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On Nemytskii operator in the space of set-valued functions of bounded p-variation in the sense of Riesz

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Abstract. We sonsider the Nemytskii operator, i.e. the composition operator defined by (Nu) $(t) = H(t, u(t))$, where H is a given set-valued function. It is shown that if the operator N maps the space of set-valued functions of bounded p-variation in the sense of Riesz into the space of set-valued functions of bounded q-variation in the sense of Riesz, there is $1 \leq q \leq p < \infty$, and if it is globally Lipschitzian, then it has to be of the form (Nu) $(t) = A(t)u(t) + B(t)$, where $A(t)$ are linear continuous set-valued and B is a set-valued function of bounded q-variation in the sense of Riesz. This generalizes results of G. Zawadzka [8], A. Smajdor and W. Smajdor [7], N. Merentes and K. Nikodem [3].

Introduction

In [7] A. Smajdor and W. Smajdor proved that every Nemytskii operator N, i.e. (Nu) $(t) = H(t, u(t))$ mapping the space $\mathrm{Lip}([a, b], cc(Y))$ into itself and globally Lipschitzian has to be of the form

$$(Nu)(t) = A(t)u(t) + B(t), \quad u \in \mathrm{Lip}([a, b], cc(Y)), \quad t \in [a, b],$$

where $A(t)$ are linear continuous set-valued functions and B is a set-valued function belonging to the space $\mathrm{Lip}([a, b]), cc(Y))$. For the first time a theorem of such a type for single-valued functions was proved by J. Matkowski [1] in the space of Lipschitz functions. Similar characterizations of the Nemytskii operator have been also obtained by G. Zawadzka (see [8]) in the space of set-valued functions of bounded variation in the classical Jordan sense. For single-valued functions it was proved by

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J. Matkowski and J. Miś [2]. Recently N. Merentes and K. Nikodem (see [3]) proved an analogous theorem in the space of set-valued functions of bounded p-variation in the sense of Riesz. The aim of this paper is prove an analogous result in the case when the Nemytskii operator N maps the space of set-valued functions of bounded p-variation in the sense of Riesz into the space of set-valued functions of bounded q-variation in the sense of Riesz, where $1 \leq q \leq p < \infty$ and N is globally Lipschitzian. The particular cases $p = q$ has been already considered by N. Merentes and K. Nikodem (see [3]), but the present case of possibly different spaces requires a different proof technique, and this extension may turn out to be useful in some applications.

1. Preliminary results

Let $(X, \|\cdot\|)$ be a normed space and $p \geq 1$ be a fixed number. Given a function $u : [a, b] \to X$ and a partition $\pi : a = t_0 < \cdots < t_n = b$ of the interval $[a, b]$, we define:

$$\sigma_p(u; \pi) := \sum_{i=1}^{n} \frac{\|u(t_i) - u(t_{i-1})\|^p}{|t_i - t_{i-1}|^{p-1}}.$$

The number:

$$V_p(u, [a, b]; X) := \sup_{\pi} \sigma_p(u, \pi),$$

where the supremum is taken over all partitions π of $[a, b]$, is called the p-variation of u in $[a, b]$. A function u is said to be of bounded p-variation if $V_p(u, [a, b]; X) < \infty$. Denote by $RV_p([a, b], X)$ the space of all functions $u : [a, b] \to X$ of bounded p-variation equipped with the norm

$$\|u\|_p := \|u(a)\| + (V_p(u, [a, b]; X))^{\frac{1}{p}}.$$

Clearly, for $p = 1$ the space $RV_1([a, b], X)$ coincides with the classical space $BV([a, b], X)$ of functions of bounded variation. In the particular case when $X = \mathbb{R}$ and $1 < p < \infty$, then we have the space $RV_p[a, b]$ of functions of bounded Riesz p-variation, and the following characterization is well-known:

Lemma 1 (see [5]). *$u \in RV_p([a, b], \mathbb{R})$ if and only if u is absolutely continuous on $[a, b]$ and its derivative $u' \in L_p([a, b]; \mathbb{R})$. In that case we also have the equality*

$$V_p(u, [a, b]; \mathbb{R}) = \int_a^b |u'(t)|^p dt.$$

Let $cc(X)$ be the family of all non-empty convex compact subsets of X and D be the Hausdorff metric in $cc(X)$, i.e.

$$D(A,B) := \inf\{t > 0 : A \subseteq B + tS,\ B \subseteq A + tS\},$$

where $S = \{y \in X : \|y\| \leq 1\}$.

We say that a set-valued function $F : [a,b] \to cc(X)$ has bounded p-variation $(1 \leq p < \infty)$ if

$$W_p(F,[a,b];cc(X)) := \sup_{\pi} \sum_{i=1}^{n} \frac{(D(F(t_i),F(t_{i-1}))^p}{|t_i - t_{i-1}|^{p-1}} < \infty,$$

where the supremum is taken over all partitions π of $[a,b]$.

Denote by $RW_p([a,b];cc(X))$ the space of all set-valued functions $F : [a,b] \to cc(X)$ of bounded p-variation equipped with the metric

$$D_p(F_1,F_2) :=$$
$$D(F_1(a),F_2(a)) + \left(\sup_{\pi} \sum_{i=1}^{n} \frac{(D(F_1(t_i)+F_2(t_{i-1}),F_1(t_{i-1})+F_2(t_i)))^p}{|t_i - t_{i-1}|^{p-1}}\right)^{\frac{1}{p}}.$$

Clearly, for $p = 1$ the space $RW_1([a,b];cc(X))$ coincides with the space $BV([a,b];cc(X))$ of set-valued functions of bounded variation.

Now, let $(X\|\cdot\|)$, $(Y,\|\cdot\|)$ be two normed spaces and K be a convex cone in X. Given a set-valued function $H : [a,b] \times K \to cc(Y)$ we consider the Nemytskii operator N genereted by H, that is the composition operator defined by:

$$(Nu)(t) := H(t,u(t)), \quad u : [a,b] \to K, \quad t \in [a,b].$$

We denote by $L(K;cc(Y))$ the space of all set-valued function $A : K \to cc(Y)$ additive and positively homogeneous. We say that A is linear if $A \in L(K;cc(Y))$.

In the proof of the main results of this paper we will use some facts which we list here as lemmas.

Lemma 2 (see [6], Lemma 3)**.** *Let $(X,\|\cdot\|)$ be a normed space and let A, B, C be subsets of X. If A, B are convex compact and C is non-empty and bounded, then*

$$D(A+C,B+C) = D(A,B).$$

Lemma 3 (see [4], Th. 5.6)**.** *Let $(X, \|\cdot\|)$, $(Y\|\cdot\|)$ be normed spaces and K be a convex cone in X. A set-valued function $F : K \to cc(Y)$ satisfies the Jensen equation*

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2}(F(x) + F(y)), \quad x, y \in K,$$

if and only if there exists an additive set-valued function $A : K \to cc(Y)$ and a set $B \in cc(Y)$ such that $F(x) = A(x) + B$, $x \in K$.

Lemma 4. *If $F \in RW_p([a,b], cc(Y))$ with $p > 1$, then F is continuous. In the case $p = 1$, we have $F^-(\cdot, x) \in BW([a,b], cc(Y))$ for all $x \in K$, where*

$$F^-(t,x) := \begin{cases} \lim\limits_{s\uparrow t} F(s,x), & t \in (a,b],\ x \in K, \\ F(a,x), & t = a,\ x \in K. \end{cases}$$

Proof. For $1 < p < \infty$, this follows immediately from the inequality

$$\begin{aligned} D(F(t), F(t_0)) &= \left(\frac{(D(F(t), F(t_0)))^p |t - t_0|^{p-1}}{|t - t_0|^{p-1}}\right)^{\frac{1}{p}} \\ &\le W_p(F, [a,b]; cc(Y))|t - t_0|^{1-\frac{1}{p}}. \end{aligned}$$

For the case $p = 1$, see [8].

2. Main results

In this section we shall present a characterization of functions $H : [a,b] \times K \to cc(Y)$ for which the Nemytskii operator N generated by H maps the space $RV_p([a,b], K)$ into the space $RW_q([a,b], cc(Y))$, where $1 < q < p$, and it is globally Lipschitzian. On the other hand if $1 < p < q$, then the Nemytskii operator N is constant.

Theorem 1. *Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ be normed spaces and K be a convex cone in X and $1 < q < p$. If the Nemytskii operator N generated by a set-valued function $H : [a,b] \times K \to cc(Y)$ maps the space $RV_p([a,b], K)$ into the space $RW_q([a,b], cc(Y))$ and if it is globally Lipschitzian, then the set-valued function H satisfies the following conditions:*

a) *For all $t \in [a,b]$ there exists $M(t)$, such that*

$$D(H(t,x), H(t,y)) \le M(t)\|x - y\| \quad (x, y \in X) \tag{1}$$

b) $H(t,x) = A(t)x + B(t)$ $(t \in [a,b],\ x \in K)$,
where $A : [a,b] \to L(K, cc(Y))$ *and* $B \in RW_q([a,b], cc(Y))$.

Proof. The Nemytskii operator N is globally Lipschitzian, then there exists a constant M, such that

$$D_q(Nu_1, Nu_2) \leq M\|u_1 - u_2\|_p \quad (u_1, u_2 \in RV_p([a,b], K)).$$

Let $t \in (a,b]$. Using the definition of the operator N and of the metric D_q we have

$$\begin{aligned} &D_q(H(t,u_1(t))) + H(a,u_2(a)), H(a,u_1(a)) + H(t,u_2(t))) \leq \\ &\leq M|t-a|^{1-\frac{1}{q}}\|u_1 - u_2\|_p, \quad (u_1, u_2 \in RV_p([a,b], K)). \end{aligned} \tag{2}$$

Define the function $\alpha : [a,b] \to [0,1]$ by:

$$\alpha(\tau) := \begin{cases} \dfrac{\tau - a}{t - a}, & a \leq \tau \leq t, \\ 1, & t \leq \tau \leq b. \end{cases}$$

The function $\alpha \in RV_p[a,b]$ and

$$V_p(\alpha; [a,b]) = \frac{1}{|t-a|^{p-1}}.$$

Let us fix $x, y \in K$ and define the functions $u_i : [a,b] \to K$ $(i = 1,2)$ by:

$$u_1(\tau) := x, \qquad \tau \in [a,b], \qquad u_2(\tau) := \alpha(\tau)(y - x) + x,\ \tau \in [a,b]. \tag{3}$$

The functions $u_i \in RV_p([a,b], K)$ $(i = 1,2)$ and

$$\|u_1 - u_2\|_p = (V_p(\alpha; [a,b]))^{\frac{1}{p}}\|x - y\| = \frac{\|x - y\|}{|t-a|^{1-\frac{1}{p}}}.$$

Hence, substituting in inequality (2) the particular functions u_i $(i = 1,2)$ defined by (3), we obtain

$$D(H(t,x) + H(a,x), H(a,x) + H(t,y)) \leq M\frac{|t-a|^{1-\frac{1}{q}}}{|t-a|^{1-\frac{1}{p}}}\|x - y\|, \tag{4}$$

for all $t \in [a,b]$, $x, y \in K$.

By Lemma 2 and the inequality (4) we have

$$D(H(t,x), H(t,y)) \leq M\frac{|t-a|^{1-\frac{1}{q}}}{|t-a|^{1-\frac{1}{p}}}\|x - y\|,$$

for all $t \in (a,b]$, $x, y \in K$.

Now, let $t = a$. Define the function $\beta : [a,b] \to [0,1]$ by

$$\beta(\tau) := \frac{\tau - a}{b - a}, \quad (\tau \in [a,b]).$$

The function $\beta \in RV_p[a,b]$ and

$$\beta(\tau) = \frac{1}{|b-a|^{p-1}}.$$

Let us fix $x, y \in K$ and define the functions $u_i : [a,b] \to K$ $(i = 1,2)$ by

$$u_1(\tau) := x \quad \tau \in [a,b], \quad u_2(\tau) := \beta(\tau)(x - y) + y, \ \tau \in [a,b]. \tag{5}$$

The functions $u_i \in RV_p([a,b], K)$ $(i = 1,2)$ and

$$\|u_1 - u_2\|_p = \left(1 + (V_p(\beta;[a,b]))^{\frac{1}{p}}\right)\|x - y\| = \left(1 + \frac{1}{|b-a|^{1-\frac{1}{p}}}\right)\|x - y\|.$$

Hence, substituting in the inequality (2), the particular functions u_i $(i = 1,2)$ defined by (5), we obtain

$$D(H(b,x) + H(a,y), H(a,x) + H(b,x)) \leq$$
$$\leq M|b-a|^{1-\frac{1}{q}}\left(1 + \frac{1}{|b-a|^{1-\frac{1}{p}}}\right)\|x - y\|.$$

By Lemma 2 and the above inequality, we have

$$D(H(a,y), H(a,x)) \leq M|b-a|^{1-\frac{1}{q}}\left(1 + \frac{1}{|b-a|^{1-\frac{1}{p}}}\right)\|x - y\|.$$

Define the function $M : [a,b] \to \mathbb{R}$ by

$$M(t) := \begin{cases} M\dfrac{|t-a|^{1-\frac{1}{q}}}{|t-a|^{1-\frac{1}{p}}}, & a < t \leq b, \\ M|b-a|^{1-\frac{1}{q}}\left(1 + \frac{1}{|b-a|^{1-\frac{1}{p}}}\right), & t = a. \end{cases}$$

Hence

$$D(H(t,x), H(t,y)) \leq M(t)\|x - y\| \quad (x, y \in X, \ t \in [a,b]),$$

and, consequently, for every $t \in [a,b]$ the function $H(t,\cdot) : K \to cc(Y)$ is continuous.

Next we shall prove that H satisfies equality b).

Let us fix $t, t_0 \in [a,b]$ such that $t_0 < t$. Since the Nemytskii operator N is globally Lipschitzian, there exists a constant M, such that

$$(6) \qquad D(H(t,u_1(t)) + H(t_0,u_2(t_0)), H(t_0,u_1(t_0)) + H(t,u_2(t))) \leq \\ \leq M\|u_1 - u_2\|_p \, |t-t_0|^{1-\frac{1}{q}}.$$

Define the function $\gamma : [a,b] \to [0,1]$ by

$$\gamma(\tau) := \begin{cases} \dfrac{\tau - a}{t_0 - a}, & a \leq \tau \leq t_0, \\ -\dfrac{\tau - t}{t - t_0}, & t_0 \leq \tau \leq t, \\ 0, & t \leq \tau \leq b. \end{cases}$$

The function $\gamma \in RV_p[a,b]$.

Let us fix $x, y \in K$ and define the functions $u_i : [a,b] \to K$ by

$$(7) \qquad \begin{aligned} u_1(\tau) &:= \frac{\gamma(\tau)}{2}x + \left(1 - \frac{\gamma(\tau)}{2}\right)y, \quad (\tau \in [a,b]) \\ u_2(\tau) &:= \frac{1+\gamma(\tau)}{2}x + \frac{1-\gamma(\tau)}{2}y, \quad (\tau \in [a,b]). \end{aligned}$$

The functions $u_i \in RV_p([a,b],K)$ $(i = 1,2)$ and

$$\|u_1 - u_2\|_p = \frac{\|x-y\|}{2}.$$

Hence, substituting in the inequality (6) the particular functions u_i $(i = 1,2)$ defined by (7), we obtain

$$(8) \qquad D\left(H(t_0,x) + H(t,y), H\left(t_0, \frac{x+y}{2}\right) + H\left(t, \frac{x+y}{2}\right)\right) \leq \\ \leq \frac{M}{2}|t-t_0|^{1-\frac{1}{q}}\|x-y\|.$$

Since N maps $RV_p([a,b],K)$ into $RW_q([a,b],cc(Y))$ $(1 < q < p)$, then $H(\cdot,z)$ is continuous for all $z \in K$. Hence, letting $t_0 \uparrow t$ in the inequality (8), we get

$$D\left(H(t,x) + H(t,y), H\left(t, \frac{x+y}{2}\right) + H\left(t, \frac{x+y}{2}\right)\right) = 0,$$

for all $t \in [a,b]$ and $x, y \in K$.

Thus for all $t \in [a,b]$, $x, y \in K$, we have

$$H\left(t, \frac{x+y}{2}\right) + H\left(t, \frac{x+y}{2}\right) = H(t,x) + H(t,y).$$

Since that values of H are convex, we have

$$H\left(t, \frac{x+y}{2}\right) = \frac{1}{2}(H(t,x) + H(t,y)), \tag{9}$$

for all $t \in [a,b]$, $x, y \in K$. Thus for all $t \in [a,b]$, the set-valued function $H(t,\cdot) : K \to cc(Y)$ satisfies the Jensen equation (9). Now by the Lemma 3, there exists an additive set-valued function $A(t) : K \to cc(Y)$ and a set $B(t) \in cc(Y)$, such that

$$H(t,x) = A(t)(x) + B(t), \quad (x \in K,\ t \in [a,b]).$$

Substituting $H(t,x) = A(t)(x) + B(t)$ into inequality (1), we obtain, for all $t \in [a,b]$ that there exists $M(t)$, such that

$$D(A(t)(x), A(t)(y)) \le M(t)\|x - y\| \quad (x, y \in K),$$

consequently, the set-valued function $A(t) : K \to cc(Y)$ is continuous, and $A(t)(\cdot) \in L(K, cc(Y))$.

$A(t)(\cdot)$ is additive and $0 \in K$, then $A(t) = \{0\}$, thus $H(\cdot, 0) = B(\cdot)$.

The Nemytskii operator N maps the space $RV_p([a,b], K)$ into the space $RW_q([a,b], c(Y))$, then $H(\cdot, 0) = B(\cdot) \in RW_q([a,b], K)$. Consequently the set-valued function H has to be of the form

$$H(t,x) = A(t)(x) + B(t),$$

for all $t \in [a,b]$, $x \in K$, where $A(t) \in L(K, cc(Y))$ and $B \in RW_q([a,b], cc(Y))$.

Theorem 2. *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be normed spaces, K a convex cone in X and $1 < p < q$. If the Nemytskii operator N generated by a set-valued function $H : [a,b] \times K \to cc(Y)$ maps the space $RV_p([a,b], K)$ into the space $RW_q([a,b], cc(Y))$ and if it is globally Lipschitzian, then the set-valued function H satisfies the following condition*

$$H(t,x) = H(t,0) \quad (t \in [a,b],\ x \in K);$$

i.e. the Nemytskii operator is constant.

Proof. Since the Nemytskii operator N is globally Lipschitzian between $RV_p([a,b], K)$ and the space $RW_q([a,b], cc(Y))$, $1 < p < q$, then there exists a constant M, such that

$$D_q(Nu_1, Nu_2) \le M\|u_1 - u_2\|_p \quad (u_1, u_2 \in RV_p([a,b], K)).$$

Let us fix $t, t_0 \in [a,b]$ such that $t_0 < t$. Using the definitions of the operator N and of the metric D_q, we have

$$\begin{aligned}(10)\qquad & D(H(t,u_1(t)) + H(t_0,u_2(t_0)), H(t_0,u_1(t_0)) + H(t,u_2(t)) \le \\ & \le M|t-t_0|^{1-\frac{1}{q}}\|u_1-u_2\|_p, \quad (u_1,u_2 \in RV_p([a,b],K).\end{aligned}$$

Define the function $\alpha : [a,b] \to [0,1]$ by

$$\alpha(\tau) := \begin{cases} 1, & a \le \tau \le t_0, \\ -\dfrac{\tau - t}{t-t_0}, & t_0 \le \tau \le t, \\ 0, & t \le \tau \le b. \end{cases}$$

The function $\alpha \in RV_p[a,b]$ and

$$V_p(\alpha; [a,b]) = \frac{1}{|t-t_0|^{p-1}}.$$

Let us fix $x \in K$ and define the functions $u_i : [a,b] \to K$ $(i = 1,2)$ by

$$(11)\qquad u_1(\tau) := x \quad \tau \in [a,b], \quad u_2(\tau) := \alpha(\tau)x \quad \tau \in [a,b].$$

The functions $u_i \in RV_p([a,b],K)$ $(i = 1,2)$ and

$$\|u_1 - u_2\|_p = \frac{\|x\|}{|t-t_0|^{1-\frac{1}{p}}}.$$

Hence, substituting in the inequality (10) the particular functions u_i $(i = 1,2)$ defined by (11), we obtain

$$D(H(t,x) + H(t_0,x), H(t_0,x) + H(t,0)) \le M\frac{|t-t_0|^{1-\frac{1}{q}}}{|t-t_0|^{1-\frac{1}{p}}}\|x\|.$$

By Lemma 2 and the above inequality, we get

$$D(H(t,x), H(t,0)) \le M\frac{|t-t_0|^{1-\frac{1}{q}}}{|t-t_0|^{1-\frac{1}{p}}}\|x\|.$$

Since $q > p$. Letting $t_0 \uparrow t$ in the above inequality, we have $D(H(t,x), H(t,0)) = 0$, thus for all $t \in [a,b]$ and for all $x \in K$, we get

$$H(t,x) = H(t,0).$$

Theorem 3. *Let* $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ *be normed spaces,* K *be a convex cone in* X *and* $1 < p < \infty$. *If the Nemytskii operator* N *generated by a set-valued function* $H : [a, b] \times K \to cc(Y)$ *maps the space* $RV_p([a, b], K)$ *into the space* $BW([a, b], cc(Y))$ *and if it is globally Lipschitzian, then the left regularization* $H^* : [a, b] \times K \to cc(Y)$ *of the function* H *defined by*

$$H^*(t, x) := \begin{cases} H^-(t, x), & t \in (a, b],\ x \in K, \\ \lim\limits_{s \downarrow a} H(s, x), & t = a,\ x \in K, \end{cases}$$

satisfies the following conditions:

a) *for all* $t \in [a, b]$ *there exists* $M(t)$, *such that*

$$D_1(H^*(t, x), H^*(t, y)) \leq M(t)\|x - y\| \quad (x, y \in X)$$

b) $H^*(t, x) = A(t)x + B(t)$ $(t \in [a, b],\ x \in K)$, *where* $A(t)$ *is linear continuous set-valued function, and* $B \in BW([a, b], cc(Y))$.

Proof. Take $t \in [a, b]$, and define the function $\alpha : [a, b] \to [0, 1]$ by:

$$\alpha(t) := \begin{cases} 1, & a \leq \tau \leq t, \\ \dfrac{\tau - b}{t - b}, & t \leq \tau \leq b. \end{cases}$$

The function $\alpha \in RV_p[a, b]$ and

$$V_p(\alpha, [a, b]) = \frac{1}{|b - t|^{p-1}}.$$

Let us fix $x, y \in K$ and define the functions $u_i : [a, b] \to K$ $(i = 1, 2)$ by

(12) $$u_1(\tau) := x \quad \tau \in [a, b], \quad u_2(\tau) := \alpha(\tau)(y - x) + x, \quad \tau \in [a, b].$$

The functions $u_i \in RV_p([a, b], K)$ $(i = 1, 2)$ and

$$\|u_1 - u_2\|_p = (V_p(\alpha; [a, b]))^{\frac{1}{p}} \|x - y\| = \left(1 + \frac{1}{|b - t|^{1 - \frac{1}{p}}}\right) \|x - y\|.$$

Since the Nemytskii operator N is globally Lipschitzian between $RV_p([a, b], K)$ and $BW([a, b], cc(Y))$, then there exists a constant M, such that

$$D(H(b, u_1(b)) + H(t, u_2(t)), H(t, u_1(t)) + H(b, u_2(b)) \leq M\|u_1 - u_2\|_p.$$

By Lemma 2, substituting the particular functions u_i $(i = 1, 2)$ defined by (12) in the above inequality, we obtain

(13) $$D(H(t, x), H(t, y)) \leq M(t)\|x - y\| \quad (x, y \in K,\ t \in [a, b]),$$

where

$$M(t) := M\left[1 + \frac{1}{|b-t|^{1-\frac{1}{p}}}\right].$$

In the case where $t = b$, by a similar reasoning as above, we obtain that there exists a constant $M(b)$, such that

$$D(H(b,x), H(b,y)) \le M(b)\|x-y\| \quad (x, y \in K). \tag{14}$$

Hence, passing to the limit in the inquality (13) by the inequality (14) and the definition of H^* we have for all $t \in [a,b]$ that there exists $M(t)$, such that

$$D(H^*(t,x), H^*(t,y)) \le M(t)\|x-y\| \quad (x, y \in K).$$

Now we shall proof that H^* satisfies the following equality

$$H^*(t,x) = A(t)x + B(t) \quad (t \in [a,b],\ x \in K),$$

where $A(t)$ is linear continuous set-valued functions, and
$B \in BW([a,b], cc(Y))$.

Let us fix $t, t_0 \in [a,b]$, $n \in \mathbb{N}$ such that $t_0 < t$. Define the partition π_n of the interval $[t_0, t]$ by $\pi_n : a < t_0 < t_1 < \cdots < t_{2n-1} < t_{2n} = t$, where

$$t_i - t_{i-1} = \frac{t - t_0}{2n}, \quad i = 1, 2, \ldots, 2n.$$

The Nemytskii operator N is globally Lipschitzian between
$RVp([a,b], K)$ and $BW([a,b], cc(Y))$, then there exists a constant M, such that

$$\begin{gathered}\sum_{i=1}^{n} D(H(t_{2i}, u_1(t_{2i})) + H(t_{2i-1}, u_2(t_{2i-1})), \\ H(t_{2i-1}, u_1(t_{2i-1})) + H(t_{2i}, u_2(t_{2i}))) \le M\|u_1 - u_2\| \\ (u_1, u_2 \in BV_p([a,b], K)).\end{gathered} \tag{15}$$

Define the function $\alpha : [a,b] \to [0,1]$ in the following way:

$$\alpha(\tau) := \begin{cases} 0, & a \le \tau \le t_0, \\ \dfrac{\tau - t_{i-1}}{t_i - t_{i-1}}, & t_{i-1} \le \tau \le t_i,\ i = 1, 3, \ldots, 2n-1, \\ -\dfrac{\tau - t_i}{t_i - t_{i-1}}, & t_{i-1} \le \tau \le t_i,\ i = 2, 4, \ldots, 2n, \\ 0, & t \le \tau \le b. \end{cases}$$

The function $\alpha \in RV_p([a,b])$ and

$$V_p(\alpha;[a,b]) = \frac{2^p n^p}{|t-t_0|^{p-1}}.$$

Let us fix $x, y \in K$ and define the functions $u_i : [a,b] \to K$ by:

$$\begin{aligned} u_1(\tau) &:= \frac{\alpha(\tau)}{2}x + \left(1 - \frac{\alpha(\tau)}{2}y\right), \quad (\tau \in [a,b]) \\ u_2(\tau) &:= \frac{1+\alpha(\tau)}{2}x + \frac{1-\alpha(\tau)}{2}y, \quad (\tau \in [a,b]). \end{aligned} \tag{16}$$

The functions $u_i \in RV_p([a,b],K)$ $(i=1,2)$ and

$$\|u_1 - u_2\|_p = \frac{\|x-y\|}{2}.$$

Substituting in the inequality (15) the particular functions u_i $(i=1,2)$ defined in (16), we obtain

$$\begin{aligned} \sum_{i=1}^{n} D\left(H(t_{2i-1},x) + H(t_{2i},y), H\left(t_{2i-1}, \frac{x+y}{2}\right) + H\left(t_{2i}, \frac{x+y}{2}\right)\right) \le \\ \le \frac{M}{2}\|x-y\|, \end{aligned} \tag{17}$$

for all $x, y \in K$.

The Nemytskii operator N maps the space $RVp([a,b],K)$ into the space $BW([a,b],cc(Y))$, then for all $z \in K$, the function $H(\cdot,z) \in BW([a,b],cc(Y))$. Letting $t_0 \uparrow t$ in the inequality (17), we get

$$D\left(H^*(t,x) + H^*(t,y), H^*\left(t, \frac{x+y}{2}\right) + H^*\left(t, \frac{x+y}{2}\right)\right) \le \frac{M}{2n}\|x-y\|.$$

Passing to the limit when $n \to \infty$, we get

$$H^*\left(t, \frac{x+y}{2}\right) + H^*\left(t, \frac{x+y}{2}\right) + H^*(t,y) + H^*(t,x), \quad (t \in [a,b],\ x,y \in K).$$

$H^*(t,x)$ is a convex set, then

$$H^*\left(t, \frac{x+y}{2}\right) = \frac{1}{2}(H^*(t,x) + H^*(t,y)) \quad (t \in [a,b], x,y \in K).$$

Thus for every $t \in [a,b]$, the set-valued function $H^*(t,\cdot)$ satisfies the Jensen equation. By Lemma 3 and by the property a) previously established, we get that for all $t \in [a,b]$ there exist an additive set-valued function $A(t) : K \to cc(Y)$ and a set $B(t) \in cc(Y)$, such that

$$H^*(t,x) = A(t)x + B(t) \quad (t \in [a,b],\ x \in K).$$

By the same reasoning as in the proof of Theorem 1, we obtain that $A(t)(\cdot) \in L(K, cc(Y))$ and $B \in BW([a,b]cc(Y))$.

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