). The main purpose of this paper is to present a complementary result: the existence of a nontrivial even isosceles orthogonally additive mapping characterizes inner product spaces even if dimension ≥ 2.

More precisely, let $(X, \|\cdot\|)$ be a real normed vector space of dimension ≥ 2 and $(Y, +)$ be an Abelian group. Consider James' *isosceles orthogonality* $\perp$ in X defined by $x \perp y \iff \|x+y\| = \|x-y\|$ $(x, y \in X)$. A mapping $f : X \to Y$ is said to be *(isosceles) orthogonally additive*, if it satisfies the conditional Cauchy equation

$$f(x+y) = f(x) + f(y), \quad \text{whenever} \quad x \perp y.$$

Now we are ready to formulate our main result:

Theorem 1.1. *There exists a nontrivial, even, isosceles orthogonally additive mapping from X to Y if, and only if, X is an inner product space.*

The idea of the proof is as follows:

First we prove that an even solution f depends only on the norm of the argument, i.e., f has the same value on vectors of equal norm. Then using a characterization of inner product spaces due to Senechalle [6], we show that in a non-inner product space there are also vectors of different norm on which the solution f takes the same value. Finally, we use certain connectivity theorems to prove that f is constant in regions bounded by concentric spheres: since these regions cover the whole space (but zero), f is constant and so identically zero. For the detailed proof see Section 3 below.

2. Connectivity theorems

We start with a technical observation:

Lemma 2.1. *If $\|x\| = \|y\| = 1$ and there exists a scalar $0 < \lambda_0 < 1$ such that $\|\lambda_0 x + (1-\lambda_0)y\| = 1$, then*

$$\|\lambda x + (1-\lambda)y\| = 1$$

for all $0 < \lambda < 1$.

Proof. For any $0 < \lambda < 1$, let $u(\lambda)$ denote the convex combination $\lambda x + (1-\lambda)y$. Then obviously

$$\|u(\lambda)\| \le \lambda\|x\| + (1-\lambda)\|y\| = 1.$$

On the other hand, if $\lambda_0 < \lambda < 1$, then we have

$$u(\lambda_0) = \frac{\lambda_0}{\lambda}u(\lambda) + \left(1 - \frac{\lambda_0}{\lambda}\right)y$$

and so by $0 < \lambda_0/\lambda < 1$,

$$1 = \|u(\lambda_0)\| \le \frac{\lambda_0}{\lambda}\|u(\lambda)\| + \left(1 - \frac{\lambda_0}{\lambda}\right),$$

whence $\|u(\lambda)\| \ge 1$ follows.

The case $0 < \lambda < \lambda_0$ can be treated in a similar way.

From now on, the sphere centered at $a \in X$ with radius $\delta > 0$ is denoted by

$$S_a(\delta) = \{x \in X \mid \|x - a\| = \delta\}.$$

For $S_0(1)$ we simply write S.

Now suppose that $0 < \delta \le 1$ and $a \in X \setminus \{0\}$ with $1-\delta \le \|a\| \le 1+\delta$. Letting $u = a/\|a\| \in S$, for a vector $v \in S \setminus \operatorname{lin}\{u\}$, consider the closed halfplane

$$P_v^+ = \{\sigma u + \tau v \mid \sigma, \tau \in \mathbb{R},\ \tau \ge 0\}.$$

Then we are going to show that

Lemma 2.2. *The set*

$$K_v = S \cap S_a(\delta) \cap P_v^+$$

is non-empty, closed and convex. In fact, K_v is a closed interval.

Proof. For $\delta = 1$, $0 < \|a\| \le 2$, the statement has already been proved in [12], Lemma 2.3. Thus we may and do assume that $\delta < 1$.

The set K_v is obviously closed. On the other hand, the mapping $\varphi(x) = \|x - a\|$ is continuous on the connected semi-circle $S \cap P_v^+$ containing $-u$ and u. Since

$$\varphi(-u) = 1 + \|a\| > 1 > \delta \ge |1 - \|a\|| = \|u - a\| = \varphi(u),$$

there exists a vector $x \in S \cap P_v^+$ with $\varphi(x) = \delta$, i.e., $x \in K_v$.

Next we prove the convexity of K_v. To do this, take any vectors $x, y \in K_v$. Then all the vectors

$$x' = \frac{x-a}{\delta}, \quad y' = \frac{y-a}{\delta}, \quad x'' = -x, \quad y'' = -y, \quad x''' = -x', \quad y''' = -y'$$

are in S. We have to show that the line segment $[x, y]$ is contained in K_v. For $x = y$ there is nothing to prove, while for $b = y - x \neq 0$ the following cases may occur:

Case I. $b = \beta a$. Changing the role of x and y, if it is necessary, we may assume that $\beta > 0$. Then $x = \lambda x' + \mu y$ with

$$\lambda = \frac{\beta\delta}{\beta+1} > 0 \quad \text{and} \quad \mu = \frac{1}{\beta+1} > 0.$$

Thus

$$1 = \|x\| \leq \lambda\|x'\| + \mu\|y\| = \lambda + \mu = \frac{\beta\delta+1}{\beta+1},$$

whence the contradiction $\delta \geq 1$ follows.

Case II. Vectors a and b are linearly independent. Then x and y can be expressed as

$$x = \alpha a + \beta b \quad \text{and} \quad y = \alpha a + (\beta+1)b.$$

Without loss of generality, we may assume that $\beta \geq 0$ ($x, y \in P_v^+$ implies that β and $\beta+1$ are of the same sign, thus the role of x and y should be changed only, when $\beta < 0$).

First we claim that $\alpha > 0$ and $\alpha \neq 1$. The value $\alpha = 0$ is impossible since otherwise $1 = \|x\| = \beta\|b\| < (\beta+1)\|b\| = \|y\| = 1$ would follow. Similarly, $\alpha = 1$ would imply $1 = \|x'\| = \frac{\beta}{\delta}\|b\| < \frac{\beta+1}{\delta}\|b\| = \|y'\| = 1$. Now, on contrary, assume that $\alpha < 0$. Then $x = \lambda x' + \mu y$ with

$$\lambda = \frac{-\alpha\delta}{\beta-\alpha+1} > 0 \quad \text{and} \quad \mu = \frac{\beta}{\beta-\alpha+1} \geq 0.$$

Thus

$$1 = \|x\| \leq \lambda\|x'\| + \mu\|y\| = \lambda + \mu = \frac{\beta-\alpha\delta}{\beta-\alpha+1},$$

whence the contradiction $\alpha \geq \frac{1}{1-\delta} > 0$ follows.

Now there may occur two different cases:

(i) $0 < \alpha < 1$. Then for

$$\lambda = 1 - \frac{\alpha}{\beta+1} > 0 \quad \text{and} \quad \mu = \frac{1-\alpha}{\delta(\beta+1)} > 0,$$

we have $x' = \lambda y' + \mu y''$, whence

$$1 = \|x'\| \le \lambda\|y'\| + \mu\|y''\| = \lambda + \mu = 1 + \frac{1-\alpha-\delta\alpha}{\delta(\beta+1)}$$

and so $\alpha \le \frac{1}{1+\delta}$. On the other hand, for

$$\lambda = \frac{\delta\alpha}{\beta+1} > 0 \quad \text{and} \quad \mu = 1 + \frac{\alpha-1}{\beta+1} > 0,$$

we have $x'' = \lambda y' + \mu y''$, whence

$$1 = \|x''\| \le \lambda\|y'\| + \mu\|y''\| = \lambda + \mu = 1 + \frac{\delta\alpha+\alpha-1}{\beta+1}$$

and so $\alpha \ge \frac{1}{1+\delta}$. Thus we have proved that

$$\alpha = \frac{1}{1+\delta}.$$

This fact implies that $x' = (1-\lambda')y' + \lambda' y''$ with

$$0 < \lambda' = \frac{1}{(1+\delta)(\beta+1)} < 1,$$

and $x'' = \lambda'' y' + (1-\lambda'')y''$ with

$$0 < \lambda'' = \frac{\delta}{(1+\delta)(\beta+1)} < 1.$$

Hence by Lemma 2.1,

$$[x', y'] \subset [y', y''] \subset S \quad \text{and} \quad [x'', y''] \subset [y', y''] \subset S.$$

From the second inclusion it follows that

$$[x, y] = -[x'', y''] \subset -S = S$$

while from the first one, we have for all $0 \le \lambda \le 1$ that

$$1 = \|\lambda x' + (1-\lambda)y'\| = \|\lambda\frac{x-a}{\delta} + (1-\lambda)\frac{y-a}{\delta}\| = \frac{1}{\delta}\|(\lambda x + [1-\lambda]y) - a\|,$$

i.e., $\lambda x + (1-\lambda)y \in S_a(\delta)$. Thus

$$[x, y] \subset S \cap S_a(\delta) \cap P_v^+ = K_v$$

what was to be proved.

(ii) $\alpha > 1$. Then for

$$\lambda = \frac{\alpha - 1}{\delta(\alpha + \beta)} > 0 \quad \text{and} \quad \mu = \frac{\delta\beta}{\delta(\alpha + \beta)} \geq 0,$$

we have $x' = \lambda x + \mu y'$, whence

$$1 = \|x'\| \leq \lambda\|x\| + \mu\|y'\| = \lambda + \mu = \frac{\alpha - 1 + \delta\beta}{\delta(\alpha + \beta)}$$

and so $\alpha \geq \frac{1}{1-\delta}$. On the other hand, for

$$\lambda = \frac{\beta + 1}{\alpha + \beta} > 0 \quad \text{and} \quad \mu = \frac{\delta\alpha}{\alpha + \beta} > 0,$$

we have $y = \lambda x + \mu y'$, whence

$$1 = \|y\| \leq \lambda\|x\| + \mu\|y'\| = \lambda + \mu = \frac{\beta + 1 + \delta\alpha}{\alpha + \beta}$$

and so $\alpha \leq \frac{1}{1-\delta}$. Thus we have proved that

$$\alpha = \frac{1}{1 - \delta}.$$

This fact implies that $x' = \lambda' x + (1 - \lambda')y'$ with

$$0 < \lambda' = \frac{1}{1 + \beta(1 - \delta)} \leq 1$$

and $y = (1 - \lambda'')x + \lambda'' y'$ with

$$0 < \lambda'' = \frac{\delta}{1 + \beta(1 - \delta)} < 1.$$

This latter means by Lemma 2.1 that $[x, y'] \subset S$. Also it follows that

$$[x', y'] \subset [x, y'] \subset S \quad \text{and} \quad [x, y] \subset [x, y'] \subset S.$$

From the first inclusion, we have for all $0 \leq \lambda \leq 1$ that

$$1 = \|\lambda x' + (1 - \lambda)y'\| = \|\lambda \frac{x - a}{\delta} + (1 - \lambda)\frac{y - a}{\delta}\| = \frac{1}{\delta}\|(\lambda x + [1 - \lambda]y) - a\|,$$

i.e., $\lambda x + (1 - \lambda)y \in S_a(\delta)$. Thus

$$[x, y] \subset S \cap S_a(\delta) \cap P_v^+ = K_v$$

what was to be proved.

Finally, note that a convex subset of a plane without interior point is necessarily an interval.

Theorem 2.3. *In a real normed vector space of dimension* ≥ 3, *the intersection of two spheres is connected (or empty).*

PROOF. For $\sigma \geq \tau > 0$ and $s, t \in X$, $\sigma - \tau \leq \|s-t\| \leq \sigma + \tau$, consider the intersection

$$K' = S_s(\sigma) \cap S_t(\tau).$$

Then the continuous mapping $\varphi : X \to X$,

$$\varphi(x) = \sigma x + s$$

carries $S \cap S_a(\delta)$ onto K', where $\delta = \tau/\sigma$ and $a = (t-s)/\sigma$. So it is sufficient to prove the connectivity of

$$K = S \cap S_a(\delta)$$

for $0 < \delta \leq 1$ and $1 - \delta \leq \|a\| \leq 1 + \delta$. If $a = 0$, then $\delta = 1$ and so there is nothing to prove. Thus we may and do assume that $a \neq 0$ whence $u = a/\|a\| \in S$.

Let now $b \in K$ be arbitrarily fixed and for any $x \in K$ choose vectors $v_{x1}, v_{x2} \in S$ such that $a, b, x \in \operatorname{lin}\{u, v_{x1}, v_{x2}\} = M_x$ and $\dim M_x = 3$. Then $W_x = S \cap M_x$ is a closed, bounded sphere in the finite dimensional subspace M_x and so it is compact. Furthermore, in the two dimensional subspace $L_x = \operatorname{lin}\{v_{x1}, v_{x2}\}$, the set $V_x = S \cap L_x$ is a connected circle. Now consider the relation $F_x \subset V_x \times W_x$ defined for any $v \in V_x$ by $F_x(v) = K_v = S \cap S_a(\delta) \cap P_v^+$, the non-empty, closed, connected interval introduced in the previous lemma. It is known from [12], Corollary 2.2, that a closed relation between a connected and a compact metric space is itself connected provided it is defined everywhere with connected values. Thus as soon as it will have been shown that F_x is closed, the connectivity of

$$F_x(V_x) = \bigcup_{v \in V_x} (K \cap P_v^+) = K \cap M_x \ni x, b$$

will be clear.

For that reason take a sequence $(v_n, w_n) \in F_x$ converging to some $(v_0, w_0) \in V_x \times W_x$. It is clear that $\varepsilon = \operatorname{dist}(u, L_x) > 0$. Since $w_n \in F_x(v_n) \subset P_{v_n}^+$, it can be written as

$$w_n = \sigma_n u + \tau_n v_n \quad \text{with} \quad \tau_n \geq 0.$$

On the other hand, w_0 can be combined linearly from u and some $v \in V_x$:

$$w_0 = \sigma_0 u + \tau_0 v \quad \text{with} \quad \tau_0 \geq 0.$$

Then we have

$$\frac{\|w_n - w_0\|}{|\sigma_n - \sigma_0|} = \left\| u - \frac{\tau_0 v - \tau_n v_n}{\sigma_n - \sigma_0} \right\| \geq \varepsilon,$$

whence

$$|\sigma_n - \sigma_0| \leq \frac{\|w_n - w_0\|}{\varepsilon} \longrightarrow 0.$$

Thus

$$\tau_n v_n = w_n - \sigma_n u \longrightarrow w_0 - \sigma_0 u = \tau_0 v,$$

and by the continuity of the norm,

$$\tau_n = \|\tau_n v_n\| \longrightarrow \|\tau_0 v\| = \tau_0.$$

Hence by $v_n \longrightarrow v_0$,

$$\tau_n v_n \to \tau_0 v_0$$

holds true, as well and by the uniqueness of the limit, $\tau_0 v_0 = \tau_0 v$, i.e.,

$$w_0 = \sigma_0 u + \tau_0 v_0 \in P^+_{v_0}.$$

On the other hand, $w_n \in K_{v_n} \subset K$ and because of K is closed, $w_0 \in K$, i.e., $w_0 \in K \cap P^+_{v_0} = K_{v_0} = F_x(v_0)$. Thus $(v_0, w_0) \in F_x$ and therefore F_x is closed.

Finally, $\bigcap_{x\in K} F_x(V_x)$ is non-empty (namely it contains b), whence

$$K = \bigcup_{x\in K} F_x(V_x)$$

is connected.

Theorem 2.4. *In a real normed vector space of dimension ≥ 3, the set*

$$C_\delta = \{(u,v) \in S \times S \mid \|u - v\| = \delta\} \subset S \times S$$

is connected for any $0 \leq \delta \leq 2$.

Proof. The set C_δ is a relation on the connected sphere S and by Theorem 2.3,

$$C_\delta(u) = \{v \in S \mid \|u - v\| = \delta\} = S \cap S_u(\delta)$$

is non-empty, connected for all $u \in S$. We shall use [12], Lemma 2.1 to prove the connectivity of relation C_δ.

If $\delta = 0$, then C_0 is the diagonal of $S \times S$, so it is connected.

If $\delta = 2$, then for any sequence $u_n \in S$ converging to $u_0 \in S$, we have $-u_n \in C_2(u_n)$ for all $n \in \mathbb{N}$. Since $-u_n \longrightarrow -u_0 \in C_2(u_0)$, by virtue of [12], Lemma 2.1, the proof is complete.

Now consider the case of $0 < \delta < 2$. For a sequence $u_n \in S$ converging to $u_0 \in S$, choose $v \in S \setminus \operatorname{lin}\{u_0\}$. Then

$$C_\delta(u_n) \cap \operatorname{lin}\{u_0, v\} \neq \emptyset$$

for sufficiently large n's. Since any sequence $v_n \in C_\delta(u_n) \cap \mathrm{lin}\{u_0, v\}$ is contained in the compact metric space $S \cap \mathrm{lin}\{u_0, v\}$, the Bolzano-Weierstrass Theorem ensures the existence of a convergent selection $v_{n_k} \longrightarrow v_0 \in S$. Since $\|u_{n_k} - v_{n_k}\| = \delta$ and $u_{n_k} \longrightarrow u_0$, we have $\|u_0 - v_0\| = \delta$, i.e., $v_0 \in C_\delta(u_0)$. Therefore [12], Lemma 2.1 completes the proof.

Theorem 2.5. *Suppose that $s, t \in X$ is a normed base for the two dimensional space X: $\|s\| = \|t\| = 1$, $\mathrm{lin}\{s, t\} = X$. For a couple of vectors $x = \sigma_x s + \tau_x t$ and $y = \sigma_y s + \tau_y t$ in X, let denote*

$$\det(x, y) = \sigma_x \tau_y - \tau_x \sigma_y.$$

Then for any $0 \le \delta \le 2$, the sets

$$C_\delta^\pm = \{(u, v) \in S \times S \mid \|u - v\| = \delta,\ \det(u, v) \in \mathbb{R}_\pm\}$$

are connected.

Proof. We perform the proof for C_δ^+ which is a relation on the compact, connected circle S. For any $u \in S$, the set

$${}_uP^+ = \{x \in X \mid \det(u, x) \ge 0\}$$

is a closed halfplane, so

$$C_\delta^+(u) = S \cap S_u(\delta) \cap {}_uP^+$$

is a non-empty, connected interval.

Now we are going to show that C_δ^+ is closed. Indeed, if $(u_n, v_n) \in C_\delta^+$ converging to some $(u_0, v_0) \in S \times S$, then by the continuity of the norm, $\delta = \|u_n - v_n\| \longrightarrow \|u_0 - v_0\|$, whence $\|u_0 - v_0\| = \delta$. Furthermore, by the continuity of the determinant,

$$\det(u_n, v_n) \longrightarrow \det(u_0, v_0)$$

showing $\det(u_0, v_0) \ge 0$.

These mean that

$$v_0 \in S \cap S_{u_0}(\delta) \cap {}_{u_0}P^+ = C_\delta^+(u_0),$$

i.e., $(u_0, v_0) \in C_\delta^+$. Thus [12], Corollary 2.2 completes the proof.

3. Proof of the main theorem

Proof. Assume that X is not an inner product space and $f : X \to Y$ is an even, orthogonally additive mapping. We are going to show that f is identically zero.

First we claim that f takes the same value on vectors of equal norm. Indeed, for $x, y \in X$, $\|x\| = \|y\|$, letting $p = (x+y)/2$, $q = (x-y)/2$, we have

$$\|p+q\| = \|x\| = \|y\| = \|p-q\|,$$

i.e., $p \perp \pm q$. Hence

$$f(x) = f(p+q) = f(p) + f(q) = f(p) + f(-q) = f(p-q) = f(y).$$

This means that f depends only on the norm of x, i.e., there exists a function on the non-negative reals into Y such that

$$f(x) = \varphi(\|x\|), \quad x \in X.$$

Also, observe that

$$f(2x) = f(2p+2q) = f(2p) + f(2q) = f(x+y) + f(x-y),$$

or in another equivalent form

$$f(x+y) = f(2x) - f(x-y), \quad x, y \in X, \ \|x\| = \|y\|.$$

By a theorem of Senechalle [6], if X fails to be an inner product space, then there exist couples of vectors (u_1, v_1), $(u_2, v_2) \in S \times S$ such that

$$\|u_1 - v_1\| = \|u_2 - v_2\| = \delta \quad \text{but} \quad \|u_1 + v_1\| = \rho_1 < \rho_2 = \|u_2 - v_2\|.$$

Now consider the scalar set

$$\Gamma_\delta = \{\|u+v\| \mid u, v \in S, \ \|u-v\| = \delta\}$$

which is image of the set

$$C_\delta = \{(u,v) \in S \times S \mid \|u-v\| = \delta\}$$

through the continuous sum-norm function $\gamma : S \times S \to \mathbb{R}$, $\gamma(u,v) = \|u+v\|$.

When $\dim X \geq 3$, by Theorem 2.4, C_δ is connected and so is Γ_δ. For $\dim X = 2$, by Theorem 2.5, $C_\delta = C_\delta^+ \cup C_\delta^-$ holds with connected C_δ^+, C_δ^- and

$$(u,v) \in C_\delta^+ \iff (v,u) \in C_\delta^-,$$

whence

$$\gamma(C_\delta^+) = \gamma(C_\delta^-) = \gamma(C_\delta) = \Gamma_\delta$$

is connected. This means in both cases that Γ_δ is an interval containing $[\rho_1, \rho_2]$.

Now let $\mu > 0$ be fixed and take any $\rho_1 \leq \rho \leq \rho_2$. Then $\rho \in \Gamma_\delta$ and so $\rho = \|u + v\|$ for some $u, v \in S$ with $\|u - v\| = \delta$. Using the above observation for $x = \mu u,\ y = \mu v$, $(\mu > 0)$ we have

$$\varphi(\mu\rho) = f(\mu u + \mu v) = f(2\mu u) - f(\mu u - \mu v) = \varphi(2\mu) - \varphi(\mu\delta),$$

i.e., $\varphi(\mu\rho)$ does not depend on ρ. Thus φ is constant on all the intervals $[\mu\rho_1, \mu\rho_2],\ \mu > 0$, whence so is on the whole positive real line:

$$\varphi(\rho) = c, \quad \rho > 0.$$

Finally, choosing any $u, v \in S, \quad u \perp v$, we have $u + v \neq 0$ and so

$$c = f(u + v) = f(u) + f(v) = c + c.$$

Thus $c = 0$ and so for all $x \in X \setminus \{0\}$

$$f(x) = \varphi(\|x\|) = 0,$$

which, together with the obvious equality $f(0) = 0$, means that f is identically zero.

This completes the proof.

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