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On the structure of univoque numbers

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1. Introduction

We shall continue our investigation in [1] on univoque sequences generated by Θ-adic expansion of real numbers. A method for the computation of the Hausdorff dimension of the set of univoque numbers will be presented.

Let $\frac{1}{2} \le \Theta < 1, \quad L = L(\Theta) = \Theta + \Theta^2 + \dots = \frac{\Theta}{1-\Theta}, \quad \lambda = \Theta L.$

For $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) \in \{0,1\}^{\mathbb{N}}$ let

$$\langle \varepsilon, \Theta \rangle := \sum_{n=1}^{\infty} \varepsilon_n \Theta^n.$$

A sequence ε is said to be univoque with respect to Θ if $\langle \varepsilon, \Theta \rangle = \langle \delta, \Theta \rangle$, $\delta \in \{0,1\}^{\mathbb{N}}$ implies that $\varepsilon = \delta$, i.e that $\varepsilon_j = \delta_j$ $(j = 1, 2, \dots)$.

It is known that for any $x \in [0, L(\Theta)]$ there exists an $\varepsilon \in \{0,1\}^{\mathbb{N}}$ such that $x = \langle \varepsilon, \Theta \rangle$, namely this is true for $\varepsilon_n = \varepsilon_n(x)$, where $\varepsilon_n(x)$ is defined by induction on n, as follows:

$$\varepsilon_n(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x)\Theta^i + \Theta^n \le x, \\ 0 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x)\Theta^i + \Theta^n > x. \end{cases} \tag{1.1}$$

The expansion $\langle \varepsilon(x), \Theta \rangle = x$, $\varepsilon(x) = (\varepsilon_1(x), \dots)$ is called the **regular** expansion of x.

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Every $x \in (0, L(\Theta)]$ can be expanded by the digits, $\delta_n = \delta_n(x)$ $(n = 1, 2, \dots)$ as well, where they are defined from

$$\delta_n(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \delta_i(x)\Theta^i + \Theta^n < x, \\ 0 & \text{if } \sum_{i=1}^{n-1} \delta_i(x)\Theta^i + \Theta^n \geq x. \end{cases} \tag{1.2}$$

The expansion $x = \langle \delta(x), \Theta \rangle$, $\delta(x) = (\delta_1(x), \dots)$ is called the **quasi-regular** expansion of x.

The expansions $\varepsilon(x), \delta(x)$ are the same except, if the regular expansion of x is finite (i.e. if $\varepsilon_n(x) = 0$ for all large n).

Let $R(\Theta) = \{\varepsilon(x) \mid x \in [0, L]\}$, $R_1(\Theta) = \{\varepsilon(x) \mid x \in [0, 1)\}$.
Let $l = l(\Theta) = (l_1, l_2, \dots) \in \{0, 1\}^{\mathbb{N}}$ be the quasi-regular expansion of 1, i.e. $\delta_j(1) = l_j$ $(j = 1, 2, \dots)$. If 1 has a finite regular expansion in the base Θ, and $\langle \varepsilon(1), \Theta \rangle = s_1\Theta + \dots + s_k\Theta^k$, $s_k = 1$, then $\delta(1) = (s_1, s_2, \dots, s_k - 1, 0,\ s_1, \dots, s_k - 1, 0 \dots)$, i.e. $\delta(1)$ is a periodic sequence with period k.

W. Parry [2] gave a simple characterization of the sequences $a = \{a_1, a_2, \dots\} \in \{0, 1\}^{\mathbb{N}}$ of $R_1(\Theta) : a \in R_1(\Theta)$, if and only if

$$\{a_r, a_{r+1}, \dots\} < \{l_1, l_2, \dots\} \quad (r = 1, 2, \dots) \tag{1.3}$$

holds, in the sense of the lexicographic ordering.

He proved furthermore that $l \in \{0, 1\}^{\mathbb{N}}$ is the regular expansion of 1 for a suitable $\Theta \in [\frac{1}{2}, 1)$, if and only if

$$\begin{gathered} l_1 = 1 \text{ and} \\ \{l_{k+1}, l_{k+2}, \dots\} < \{l_1, l_2, \dots\} \\ k = 1, 2, \dots \end{gathered} \tag{1.4}$$

holds.

One can prove simply that the periodic sequence $l \in \{0, 1\}^{\mathbb{N}}$ with $l_1 = 1$ is the quasi-regular expansion of 1 with a suitable $\Theta \in [\frac{1}{2}, 1)$ if and only if

$$\{l_k, l_{k+1}, \dots\} \leq \{l_1, l_2, \dots\} \quad (k = 1, 2, \dots) \tag{1.5}$$

holds. If (1.5) holds, then with the correspondting Θ as base, the regular expansion of 1 is finite. In [1] we proved the following assertions (Theorem 2.1 and 2.4 which are formulated now as Lemma 1 and 2).

Lemma 1. *The sequence $\varepsilon \in \{0, 1\}^{\mathbb{N}}$ is univoque with respect to Θ if and only if both of the sequences $\varepsilon, \underline{1} - \varepsilon \in R(\Theta)$, where $\underline{1} = \{1, 1, \dots\}$.*

Let $U(\Theta)$ be the set of univoque sequences.

Lemma 2. *If* $\frac{1}{2} \le \Theta' < \Theta < 1$, *then* $U(\Theta) \subseteq U(\Theta')$.

Definition. The number $\Theta \in (\frac{1}{2}, 1)$ is said to be stable from below, if $U(\Theta) = U(\Theta')$ holds for some $\Theta' < \Theta$. Similarly, Θ is stable from above, if $U(\Theta'') = U(\Theta)$ holds for some $\Theta'' > \Theta$.

Remark. This definition is somewhat different from that was given in [1].

Let $H(= H_\Theta)$, $H^*(= H^*_\Theta)$ be the set of univoque numbers (with respect to Θ) on the intervals $[\Theta, 1)$, $[0, 1)$, respectively. It is clear that

$$H^* = \{0\} \cup \bigcup_{n=0}^{\infty} \Theta^n H . \tag{1.6}$$

The set of univoque numbers $x \in [1, L]$ can be given as $(L - H^*) \cap [1, L]$. Let

$$U_1(\Theta) := \{\varepsilon \in U(\Theta), \langle \varepsilon, \Theta \rangle \in H\}, \tag{1.6}$$

i.e. $U_1(\Theta)$ is the set of those univoque sequences for which the represented number $\langle \varepsilon, \Theta \rangle$ falls into $[\Theta, 1)$.

2. A new notation for univoque sequences

First of all, let $\mathcal{K}_h$ denote the set of words of length h over $\mathbb{N}$, and $\mathcal{M}$ be the set of infinite words over $\mathbb{N}$, i.e. let

$$\mathcal{K}_h := \{m_1 m_2 \dots m_h; m_j \in \mathbb{N}\}$$
$$\mathcal{M} := \{\underline{m} = m_1 m_2 \dots; m_j \in \mathbb{N}\} .$$

Let $F_h : \mathcal{M} \to \mathcal{K}_h$ be the mapping $F_h(\underline{m}) = m_1 \dots m_h$; let σ be the shift operator acting as $\sigma(m_1 m_2 \dots) = m_2 m_3 \dots$.

Let us define the ordering relations in $\mathcal{K}_h$ and in $\mathcal{M}$ by the following relations:

(1) in $\mathcal{K}_1 (= \mathbb{N})$: the common ordering

(2) in $\mathcal{K}_2$: $n_1 n_2 < m_1 m_2$ holds if $n_1 < m_1$, or if

$$n_1 = m_1 \text{ and } n_2 > m_2 .$$

(h) in $\mathcal{K}_h$: $n_1 n_2 \dots n_h < m_1 \dots m_h$, *if*

$$n_1 < m_1, \text{ or if } n_1 = m_1 \text{ and } n_2 \dots n_h > m_2 \dots m_h .$$

In other words, if $n_1 \dots n_h \ne m_1 \dots m_h$ and k is the smallest index for which $n_k \ne m_k$, then

for odd k : $n_1 \dots n_h < m_1 \dots m_h$, if $n_k < m_k$
for even k : $n_1 \dots n_h < m_1 \dots m_h$, if $n_k > m_k$.
Let $\underline{m}, \underline{n}$ be two distinct words in $\mathcal{M}$. We say shat $\underline{m} < \underline{n}$, if $F_h(\underline{m}) \neq F_h(\underline{n})$ implies that $F_h(m) < F_h(n)$ in $\mathcal{K}_h$. It is clear that this definition is correct.
Let $E \subseteq \{0,1\}^{\mathbb{N}}$ be the set of those sequences $\varepsilon = \{\varepsilon_1, \varepsilon_2, \dots\}$ in which both of 0 and 1 occurs infinitely often, and $\varepsilon_1 = 1$. Let $\varphi : E \to \mathcal{M}$ be the one to one mapping defined as follows: Let ε (considered as an infinite word over $\{0,1\}$) of form $1^{a_1}0^{b_1}1^{a_2}0^{b_2}\dots$. Then $\varphi(\varepsilon) = a_1 b_1 a_2 b_2 \dots$.

It is clear that, if $\varepsilon, \delta \in E$, then $\varepsilon < \delta$ holds in E (in the sense of the lexicograpic ordering) if and only if $\varphi(\varepsilon) < \varphi(\delta)$ in $\mathcal{M}$.

We have $U_1(\Theta) \subseteq E$. Let $\mathcal{M}^{(0)} = \mathcal{M}_\Theta^{(0)} = \varphi(U_1(\Theta))$.
Let furthermore

$$\underline{t} = t_1 t_2 \dots = \varphi(l(\Theta)),$$

where $l(\Theta)$ is the sequence getting as the quasiregular expansion of 1 in the base Θ.

From the Parry condition and Lemma 1 we have

Lemma 3. *$\alpha \in \mathcal{M}$ belongs to $\mathcal{M}_\Theta^{(0)}$ if and only if*

$$\sigma^l(\alpha) < t \quad (l = 0, 1, 2, \dots). \tag{2.1}$$

Proof. Clear.

Let $Y = y_1 y_2 \dots \in \mathcal{M}$, $Y_h := F_h(Y) = y_1 \dots y_h$;

$$S(Y) := \{\alpha \in \mathcal{M} : \ \sigma^l(\alpha) < Y, \ l = 0, 1, 2, \dots\}, \tag{2.2}$$

$$U_k(Y) := \{\alpha \in \mathcal{M} : \ F_k(\sigma^l(\alpha)) < Y_k, \ l = 0, 1, 2, \dots\}, \tag{2.3}$$

$$V_k(Y) := \{\alpha \in \mathcal{M} : \ F_k(\sigma^l(\alpha)) \leq Y_k, \ l = 0, 1, 2, \dots\}. \tag{2.4}$$

It is clear that $U_1(Y) \subseteq U_2(Y) \subseteq \dots$ and $V_1(Y) \supseteq V_2(Y) \supseteq \dots$.

Lemma 4. *For each $k, l \in \mathbb{N}$ we have*

$$U_k(Y) \subseteq S(Y) \subseteq V_l(Y). \tag{2.5}$$

Proof. Clear.

Lemma 5. *Let p be the smallest integer, if any, for which there exist $u, r \geq 1$, $u + r = p$ such that*

$$y_{u+1} \dots y_{u+r} > Y_r \tag{2.6}$$

in the sense of ordering introduced in $\mathcal{K}_r$. Then

$$S(Y) = U_p(Y)\,. \tag{2.7}$$

Proof. If there is an $\alpha \in V_{u+r}(Y) \setminus U_{u+r}(Y)$, then $F_p(\sigma^j(\alpha)) = Y_p$ holds for some j. Then

$$F_r(\sigma^{j+r}(\alpha)) > Y_r,$$

i.e. $\alpha \notin V_r(Y)$. Hence $V_{u+r}(Y) = U_{u+r}(Y)$, and (2.7) follows from (2.5).

Lemma 6. *If $y_2 > y_1$, then*

$$S(Y) = \{\alpha = a_1 a_2 \dots \ \mid \ 1 \leq a_j \leq y_1 - 1\}\,. \tag{2.8}$$

Let $y_2 = y_1$ and denote $z = y_1 y_1 y_1 \dots$. If $z \geq Y$, then $S(Y)$ as in (2.8). If $z < Y$, then

$$\begin{aligned} S(Y) = \{\alpha = a_1 a_2 \dots \ &\mid \ 1 \leq a_j \leq y_1 - 1, j = 1, 2, \dots\} \cup \\ \cup \{\alpha = \beta z \ \mid \ \beta = b_1 \dots b_h,\ &1 \leq b_j \leq y_1 - 1, h = 0, 1, 2, \dots\} \end{aligned} \tag{2.9}$$

($h = 0$ is for the empty word!)

Proof. The first assertion comes from Lemma 5 immediately. Assume that $y_2 = y_1$. If $\alpha \in S(Y)$, and the first occurrence of y_1 in the sequence is $a_{h+1} = y_1$, then $\alpha = a_1 a_2 \dots a_h z$, thus $\sigma^h(\alpha) = z$, $\sigma^h(\alpha) < Y$, this may occur only if $z < Y$. The further part of the lemma is clear.

Lemma 7. *Let $\underline{t} = t_1 t_2 \cdots = \varphi(l(\Theta))$, and assume that $t_2 \geq t_1$. Then*

$$\mathcal{M}_\Theta^{(0)} = \{\alpha = a_1 a_2 \dots \ \mid \ 1 \leq a_j \leq t_1 - 1, j = 1, 2, \dots\}.$$

Proof. The assertion immediately follows from Lemma 4 and 6. The only critical element is $z = t_1 t_1 \dots$ in the case $t_2 = t_1$. Since $\underline{t}$ comes from a quasi regular expansion of 1, therefore $t_{2j+1} \leq t_1$ and in the case $t_{2j+1} = 1$ $t_{2j+2} > t_2 = t_1$, since $\sigma^{2j}(\underline{t}) \leq \underline{t}$. If $t_k = t_1$ for each k, then $z = \underline{t}$, and $z < \underline{t}$ does not hold. Let k be the smallest index for which $t_k \neq t_1$. If k odd, then $t_k < t_1$, but then $z > \underline{t}$. If k even, then $t_k > t_1$, and similarly we have $z > \underline{t}$. Thus $z \notin \mathcal{M}_\Theta^{(0)}$.

3. The structure of $M_\Theta^{(0)}$ in the case $t_2 \geq t_1$

Theorem 1. *Assume that the condition stated in Lemma 7 holds. Then H is self-similar, it is the attractor of the iterated function system*

$$H = \bigcup_{a_1=1}^{t_1-1} \bigcup_{a_2=1}^{t_1-1} f_{a_1,a_2}(H)\,, \tag{3.1}$$

where $f_{a_1,a_2}(x) = \Theta^{a_1} + \Theta^{a_1+a_2}x$. The components on the right hand side of (3.1) are disjoint sets.
Let ξ denote the positive root of the polynomial $x^{t_1-1} + \cdots + x - 1$; let

$$s := \frac{\log 1/\xi}{\log 1/\Theta} (< 1)\,. \tag{3.2}$$

Then the Hausdorff dimension of H equals to its similarity dimension, $= s$.

Proof. (3.1) is a consequence of Lemma 7. From the definition follows that the components are disjoint. H is closed and bounded. The further assertion follows from a theorem of Hutchinson (see G. Edgar [3]).

4. On the set $\mathcal{F}$

Let $\mathcal{F}$ denote the set of those Θ for which 1 is univoque with respect to Θ. If the regular expansion of 1 is finite, then clearly $\Theta \notin \mathcal{F}$, since then 1 has another expansion. If $\Theta \in \mathcal{F}, \underline{t} = \varphi(l(\Theta))$, then $\left\langle \varphi^{-1}(\sigma^j(\underline{t})), \Theta \right\rangle \in H_\Theta$ for each $j \geq 1$, therefore

$$\sigma^j(\underline{t}) < \underline{t} \quad (j = 1, 2, \dots) \tag{4.1$_j$}$$

holds.
Let now $\underline{t} \in \mathcal{M}$ be an arbitrary sequence for which $(4.1)j$ $(j = 1, 2, \dots)$ holds. The fulfilment of the conditions $(4.1)_{2l}$ $(l = 1, 2, \dots)$ guarantee the existence of a Θ for which $\varphi^{-1}(\underline{t}) = l(\Theta)$, $\Theta \in (\frac{1}{2}, 1)$. Then $(4.1)j$ implies that $\left\langle \varphi^{-1}(\sigma^j(t)), \Theta \right\rangle \in H$ for $j \geq 1$ (see Lemma 3), thus 1 is univoque with respect to Θ. We have proved

Theorem 2. *$\underline{t} \in \mathcal{M}$ is the image of the regular (quasi-regular) expansion of* 1 *in the base of a suitable* $\Theta \in \mathcal{F}$ *if and only if* $(4.1)j$ $(j = 1, 2, \dots)$ *holds.*

Theorem 3. *The Lebesgue measure of $\mathcal{F}$ is zero, its Hausdorff dimension is* 1.

Proof. **I.** In [1] we proved that $\Theta \in \mathcal{F}$ implies that $\Theta \le \frac{\sqrt{5}-1}{2}$.
Let Θ and Θ' be such numbers for which $l(\Theta) = \{l_1, \dots, l_k, l_{k+1}, \dots\}$, $l(\Theta') = \{l_1, \dots, l_k, l'_{k+1}, l'_{k+2}, \dots,\}$ $l_{k+1} = 0, l'_{k+1} = 1$.
Let $P_1(z) = \sum_{j=1}^{\infty} l_j z^j - 1$, $P_2(z) = \sum_{j=1}^{k} l_j z^j + \sum_{j=k+1}^{\infty} l'_j z^j - 1$. Since $P_1(\Theta) = 0, P_2(\Theta') = 0, P_2(\Theta) = |P_2(\Theta) - P_1(\Theta)| \le c\Theta^k, P_2(\Theta) - P_2(\Theta') = = (\Theta - \Theta')P_2''(\xi), \xi \in (\Theta, \Theta')$, and $(0 <)c_1 < P_2''(\xi) < c_2$ with numerical constants c_1, c_2, therefore

$$0 < \Theta - \Theta' < c_3 \Theta^k \,. \tag{4.2}$$

Let $\mathcal{F}_K = \{\Theta \mid \Theta \in \mathcal{F},\ t_1 = K\}$. If $\Theta \in \mathcal{F}_K$, then
$1 = \Theta + \dots + \Theta^{t_1} + \Theta^{t_1+t_2+1} + \cdots,\ t_1 = K,\ t_2 \le K$, consequently $\Theta_K = \max_{\Theta \in \mathcal{F}_K} \Theta$ satisfies

$$\sum_{j=1}^{K} \Theta_K^j \le 1 - \Theta_K^{2K+1} \,. \tag{4.3}$$

Let R be an arbitrary large integer. Let us classify the elements of $\mathcal{F}_K$ according to the sequence $t_1, t_2, \dots, t_R$. The distance of two numbers, $\Theta_1, \Theta_2 \in \mathcal{F}_K$ with $F_R(\varphi(\Theta_1)) = F_R(\varphi(\Theta_2)) = t_1 t_2 \dots t_R$ is less than $c_3 \Theta_K^{t_1 + \dots + t_R}$, due to (4.2). Thus $\mathcal{F}_K$ can be covered by finitely many intervals the total length of which is less than

$$c_3 \Big(\sum_{j=1}^{K} \Theta_K^j \Big)^R \le c_3 \big(1 - \Theta_K^{2K+1}\big)^R .$$

The right hand side tends to zero as $R \to \infty$. Thus meas $(\mathcal{F}_K) = 0$, whence meas $(\mathcal{F}) = \sum$ meas $(\mathcal{F}_K) = 0$. The first part of the theorem is proved.

II. Let $\mathcal{F}_K^{(0)}$ be the subset of $\mathcal{F}_K$ defined by the conditions $\mathcal{F}_K^{(0)} = \{\Theta \mid \varphi(l(\Theta)) = t_1 t_2 \dots;\ t_1 = K; 1 \le t_j \le K-1, j \ge 2\}$. We shall show that for any given $\sigma < 1$ there is a K such that the Hausdorff dimension of $\mathcal{F}_K^{(0)}$ is larger than σ.
Let $\Theta_{\min},\ \Theta_{\max}$ denote the smallest and the largest elements of $\mathcal{F}_K^{(0)}$, respectively. Assume that $K \ge 3$. Then $\varphi(l(\Theta_{\min}))=K\,1\,(K-1)1(K-1)\dots$.
Let Ψ_K be the positive root of the polynomial $1 - (z + \dots + z^{K-1})$. Then

$\Theta_{\min} < \Theta_{\max} < \Psi_K$, furthermore $\Psi_K - \Theta_{\min} < c\,\Psi_K^K$ holds with a suitable numerical constant c. The last inequality follows from (4.2).

Let $\Theta' < \Theta$, $\Theta', \Theta \in \mathcal{F}_K^{(0)}$ with $l(\Theta) = l_1 l_2 \dots, l(\Theta') = l_1' l_2' \dots$ such that $l_s = 0$, $l_{s+1} = 1$ and $l_j = l_j'$ for $1 \le j \le s$. Assume that $s > K$. Then $l_1\Theta + \dots + l^{s-1}\Theta^{s-1} + \Theta^s > 1$, $l_1\Theta' + \dots + l_{s-1}\,\Theta'^{s-1} + \Theta'^s < 1 - \Theta'^{s+K}$. The polynomial $h(z) := l_1 z + \dots + l_{s-1}z^{s-1} + z^s$ satisfies $(1 \le) h'(z) \le 9$ for $z \le 0,9$, say, whence

$$\Theta'^{s+K} < h(\Theta) - h(\Theta') = (\Theta - \Theta')h'(\xi)\ ,\ \xi \in (\Theta', \Theta)$$

thus

$$\Theta - \Theta' \ge \frac{1}{9}\Theta'^{s+K} \ge \frac{1}{9} \cdot \Theta_{\min}^{s+K}\,. \tag{4.4}$$

Let $f(V)$ be the number of the sequences $l_1, \dots, l_V$ which occur as the first V elements of $l(\Theta) = \{l_1, l_2, \dots\}$ for some $\Theta \in \mathcal{F}_K^{(0)}$. For $V > U$ and given $l_1^*, \dots l_U^*$ let $g(V \mid l_1^*, \dots, l_U^*)$ be the number of the distinct $l_1, \dots, l_V$ occuring in the beginning of $l(\Theta) = \{l_1, l_2, \dots\}$ for which the first U elements are fixed, $l_1 = l_1^*, \dots, l_U = l_U^*$.

Lemma 8. *With suitable positive constants $c(K), c_1(K), c_2(K)$ we have*

$$f(V) = c(K)\Psi_K^{-V}(1 + o(1)) \quad \text{as} \quad V \to \infty\,, \tag{4.5}$$

$$c_1(K) < g(V \mid l_1^*, \dots, l_U^*)\Psi_K^{V-U} < c_2(K) \quad \text{if} \quad U < V\,. \tag{4.6}$$

First we continue the proof of the theorem assuming the validity of Lemma 8, then we prove it.

Assume in contrary that dimension $(\mathcal{F}_K^{(0)}) < \sigma$. Then, for arbitrary choice of $\varepsilon, \delta > 0$, there is a covering $\mathcal{F}_K^{(0)} \subseteq \bigcup_{j=1}^{\infty} E_j$, such that $\operatorname{diam} E_j < \delta$ and

$$\sum_{j=1}^{\infty} (\operatorname{diam} E_j)^\sigma < \varepsilon\,.$$

Then there is such a covering with open intervals I_j, and even we may assume that the set of lengths of I_j belongs to the set $\{\Theta_{\min}^r \mid r = 1, 2, \dots\}$.

$\mathcal{F}_K^{(0)}$ is a closed set. Let $\mathcal{F}_K^{(0)} \subseteq \bigcup I_j$,

$$\sum_{j=1}^{\infty} (\operatorname{diam} I_j)^\sigma < \varepsilon, \quad \operatorname{diam} I_j < \delta\,. \tag{4.7}$$

From the Heine-Borel theorem we obtain that there is a finite subcover, $\mathcal{F}_K^{(0)} \subseteq \bigcup_{j=1}^{p} I_j$. Let M_r be the number of intervals I_j with length $\Theta_{\min}^r$. Let $r_o = \left[\frac{\log 1/\delta}{\log 1/\Theta_{\min}}\right]$, and r_1 be the largest j for which $M_j \neq 0$. Then $M_j = 0$ for $j < r_o$. We have

$$\sum_{j=1}^{p} (\operatorname{diam} I_j)^\sigma \leq \sum_{r_o \leq j \leq r_1} M_j \Theta_{\min}^{j\sigma} < \varepsilon \,. \tag{4.8}$$

Let $V > r_1$ and $\mathcal{F}_K^{(0)}(l_1, \ldots, l_V)$ be the set of those $\Theta \in \mathcal{F}_K^{(0)}$, for which the first V elements of $l(\Theta)$ is the given sequence $l_1, \ldots, l_V$.

If $\Theta_1, \Theta_2 \in \mathcal{F}_K^{(0)}$ are covered with the same interval I_j of length $\Theta_{\min}^r$, then $\Theta_{\min}^r > |\Theta_1 - \Theta_2|$ and by (4.4) we obtain that the first $r - K - 4$ digits of $l(\Theta_1)$ and of $l(\Theta_2)$ coincide.

Due to (4.6), the number of that sets among $\mathcal{F}_K^{(0)}(l_1, \ldots, l_V)$ which have nonempty intersection with I_j, is less than

$$c_2(K) \Psi_K^{r-K-4} \cdot \Psi_K^{-V} \,.$$

Since any of $\mathcal{F}_K^{(0)}(l_1, \ldots, l_V)$ has a nonempty intersection with at least one I_j, therefore

$$f(V) \leq c_2(K) \Psi_K^{-V} \sum_{r_o \leq r \leq r_1} M_r \Psi_K^{r-K-4} \,.$$

Then, from (4.5), taking the limit $V \to \infty$, we have

$$(A :=) \frac{1}{\Psi_K^{K+4}} \, \frac{c(K)}{c_2(K)} \leq \sum_{r_o \leq r \leq r_1} M_r \cdot \Psi_K^r \,.$$

From (4.8) it follows that $M_r < \varepsilon \Theta_{\min}^{-r\sigma}$, thus

$$A \leq \varepsilon \sum_{r_o \leq r < r_1} \left(\frac{\Psi_K}{\Theta_{\min}^\sigma} \right)^r \,. \tag{4.9}$$

If K is large enough, then $\Psi_K < \Theta_{\min}^\sigma$. For such choice of K the inequality (4.9) cannot be held if ε is small enough. This finishes the proof of the theorem.

Proof of Lemma 8. Let $f_1(n)$ be the number of that sequences in $\{0,1\}^n$, which do not contain K consecutive $1's$ and $0's$.

Then $f_1(n) = f_1(n-1) + \cdots + f_1(n-(K-1))$ for $n \geq K$. The characteristic polynomial $x^{K-1} - (x^{K-2} + \cdots + x + 1)$ of this recursion has only one root, namely Ψ_K^{-1} in the domain $|z| \geq 1$, therefore $f_1(n) = C\Psi_K^{-n}(1+\sigma(1))$ $(n \to \infty)$, $C > 0$ holds since $f(V) = f_1(V-k)$, (4.5) holds.

Since $g(V \mid l_1^*, \dots, l_U^*) \geq f_1(V-U)$, $g(V \mid l_1^*, \dots, l_U^*) \geq f_1(V-U-K)$ clearly hold, (4.6) is true.

5. On stable numbers

Theorem 4. $\Theta \in (\frac{1}{2}, 1)$ *is stable from both sides if and only if*

$$\sigma^j(\underline{t}) > \underline{t} \quad , \; \underline{t} = \varphi(l(\Theta)) \tag{5.1}$$

holds for at least one j. If (5.1) *fails, then Θ is instable from below.*

Proof. Assume that (5.1) holds with $j = u$. Then for a suitable $r \geq 1$ we have

$$t_{u+1} \dots t_{u+r} > T_r \, , \tag{5.2}$$

where in general $T_s := F_s(\underline{t}) = t_1 \dots t_s$. Then u is an even number due to (1.4), (1.5).

Then, from Lemma 5., applying it with $Y = \underline{t}$,

$$\varphi(U_1(\Theta)) = \mathcal{M}_\Theta^{(0)} = \{\alpha : \; F_p(\sigma^j(\alpha)) < T_p\} \, , \tag{5.3}$$

where $p = u+r$. (5.3) remains true for all those $\tilde{\Theta}$ for which in the notation $\tilde{t} = \varphi(l(\tilde{\Theta}))$ the relation $F_p(\tilde{t}) = T_p$ holds: Hence

$$\mathcal{M}_{\tilde{\Theta}}^{(0)} = \mathcal{M}_\Theta^{(0)}, \quad U_1(\Theta) = U_1(\tilde{\Theta}) \tag{5.4}$$

is valid in an open interval J around Θ. We may assume that $\Theta \leq \dfrac{\sqrt{5}-1}{2}$, since for bigger Θ, $U(\Theta) = \{\underline{0}, \underline{1}\}$. Then $L(\Theta) < 2$. The whole set of the univoque sequences, written as infinite words over $\{0,1\}$ can be given by the relation

$$U(\Theta) = \quad \{\underline{0}\} \cup \{\underline{1}\} \cup \bigcup_{\substack{k=0 \\ l=0}}^{\infty} (1^k 0^l U_1(\Theta)) \, . \tag{5.5}$$

(5.5) it follows immediately from Lemma 1 and from (1.6). (5.4) and (5.5) implies $U(\Theta) = U(\Theta')$. The first part of the theorem is proved.
Assume that

$$\sigma^j(\underline{t}) \leq \underline{t} \quad (j = 1, 2, \dots) \tag{5.6}$$

holds.

If $\underline{t}$ is periodic, then the regular expansion of 1 in the base Θ is finite, therefore 1 is not univoque with respect to Θ, $\underline{t} \notin \mathcal{M}(U(\Theta))$. For an arbitrary $\Theta' < \Theta$, the sequence $\underline{t}' = \varphi(l(\Theta'))$ is larger than $\underline{t}$, thus by (5.6),

$$\sigma^j(\underline{t}) < \underline{t}' \quad (j = 0, 1, 2, \dots)\,.$$

Consequently $\underline{t} \in \mathcal{M}(U(\Theta'))$.

Assume that $\underline{t}$ is not periodic. Then $\sigma_j(\underline{t}) < \underline{t}$ $(j = 1, 2, \dots)$. We may assume that $t_1 \geq 2$. Let $\alpha = 11\underline{t}$. It is not the image of the quasi regular expansion of any number with respect to Θ, therefore $\alpha \notin \mathcal{M}(U(\Theta))$.

The theorem is completely proved.

Let $\mathcal{F}_0$ be the set of those Θ for which $\underline{t} = \varphi(l(\Theta))$ is periodic and (5.6) holds. Thus Θ is unstable from below if $\Theta \in \mathcal{F} \cup \mathcal{F}_0$.

Let $\Theta \in \mathcal{F} \cup \mathcal{F}_0$ and $w = \sup\limits_{\eta \in H^0_\Theta} \eta.$. If $\eta \in H^{(0)}_\Theta$, then $\varepsilon(\eta) \leq \delta(w)$, where $\varepsilon(\eta)$ is the regular expansion of η, and $\delta(w)$ is the quasi regular expansion of w (≤ 1) in the base Θ. If $\eta = \sum\limits_{k=1}^{\infty} \varepsilon_k(\eta)\Theta^k$ is univoque, then so is $\eta_l = \sum\limits_{k=1}^{\infty} \varepsilon_{k+l}(\eta)\Theta^k$, and thus $\eta_l = w$. Furthermore, in the case $\varepsilon_l(\eta) = 1$ we have $L(\Theta) - \eta_l \leq w$. Since w can be approximated by η, hence we have

$$\sigma^j(\varphi(\varepsilon(\eta))) < \kappa = \varphi(\delta(w))\,,$$

and even

$$\sigma^j(\kappa) \leq \kappa \quad (j = 0, 1, 2, \dots)$$

holds.

Let us assume first that $w = 1$. Let $\eta_\nu \in H_\Theta$ $\eta_\nu \uparrow 1$, $\kappa^{(\nu)} = \varphi(\varepsilon(\eta_\nu))$. The sequence $11\kappa^{(\nu)} \in \mathcal{M}^{(0)}_\Theta$ due to the fact that $\sigma^j(11\kappa^{(\nu)}) < \kappa = \bar{t}$. Furthermore, for an arbitrary $\Theta' > \Theta$, if $t' = \varphi(l(\Theta'))$, then $t' < t$, and $\sigma^2(11\kappa^{(\nu)}) < t'$ does not hold for at least one ν. Thus $11\kappa^{(\nu)}$ is not the image of a univoque number with respect to Θ'. Consequently Θ is unstable from above.

Let now $w < 1$. Let $\Theta' (> \Theta)$ be close to Θ so that for $t' = \varphi(l(\Theta'))$, $\kappa < \underline{t}' < \underline{t}$. If $Y = y_1 y_2 \dots \in \mathcal{M}^{(0)}_\Theta$, then $\sigma^j(Y) \leq \kappa < t'$, consequently $Y \in \mathcal{M}^{(0)}_{\Theta'}$. Thus one get immediately that $U(\Theta') = U(\Theta)$, i.e. Θ is stable from above.

6. On the structure of $\mathcal{M}_\Theta^{(0)}$ for stable numbers

6.1 Due to Lemma 5 and Theorem 4, if Θ is stable (from both sides), then

$$\mathcal{M}_\Theta^{(0)} = \{\alpha : \ F_k(\sigma^j(\alpha)) < T_k \ , \ j = 0, 1, 2, \dots\} , \tag{6.1}$$

where $T_k = t_1 \dots t_k$, k is the least index for which $t_{u+1} \dots t_k > T_{k-u}$ holds for some $u \le k$. Starting from (6.1), we can compute such an $Y = y_1 \dots y_s$ for which

$$\mathcal{M}_\Theta^{(0)} = \{\alpha : \ F_s(\sigma^j(\alpha)) \le Y_s \ , \ j = 0, 1, 2, \dots\} . \tag{6.2}$$

We may assume furthermore that Y_s cannot be substituted by a smaller sequence $Y_s'(< Y_s)$, and with a shorter one. If Y_s is so chosen then there is an element $\alpha(\in \mathcal{M}_\Theta^{(0)})$ with prefix Y_s.

Let $Y_v = y_1 \dots y_v \quad (v = 1, \dots, s)$. If Y_s is such a sequence, then

$$\begin{cases} y_{u+1} \dots y_{u+r} \le Y_r \\ 0 \le u < u + r \le s \end{cases} \tag{6.3}$$

We assume the fulfilment of (6.3) for the whole section 6.
Notations: If $w \in \mathcal{K}_h$, $Z \subseteq \mathcal{M}$, then $wZ = \{wz \ : \ z \in \mathbb{Z}\}$.
The union of the sets $B_r(\subseteq \mathcal{M})$ is denoted as $\sum B_r$. Let $(\mathcal{M}_\Theta^{(0)} =)X$ be the set of sequences α determined by the inequalities in the right hand side of (6.2).
$\lambda(w)$ denotes the length of w. Thus $\lambda(w) = h$ for $w \in \mathcal{K}_h$.
Let $w^k = w \dots \overset{k}{w}$, and $w^{\mathbb{N}} = ww \dots$, $w^0 =$ empty word.
Let $X_w := \ \{\alpha \ : \ \alpha \in X, F_{\lambda(w)}(\alpha) = w\}$.

Lemma 9. *Let $w = r_1 \dots r_h$, $1 \le h < s$ be such a sequence for which*

$$r_{u+1} \dots r_h \le Y_{h-u} \quad (u = 0, \dots, h-1) \tag{6.4}$$

holds. If u^* is the smallest integer u for which $r_{u^*+1} \dots r_h = Y_{h-u^*}$, then

$$X_w = r_1 \dots r_{u^*} X_{Y_{h-u^*}} . \tag{6.5}$$

If (6.4) *holds with the strict inequality for every u, then*

$$X_w = w . \tag{6.6}$$

If (6.4) *fails to hold for some u, then X_w is empty.*

Proof. Clear. $w\alpha \in X$ if $X \ni \alpha = a_1 a_2 \dots$, and

$$(6.7)_u \qquad r_{u+1} \dots r_h \ a_1 \dots a_{s-(h-u)} \le Y_s, \ u = 0, \dots, h-1$$

holds. If $r_{u+1}\dots r_h < Y_{h-u}$, then u is not a critical value, $(6.7)_u$ is valid for each α. The least critical value is $u=u^*$. It means that $w\alpha\in X$ if and only if $Y_{h-u^*}\alpha\in X$. Thus (6.5), (6.6) holds. The last assertion is obvious.

Lemma 10. *Let k be an odd integer.*
1. If $y_k=1$ and $Y_{2k-1}=Y_kY_{k-1}$, $s\ge 2k-1$, then

$$X_{Y_{k-1}} = X_{Y_k} = Y_k X_{Y_k} = \{Y_k^{\mathbb{N}} = Y_kY_k\dots\}\,. \tag{6.8}$$

2. If $y_k>1$ and $s\ge 2k+h(k+1)$, $h\ge 0$ and

$$Y_{2k+h(k+1)} = Y_k\ (Y_{k-1}(y_k-1)1)^h Y_k\,, \tag{6.9}$$

then

$$X_{Y_k} = \sum_{j=0}^{h} Y_k(Y_{k-1}(y_k-1)1)^j X_{Y_k}\,. \tag{6.10}$$

Here Y_0 is thought to be the empty word.

Proof. **1.** Let $\alpha\in X$, $\alpha=a_1a_2\dots$, $F_{k-1}(\alpha)=Y_{k-1}$. Then $F_{2k-1}(\alpha)\le Y_{2k-1}$, whence $a_ka_{k+1}\dots a_{2k-1}\le 1Y_{k-1}$. Thus $a_k=1$ and $(Y_{k-1}\ge)a_{k+1}\dots a_{2k-1}\ge Y_{k-1}$, i.e. $a_{k+1}\dots a_{2k-1}=Y_{k-1}$, and (6.8) holds.

2. Assume the fulfilment of (6.9). Let $\alpha=a_1a_2\dots\in X$, $F_k(\alpha)=Y_k$. Then $a_{k+1}\dots\ a_{2k}a_{2k+1}\ge Y_{k-1}(y_k-1)1$. Hence $a_{k+1}\dots a_{2k}=Y_{k-1}$ and $a_{2k}>y_k-1$. Then either $a_{2k}=y_k$, or $a_{2k}=y_k-1$ and $a_{2k+1}=1$. In the first case $\alpha=Y_kY_k\dots$, in the second $\alpha=Y_kY_{k-1}(y_k-1)1\alpha_1$ and

$$F_{h(k+1)-1}(\alpha_1)\ge (Y_{k-1}(y_k-1)1)^{h-1}Y_k\,.$$

Similarly as above we obtain that either $\alpha_1=Y_k\alpha_2$, or $\alpha_1=Y_{k-1}(y_k-1)1\alpha_2$, and in the latter case

$$F_{(h-1)(k+1)-1}(\alpha_2)\ge (Y_{k-2}(y_k-1)1)^{h-2}Y_k\,.$$

Iterating this argument at most h times we obtain (6.10).

Lemma 11. *Let k be even:*
1. If $y_k=y_1$, $s\ge 2k-2$, and

$$Y_{2k-2}=Y_{k-1}Y_{k-1}\,, \tag{6.11}$$

then

$$X_{Y_{k-1}} = X_{Y_k} = Y_{k-1}X_{Y_{k-1}} = \{Y_{k-1}^{\mathbb{N}}\}\,. \tag{6.12}$$

2. Let $y_k<y_1$, $s\ge 2k+(k+1)h$, $h\ge 0$, and

$$Y_{2k+(k+1)h} = Y_k(1Y_{k-1}(y_k+1))^hY_k\,, \tag{6.13}$$

then

$$X_{Y_k} = \sum_{j=0}^{h} Y_k(1Y_{k-1}(y_k+1))^j X_{Y_k}\,. \tag{6.14}$$

Proof. It is very similar to that of Lemma 10. We leave it for the reader.

Remark. If (6.8) or (6.11) holds, then Y_s can be reduced to Y_k. We can exclude these cases.

6.2. Our purpose is to find an appropriate partition of X, the components of which are characterized by the prefixes of their elements, such that the relations among them allow to define a strongly connected Mauldin Williams multigraph.

Definition. **1.** We say that Y_s is of type A if for each $k \le s$ there exists a suitable finite word w for which

$$wX \subseteq X_{Y_k} \tag{6.15}$$

holds.

2. We say that Y_s is of type B_l if l is the smallest integer for which no finite word w exists with the property $wX \subseteq X_{Y_l}$.

Theorem 5. *Let Y_s be of type B_k. Then, for odd k* (6.9), *for even k* (6.13) *holds.*

Proof. *1.* Let $k=1$. If $s=1$, then $X_{Y_1} = Y_1X$, i.e. (6.15) holds. Let $s \ge 2$. $y_2 \le y_1$ follows from (6.3). If $y_2 = y_1$, then Y_s can be reduced to Y_1 in (6.2), but we assumed that Y_s is the shortest which gives (6.2). If $y_2 \le y_1 - 2$, then $X_{Y_1} \supseteq X_{Y_1(y_2+1)} = y_1(y_2+1)X$. It remains the case $y_2 = y_1 - 1$. If $s=2$, then $X_{Y_1} \supseteq X_{Y_2} = Y_2X$. Let $s \ge 3$. If $y_3 > 1$, then

$$Y_2 1X = X_{Y_2 1} \subseteq X_{Y_2} \subseteq X_{Y_1}\,.$$

Let $y_3 = 1$. If $s=3$, then $X_{Y_1} \supseteq X_{Y_3} = Y_3X$. Let $s \ge 4$. If $y_4 < y_2$ then $X_{Y_1} \supseteq X_{Y_3 y_2} = Y_3 y_2 X$. If $y_4 = y_1$, then Y_s is of type (6.9). It remains the case $y_4 = y_2$. Continuing this argument, since s is finite, we conclude that Y_s is of form (6.9).

2. Let $k > 1$, k odd, Y_s be of type B_k.

Due to the minimality of k $\quad X_{Y_{k-1}} \ne X_{Y_k}$, thus there is an l, $l \ne y_k$ such that $X_{Y_{k-1}l} \ne \emptyset$. Since k odd, therefore $l < y_k$, and so $y_k > 1$.

From (6.3) we obtain that $y_{u+1}\dots y_{k-1}\,y_k \le Y_{k-u}$, $y_{u+1}\dots y_{k-1}\,l \le Y_{k-u}$, which for odd u implies that

$$y_{u+1}\dots y_{k-1} < Y_{k-u-1}\ (u \text{ odd})\,.$$

Let

$$\alpha = (Y_{k-1}(y_k-1)1)^{\mathbb{N}}\,.$$

Let us observe first that

$$F_K(\sigma^j(\alpha)) < Y_k \quad (j = 0,1,2,\dots) \tag{6.17}$$

holds. The sequence $\beta := Y_k\alpha$ cannot go through all the tests

$$F_s(\sigma^j(\beta)) \le Y_s \tag{6.18}$$

Assume in contrary that (6.18) holds.

Let $\triangle = \beta(Y_{k-1}(y_k-1)1)^r$, r large. Then $X_\triangle$ is non empty, $\beta\alpha \in X_\triangle$. Hence, similarly as in the proof of Lemma 9, we get that $\triangle = \triangle_1 Y_j$ with an appropriate $j \in \{0,1,\dots,k-1\}$ such that $X_\triangle = \triangle_1 X_{Y_j}$. ($Y_0$ = empty word, $X_{Y_0} = X$). Since $j < k$, therefore $w^*X \subseteq X_{Y_j}$ would imply that $X_{Y_k} \supseteq X_\triangle \supseteq \triangle_1 w^* X$, thus Y_s cannot be of type B_k.

Let v be the smallest j for which (6.18) fails to hold. Since $v \le k$, taking into account (6.16), we get that v is an even number. Let $s_0 + 1 (\le s)$ be the smallest number for which $F_{s_0+1}(\sigma^V(\beta)) > Y_{s_0+1}$. Then $F_{s_0}(\sigma^V(\beta)) = Y_{s_0}$. Furthermore $s_0 \ge k$. We prove that $v = 0$. If $v \ne 0$, then $y_{v+1}\dots y_k y_1 \dots y_v = Y_k$, i.e. $Y_k = Y_{k-v}Y_v = Y_v Y_{k-v}$. Consequently, if in a sequence $X \ni \gamma = c_1c_2\dots\ , c_1\dots c_{k-v} = Y_{k-v}$, $k-v$ is odd, then $F_k(\gamma) \le Y_k$ implies that $c_{k-v-1}\dots c_k = Y_v$. Hence we obtain that $X_{Y_{k-v}} = X_{Y_k}$; $k-v<k$, which contradicts to the minimality of k. We obtained that $v=0$,

$$F_{s_0}(\beta) = Y_{s_0}, \quad F_{s_0+1}(\beta) > Y_{s_0+1}\,.$$

Hence we obtain that $F_{s_0-k}(\alpha) = y_{k+1}\dots y_{s_0}$ and $F_{s_0+1-k}(\beta) < y_{k+1}\dots y_{s_0+1}$. Let r be the largest integer for which $r(k+1) \le s_0 - k$. Since $\sigma^{k+1}(\alpha) = \alpha$, we obtain that $F_{s_0-k-r(k+1)}(\alpha) = y_{k+r(k+1)+1}\cdots\ y_{s_0}$ and that

$$F_{s_0-l}(\alpha) = y_{l+1}\dots y_{s_0}, \quad F_{s_0+1-l}(\alpha) < y_{l+1}\dots y_{s_0+1}, \quad l = k + r(k+1)\,.$$

This can occur only if $y_{l+1}\dots y_{s_0+1} = Y_k$. This proves the theorem for odd k.

3. The case k =even can be proved similarly. We omit the details.

6.3. Assume that Y_s is of type A. Let W be the set of the following finite words:

(1) $i \in W$, if $i \in \{1, \dots, y_1 - 1\}$.

(2) for every k, $1 \le k < s$, $w = y_1 \dots y_k i \in W$, if $y_{u+1} \dots y_k i \le Y_{k+1-u}$ $(u = 0, \dots, k-1)$ and $i \ne y_{k+1}$, $i \le y_1$ hold;

(3) $Y_s \in W$.

Then X_w $(w \in W)$ are mutually disjoint sets, $\sum X_w = X$.

Assume now that Y_s is of type B_k. Then W $(= W^{(k)})$ is defined as follows:

(1) $i \in W^{(k)}$, if $i \in \{1, \dots, y_1 - 1\}$

(2) for every j, $1 \le j < k$, $w = y_1 \dots y_j l$ for which $y_{u+1} \dots y_j l \le Y_{j+1-u}$ $(u = 0, \dots, j-1)$ $l \ne y_{j+1}$, $l \le y_1$ hold, let $w \in W^{(k)}$

(3) $Y_k \in W^{(k)}$.

Then $\{X_w, w \in W^{(k)}\}$ is a subdivision of X into the mutually disjoint sets X_w.

Now we define the directed multigraph $G(W)$ (resp. $G(W^{(k)})$ over the set W (resp. $W^{(k)}$) as the set of nodes by the following relation.
Let first Y_s is of type A.

For $1 \le i < y_1$ we have $X_i = iX$, thus

$$X_i = \sum_{w \in W} i \; X_w \,. \tag{6.19}$$

Let $z = Y_r i \in W$, $1 \le r < s$. If h is the largest number ($h = 0$ is included with $X_{Y_0} = X$) for which $z = Y_{r-h} Y_h$, then

$$X_z = Y_{r-h} X_{Y_h} = \sum_{\substack{w \in W \\ \lambda(w) > h}} Y_{r-h} X_w \,, \tag{6.20}$$

(see Lemma 9). Especially in the case $h = 0$ we have

$$X_z = \sum_{w \in W} z \; X_w \,. \tag{6.21}$$

Finally we give a formula for X_{Y_s}. We have $X_{Y_s} = \sum_{1 \le l \le y_1}{}^{*} X_{Y_s l}$, where the asterisk means that we sum only for those l for which additionally $y_{u+1} \dots y_s l \le Y_{s+l-u}$ $(u = 1, 2, \dots, s-1)$ holds. Let u_l^* be the smallest value, if any, for which $y_{u_l^*+1} \dots y_s \; l = Y_{s+l-u_l^*}$. For such an l we have $X_{Y_s l} = Y_{u_l^*} X_{Y_s+1-u_l^*}$. Such an l will be called of first kind. If such a

u does not exist (we say l is of second kind), then clearly $X_{Y_s l} = Y_s l X$. Consequently

$$X_{Y_s} = {\sum_l}' Y_{u_l^*} X_{Y_{s+1-u_l^*}} + {\sum_l}'' Y_s l X \,, \tag{6.22}$$

where in $\sum'$ we sum over the l of first kind, and in $\sum''$ over the others. At least one of the sums on the right hand side is non-empty.

Since $y_2 < y_1$ ($y_2 = y_1$ leads to the reducible case $s = 2$, $Y_2 = y_1 y_1$), therefore $s + 1 - u_l^* \le s - 1$. Thus, by Lemma 9 we obtain that

$$X_{Y_s} = {\sum_l}' \sum_{\substack{w\in W \\ \lambda(w)>s+1-u_s^*}} Y_{u_l^*} X_w + {\sum_l}'' \sum_{w\in W} Y_s l X_w \,. \tag{6.23}$$

Construction of $G(W)$:

Let $z \in W$, $z \ne Y_s$. Then direct edges to that $w \in W$ which occur in the formula (6.19), (6.20), (6.21) respectively. The edge is labeled by the corresponding "coefficient" standing before X_w. For example, if z is subjected to (6.20), then we direct one edge to a $w \in W$ if $\lambda(w) > h$, and label this with Y_{r-h}. For $z = Y_s$ and $w \in W$ we direct as many edges from z to w as many times X_w occurs in the right hand side of (6.23), and label them with the corresponding coefficients $Y_{u_l^*}$ or $Y_s l$.

Theorem 6. *If Y_s is of type A then $G(W)$ is strongly connected.*

Proof. The assertion is an immediate consequence of Lemma 9 and (6.22), whence we obtain that for each $w \in W$, $X_w \supseteq zX$ with an appropriate finite word z holds.

Assume now that Y_s if of type $B^{(k)}$. The construction of $G(W^{(k)})$ is similar as earlier. The relations (6.19), (6.20), (6.21) are valid. Instead of (6.22) we use the relation, (6.10), (6.14). Thus for odd k, from the point Y_k $h+1$ loops are going out which are labelled by
$Y_k(Y_{k-1}(y_k - 1)1)^j$ $(j = 0, \dots, h)$. Thus from Y_k we cannot reach any element of $W^{(k)} \setminus \{Y_k\}$. Furhermore the graph $G(W^{(k)}) \setminus \{Y_k\})$ is strongly connected, due to the minimality condition in the definition B_k.

Theorem 7. *If Y_s is of type $B^{(k)}$ then $G(W^{(k)} \setminus \{Y_k\})$ is strongly connected.*

Example 1. Let $Y = Y_4 = 4213$. Then $W=\{1,2,3,44,43,4214,4213\}$ We have: $X_i = \sum_{w\in W} iX_w$,

$$X_4 = 4\ X_4 = 4\ X_{44} + 4\ X_{43} + 4\ X_{4214} + 4\ X_{4213}$$

$$X_{43} = 43\ X = \sum_{w\in W} 43\ X_w$$

$$X_{4214} = 421\ X_4 = 421\ X_{44} + 421\ X_{43} + 421 X_{4214} + 421\ X_{4223}$$

$$X_{4213} = 4213\ X = \sum_{w\in W} 4213\ X_w\,.$$

We draw $G(W)$ in a simplified form. If $z \in W$ is such a node which is subjected to the formula $X_z = \sum_{w\in W} zX^w$, then the corresponding edges are not drawn and the nodes are marked with asterisk.
Then $G(W)$

Example 2. Let $Y_{21} = 322(3211)^3 322333$. Computing X we can substitute Y_{21} by $Y_{18} = 322(3211)^3 322$. Then Y is of type $B^{(3)}$. $W = W^{(3)} = \{1,2,33,321,322\}$

We have: $X_z = \sum_{w\in W} X_w$ if $w = 1; 2; 321$. We denote them with z^*. Furthermore $X_{33} = 3\ X_{33} + 3\ X_{321} + 3\ X_{322}$, $\quad X_{322} = \sum_{j=0}^{3} 322(3211)^j X_{322}$. Thus the simplified form of $G(W^{(3)})$ is the following

6.4. Let $\Theta + \Theta^2 < 1$. If V is an arbitrary subset of $H(= H_\Theta)$, then $L - V = \{L - x\ :\ x \in V\} \subseteq H^*$.
For some $\alpha \in X$ let $\Psi(\alpha) := \left\langle \varphi^{-1}(\alpha), \Theta \right\rangle$. If $x = \Psi(\alpha)$, $y = \Psi(\sigma(\alpha))$, then

$$x = \Theta + \cdots + \Theta^{a_1} + \Theta^{a_1}(L - y)\,.$$

The assumption $\Theta + \Theta^2 < 1$ guarantees that if $x \in [\Theta, 1)$ then $y \in [\Theta, 1)$. Thus, if

$$K = iQ, \quad K, Q \subseteq X\,,$$

then

$$\Psi(K) = \Theta + \cdots + \Theta^i + \Theta^i(L - \Psi(Q))\,.$$

For an arbitrary finite word z we define the similarity $f_z : \mathbb{R} \to \mathbb{R}$ recursively, by the following rules:

(1) For $z = i \in \mathbb{N}$ let $f_i(x) = \Theta + \cdots + \Theta^i + \Theta^i(L - x)$.

(2) If $f_{i_1\dots i_j}$ are defined for every $j \le r$ and every $i_1 \dots i_r \in \mathcal{K}_r$, then $f_{i_1\dots i_r i_{r+1}}(x) = f_{i_1}(f_{i_2\dots i_{r+1}}(x))$. It is clear that $f_{i_1} \dots i_r$ is a linear function with contraction factor

$$r(i_1 \dots i_r) = \Theta^{i_1+\cdots+i_r}\,.$$

Let $H(w) := \Psi(X_w)$ defined for finite words w. Then $\{H(w) \mid w \in W\}$ is a partition of $H(= \Psi(X))$ into disjoint non-empty compact sets.

The multigraph $G(W)$ $(G(W^{(k)}))$ generates the following relation among them:

$$H(z) = \bigcup f_e(H(w)) \quad (z \in W) \tag{6.24}$$

where in the right hand side we sum over all edges leaving z. e denotes the label of the edge and w the endpoint.

Assume that Y_s is of type A. Then $G(W)$ is a Mauldin-Williams graph. The open set condition (due to Moran) clearly satisfied, therefore the similarity dimension equals to the Hausdorff dimension of the components $H(z)$. All of the components have positive finite measures (with respect to the σ-dimensional Hausdorff-measure μ_σ). σ can be computed as the only nonnegative real number for which the equation system

$$q_z^\sigma = \sum r(e)^\sigma q_w^\sigma \quad (z \in W) \tag{6.25}$$

has positive $q_z (z \in W)$ solution.

Let us consider now the case when Y_s is of type $B^{(k)}$. Assume that $t_1 \ge 3$. Let $m = t_1 + \cdots + t_k$. The set $H(Y_k)$ is self-similar, it is the attractor of the iterated function system

$$H(Y_k) = \bigcup f_e(H(Y_k))\,,$$

where in the right hand side we sum over the loops coinciding Y_k. Thus its similarity dimension = Hausdorff dimension = λ can be computed from

$$1 = \sum_e r(e)^\lambda\,.$$

Since $r(e)$ run over the values $\Theta^m, \Theta^{2m}, \dots, \Theta^{hm}$, for odd k, and over the values $\Theta^{m+j(m+2)}$ $(j = 0, \dots, h)$ for even k, we have

$$(6.25)_{k\text{ odd}} \qquad 1 = \Theta^{m\lambda} + \Theta^{2m\lambda} + \cdots + \Theta^{(h+1)m\lambda}$$

$$(6.25)_{k\text{ even}} \qquad 1 = \Theta^{m\lambda} + \Theta^{m\lambda+(m+2)\lambda} + \cdots + \Theta^{m\lambda+h(m+2)\lambda}$$

Let $X^{(1)} := \{\alpha \ :\ F_k(\alpha) < Y_k\}, \Psi(X^{(1)}) = \tilde{H}$
It is clear that $\tilde{H}^* \subseteq H$.

Let $X^{(2)} = X \setminus X^{(1)}$. Then $X^{(2)}$ can be represented as the union of countable many sets of form $z\ X_{Y_k}$. Since the $\lambda+\varepsilon$ dimensional measure of $\Psi(zX_{Y_k})$ is zero for all of these subsets, therefore

$$\mu_{\lambda+\varepsilon}(X^{(2)}) = 0 \quad \text{for every} \quad \varepsilon > 0\,.$$

Let $X_w^{(1)} = X_w \setminus X^{(2)}$ defined for $w \in W^k \setminus Y_k$.
Then $X^{(1)}$ is the union of the disjoint sets $X_w^{(1)}$ the relation among them are defined by the strongly connected multigraph $G(W \setminus \{Y_k\})$. Thus the Hausdorff-dimension σ of the sets $\Psi(X^{(1)})$, $\Psi(X_w^{(1)})$ can be computed, $\mu_\sigma(\Psi(X_w^{(1)})) > 0$.

If we can prove that $\sigma > \lambda$, then we conclude that

$$\infty > \mu_\sigma(X_w) > 0 \;\text{ if }\; w \neq Y_k, 0 < \mu_\sigma(X) < \infty\,.$$

Let $Z = \{\alpha : \quad \alpha = a_1 a_2 \ldots, 0 \leq a_i < y_1\}$. Then $Z \subseteq X^{(1)}$, $D := \Psi(\mathbb{Z}) \subseteq H^*$. Furthermore D is a self-similar set,

$$D = \sum_{i=1}^{y_1 - 1} f_i(D)\,,$$

its Hausdorff dimension is that η for which

$$1 = \sum_{i=1}^{y_1 - 1} \Theta^{i\eta}$$

holds. Since $\eta \leq \sigma$, if is enough to prove that $\lambda < \eta$.

But this is clear, if $y_1 \geq 3$. λ as a function of h in (6.25) is monotonically increasing. Thus $\lambda \leq \lambda_0$, where

$$1 = \frac{\Theta^{m\lambda_0}}{1 - \Theta^{m\lambda_0}}\,, \text{ i.e. } \Theta^{\lambda_0} = \left(\frac{1}{2}\right)^{1/m}.$$

Since $m \geq 3$, therefore $\Theta^{\lambda_0} > 3/4$, $\frac{3}{4} + \left(\frac{3}{4}\right)^2 > 1$, consequently $\eta > \lambda_0$.

Finally we observe that the above method is applicable even in the case $y_1 = 2$. If $s = 1$, then this is clear. If $s \geq 2$ and $y_1 = y_2$, then Y_3 can be reduced to $Y_2 = 22$, and we get that $X_1 = 1\ X$, $X_2 = \ 2X_2$, which implies that $H(\Theta)$ is a countable set, therefore its Hausdorff dimension equals to zero. We should consider only the cases when Y_s is of type B. Let $y_2 = 1$. Assume that Y_s is of type B_k. If $k = 1$, then Y_s has the prefix $2(11)^h 2$ with some integer $h \geq 0$. Hence we obtain that

$$X_1 = 1X, \quad X_{2(11)^j 2} = 2(11)^j X_2 \quad (j = 0. \ldots, h)$$

whence

$$\Psi(X_2)=\sum_{j=0}^{h} f_{2(11)^j}(\Psi(X_2))$$

follows. Then $\Psi(X_2)$ is a self-similar set, its Hausdorff dimension λ can be computed as the solution of the equation

$$1=\sum_{j=0}^{h}\Theta^{\lambda(2+2j)}\,.$$

We have $0<\lambda<1$.
Furthermore $0<\mu_\lambda(\Psi(X_2))<\infty$. Since

$$X_1=\left\{1^{\mathbb{N}}\right\}+\sum_{l=1}^{\infty}1^l X_2\,,$$

therefore $\mu_\lambda(\Psi(X_1))=\mu_\lambda(\Psi(1^{\mathbb{N}}))+\sum_{l=1}^{\infty}\mu_\lambda(\Psi(1^l X_2))=0+\sum_{l=1}^{\infty}\Theta^{l\lambda}\mu_\lambda(\Psi(X_2))$, thus $0<\mu_\lambda(\Psi(X_1))<\infty$.
Assume that $k\geq 2$. If k is odd and $y_k=1$, or if k is even and $y_k=y_1(=2)$, then Y_s is of form (6.8) or (6.11) respectively, thus it is reducible. These cases can be excluded.

Let $k(\geq 3)$ be odd. Then k is at least so large then the index of the second occurence of 2 in $y_1y_2\dots$. Thus $Y_s=2\;1^r2\dots$ and $k\geq r+2$. Since (6.9) holds, therefore $m:=y_1+\dots+y_k\geq r+4$. The Hausdorff dimension λ of $\Psi(X_{Y_k})$ can be computed from the equation

$$\Theta^{m\lambda}+\dots+\Theta^{m\lambda(h+1)}=1\,. \tag{6.26}$$

Let $X'=\{\alpha : F_{r+2}(\sigma^j(\alpha))<21^r2\}$. If we prove that the Hausdorff dimension of $\Psi(X')$ is larger than λ, then we can compute it from the Mauldin-Williams graph omitting the node Y_k.

Let X'' be the attractor of

$$X''=\sum_{\substack{l=0\\2l<r}}2\;1^{2l}X''\,. \tag{6.27}$$

Then $X''\subseteq X'$. The dimension σ of $\Psi(X'')$ is obtained from

$$1=\sum_{\substack{l=0\\2l<r}}\Theta^{\sigma(2l+2)}\,. \tag{6.28}$$

$\sigma\leq\lambda$ would imply that $(\xi=)\Theta^\sigma\geq\Theta^\lambda(=\eta)$. From (6.26), (6.28) we can get immediately that it is impossible if $m\geq 6$, i.e. if $r\geq 2$. It remains the

case $r=1$.
Let $k=$ even of form (6.13). Then $y_k=1$. If $2\ 1^r 2$ is a prefix in Y_s, then $k\ge r-1$, and so $m=y_1+\cdots+y_k\ge r$. Now the Hausdorff dimension λ of $\Psi(X_{Y_k})$ is computed from the equation,

$$\Theta^{m\lambda}+\Theta^{m\lambda+(m+2)\lambda}+\cdots+\Theta^{m\lambda+h(m+2)\lambda}=1\,. \tag{6.29}$$

If r is even, then Y_s is of type $B^{(1)}$ which was considered earlier. Let r be odd. Let us consider the set X'' defined by (6.27). The Hausdorff dimension of $\Psi(X'')$ is given as that σ for which (6.28) holds. Let $\xi=\Theta^\sigma$, $\eta=\Theta^\lambda$. Let $r>1$. The smallest value of η is getting by for $h\to\infty$, i.e. for $1=\eta^m+\eta^{2m+2}$. Furthermore, from (6.28), $1=\xi^2+\xi^4+\cdots+\xi^{r+1}$, and this implies that $\xi<\eta$ for $m\ge 3$.

Finally we consider the case when $r=1$, $k=$even, Y_s is of form $B^{(k)}$. If $k=2$, then Y_s is of form (6.13), i.e.

$$Y_{3h+6}=21(121)^h22, \text{ and } F_3(Y_s)\ne 212.$$

Then $k\ge 4$. Consequently either $k=4$ and $Y_{8+5h}=2121(12122)^h2121$ for some $h\ge 0$ or $k\ge 6$.
Let $k=4$, $W^{(0)}=\{1,22,211\}, X'=X'_1+X'_{22}+X'_{211}$ defined by $X'_1=1X', X''_{22}=2X''_2, X'_{211}=211X'$.
The Hausdorff-dimension σ of $\Psi(X')$ can be computed from: $q_1^\sigma=\Theta^\sigma R$, $q_{22}^\sigma=\Theta^{2\sigma}(R-q_1^\sigma)$, $q_{211}^\sigma=\Theta^{4\sigma}R$, $R=q_1^\sigma+q_{22}^\sigma+q_{211}^\sigma \quad (>0)$, i.e. it is the solution of the equation $1=\Theta^\sigma+\Theta^{2\sigma}-\Theta^{3\sigma}+\Theta^{4\sigma}$. Since $m=6$, similarly as above we deduce that $\sigma>\lambda$. The case $k\ge 6$ is similar, the proof is left to the reader.

6.5. Now we summarize our result for the computation of the Hausdorff dimension of H.

Assume that Y_s defining (6.2) cannot be further reduced. Then we have:

1. If Y_s is of type A, then the Hausdorff dimension of H_Θ equals to the similarity dimension of the Mauldin-Williams graph $G(W)$, $G(W)$ is strongly connected.
2. Assume that Y_s is of type $B^{(k)}$ and that $Y_{2j+2}\ne 2\ 1^{2j}2 \quad (j=0,1,\dots)$.
   Then the Hausdorff-dimension σ of $\Psi(X)$ is the same as the similarity dimension of (the strongly connected) graph $G(W\mid\{Y_k\})$.

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