Publ. Math. Debrecen
47 / 1-2 (1995), 29–64

Double weighted mean methods equivalent to $(C,1,1)$

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Abstract. We establish sufficient conditions for double weighted mean matrices $(\overline{N},p_{ij})$ and $(C,1,1)$, the double Cesàro matrix of order $(1,1)$, to be equivalent and absolutely equivalent of order $k\ge 1$. The latter equivalence is proved only in the case where $\{p_{ij}\}$ is factorable.

1. Introduction

Let $\{p_n\}$ generate a regular weighted mean method with $p_0>0$. (The condition for regularity is that $\lim P_n=\infty$.) In 1968 Borwein and Cass [3] showed that, if

(i) either $\{p_n\}$ is nondecreasing and $np_n/P_n=\mathcal{O}(1)$,

(ii) or $\{p_n\}$ is nonincreasing and $P_n/np_n=\mathcal{O}(1)$,

then the weighted mean methods $(\overline{N},p_n)$ and $(C,1)$, the Cesàro matrix of order 1, are equivalent. Using the same boundedness conditions, but without the assumptions of monotonicity of $\{p_n\}$, Bor [1] and [2], showed the two methods to be absolutely equivalent for each $k\ge 1$.

It is a natural question to ask if analogous conditions imply the equivalence, or absolute k-equivalence, for double summability, which is the

Mathematics Subject Classification: 40B05, 40G05, 40G99.
Key words and phrases: Comparison theorems, Doubly infinite matrices, Cesàro means, Weighted means.
*This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant #T 016 393.
**This research was begun while the author was a Fulbright Scholar at the Bolyai Institute, University of Szeged, Hungary, during the fall semester of the 1992/93 academic year.

theme of this paper. The first part of this paper is a natural follow-up of [5], and the reader is referred there for basic properties of double summability not specifically described herein.

2. Equivalence of $(\overline{N}, p_{ij})$ and $(C, 1, 1)$

Let $\{p_{ij} : i, j = 0, 1, \dots\}$ be a double sequence of positive numbers,

$$P_{mn} := \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij} \quad (m, n = 0, 1, \dots),$$

and define

$$\Delta_{11} p_{ij} = p_{ij} - p_{i+1,j} - p_{i,j+1} + p_{i+1,j+1}.$$

Let A and B be two doubly infinite matrices. A is said to be stronger than B if every sequence that is assigned a limit by B is assigned the same limit by A.

We shall say that $\{p_{ij}\}$ is

(i) nondecreasing if $p_{ij} \le \min\{p_{i+1,j}, p_{i,j+1}\}$, and
(ii) nonincreasing if $p_{ij} \ge \max\{p_{i+1,j}, p_{i,j+1}\}$, for all i and j $(i, j, = 0, 1, \dots)$.

The following results were established in [5].

Proposition 1. *If $\{p_{ij}\}$ is nonincreasing, $\Delta_{11} p_{ij}$ is of fixed sign, and*

(1) $$\lim_{m,n\to\infty} \frac{1}{P_{mn}} \sum_{i=0}^{m} (j+1) p_{ij} = 0 \quad (j = 0, 1, \dots),$$

(2) $$\lim_{m,n\to\infty} \frac{1}{P_{mn}} \sum_{j=0}^{n} (i+1) p_{ij} = 0 \quad (i = 0, 1, \dots),$$

then $(\overline{N}, p_{ij})$ is stronger than $(C, 1, 1)$.

Proposition 2. *If $\{p_{ij}\}$ is nondecreasing, $\Delta_{11} p_{ij}$ is of fixed sign, and*

(3) $$\lim_{m,n\to\infty} \frac{(m+1) p_{m,j+1}}{P_{mn}} = 0 \quad (j = 0, 1, \dots),$$

(4) $$\lim_{m,n\to\infty} \frac{(n+1) p_{i+1,n}}{P_{mn}} = 0 \quad (i = 0, 1, \dots),$$

(5) $$\sup_{m,n\ge 0} \frac{(m+1)(n+1) p_{mn}}{P_{mn}} < \infty,$$

then $(\overline{N}, p_{ij})$ *is stronger than* $(C, 1, 1)$.

We note that condition (5) implies conditions (3) and (4).

To prove condition (3), using (5), for any fixed j and $n > j$,

$$\frac{(m+1)p_{m,j+1}}{P_{mn}} \leq \frac{(m+1)p_{mn}}{P_{mn}} \leq \frac{M}{n+1} \to 0 \quad \text{as } m, n \to \infty.$$

Condition (4) is established similarly.

Proposition 3. *If* $\{p_{ij}\}$ *is nondecreasing and* $\Delta_{11}(1/p_{ij}\}$ *is of fixed sign, then* $(C, 1, 1)$ *is stronger than* $(\overline{N}, p_{ij}\}$.

This result is established by using Theorem 3 of [5], which is restated here for convenience.

Lemma 1 (see [5, Theorem 3]). *If* $\{p_{ij}/q_{ij}\}$ *is nonincreasing,* $\Delta_{11}(p_{ij}\ /q_{ij})$ *is of fixed sign, and*

$$\lim_{m,n\to\infty} \frac{1}{P_{mn}} \sum_{i=0}^{m} \frac{p_{ij}}{q_{ij}} \sum_{s=0}^{j} q_{is} = 0 \quad (j = 0, 1, \dots), \tag{6}$$

$$\lim_{m,n\to\infty} \frac{1}{P_{mn}} \sum_{j=0}^{n} \frac{p_{ij}}{q_{ij}} \sum_{r=0}^{i} q_{ri} = 0 \quad (i = 0, 1, \dots), \tag{7}$$

$$\sup_{m,n\geq 0} \frac{1}{P_{mn}} \sum_{i=0}^{m} \frac{p_{in}}{q_{in}} \sum_{s=0}^{n} q_{is} < \infty, \tag{8}$$

$$\sup_{m,n\geq 0} \frac{1}{P_{mn}} \sum_{j=0}^{n} \frac{p_{mj}}{q_{mj}} \sum_{r=0}^{m} q_{rj} < \infty, \tag{9}$$

then $(\overline{N}, p_{ij})$ *is stronger than* $(\overline{N}, q_{ij})$.

Proof of Proposition 3. With $p_{ij} = 1$ for each i and j, and q_{ij} replaced by p_{ij}, condition (6) becomes

$$\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \frac{1}{p_{ij}} \sum_{s=0}^{j} p_{is} \leq \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \frac{1}{p_{ij}} (j+1) p_{ij}$$
$$= \frac{j+1}{n+1} \to 0 \quad \text{as } n \to \infty \quad (j = 0, 1, \dots).$$

Condition (7) is proved similarly.

For condition (8),

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\frac{1}{p_{in}}\sum_{s=0}^{n}p_{is} \le \frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\frac{1}{p_{in}}(n+1)p_{in} = 1.$$

Similarly, (9) is dominated by 1.

Proposition 4. *If $\{p_{ij}\}$ is nonincreasing, $\Delta_{11}(1/p_{ij})$ is of fixed sign, and*

$$\sup_{m,n\ge 0}\frac{P_{mn}}{(m+1)(n+1)p_{mn}} < \infty, \tag{10}$$

then $(C,1,1)$ is stronger than $(\overline{N},p_{ij})$.

We shall need Theorem 4 of [5], which is stated below for convenience.

Lemma 2 (see [5, Theorem 4]). *If $\{p_{ij}/q_{ij}\}$ is nondecreasing, $\Delta_{11}(p_{ij}\ /q_{ij})$ is of fixed sign, and*

$$\lim_{m,n\to\infty}\frac{p_{m,j+1}Q_{mj}}{q_{m,j+1}P_{mn}} = 0 \quad (j=0,1,\dots), \tag{11}$$

$$\lim_{m,n\to\infty}\frac{p_{i+1,n}Q_{in}}{q_{i+1,n}P_{mn}} = 0 \quad (i=0,1,\dots), \tag{12}$$

$$\sup_{m,n\ge 0}\frac{p_{mn}Q_{mn}}{q_{mn}P_{mn}} < \infty, \tag{13}$$

then $(\overline{N},p_{ij})$ is stronger than $(\overline{N},q_{ij})$.

Proof of Proposition 4. Using Lemma 2 with $p_{ij}=1$ for all i and j, and q_{ij} replaced with p_{ij}, condition (13) reduces to (10).

For (11), by using (10),

$$\frac{P_{mj}}{p_{m,j+1}(m+1)(n+1)} \le \frac{M(j+2)}{n+1} \to 0 \quad \text{as} \quad n\to\infty \quad (j=0,1,\dots).$$

Condition (12) is established in a similar manner.

Theorem 1. *If $\{p_{ij}\}$ is positive, $\Delta_{11}p_{ij}$ and $\Delta_{11}(1/p_{ij})$ are of fixed sign, and either $\{p_{ij}\}$ is nondecreasing and satisfies (5), or $\{p_{ij}\}$ is nonincreasing and satisfies (1), (2), and (10), then $(\overline{N},p_{ij})$ and $(C,1,1)$ are equivalent.*

Theorem 1 is proved by combining Propositions 1–4.

We note that, if $\{p_{ij}\}$ is nondecreasing and $\Delta_{11}p_{ij} \le 0$, then we necessarily have $\Delta_{11}(1/p_{ij}) \ge 0$ for all i and j. This follows immediately from the inequality

$$\Delta_{11}\frac{1}{p_{ij}} = \frac{p_{i+1,j}-p_{ij}}{p_{ij}p_{i+1,j}} - \frac{p_{i+1,j+1}-p_{j,k+1}}{p_{i,j+1}p_{i+1,j+1}} \ge -\frac{\Delta_{11}p_{ij}}{p_{i,j+1}p_{i+1,j+1}}.$$

Similarly, if $\{p_{ij}\}$ is nonincreasing, then we have

$$\Delta_{11}\frac{1}{p_{ij}} \ge -\frac{\Delta_{11}p_{ij}}{p_{ij}p_{i+1,j}},$$

from which it follows again that $\Delta_{11}(1/p_{ij}) \ge 0$ if $\Delta_{11}p_{ij} \le 0$ for all i and j.

3. Absolute equivalence of $(\overline{N},p_{ij})$ and $(C,1,1)$

We now consider absolute equivalence. A double series $\sum_i\sum_j a_{ij}$ with partial sums s_{mn}, is absolutely k-summable, $k \ge 1$, by a double weighted mean matrix $(\overline{N},p_{ij})$ if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}|\Delta_{11}T_{m-1,n-1}|^k < \infty,$$

where

$$T_{mn} := \frac{1}{P_{mn}}\sum_{i=0}^{m}\sum_{j=0}^{n}p_{ij}s_{ij} \quad (m,n=0,1,\dots).$$

A weighted mean matrix is said to be factorable if $p_{ij} = p_iq_j$ for single sequences $\{p_i\},\{q_j\}$.

Theorem 2. *If $\{p_{ij}\}$ is positive, $\Delta_{11}p_{ij}$ and $\Delta_{11}(1/p_{ij})$ are of fixed sign, either $\{p_{ij}\}$ is nondecreasing and satisfies* (5), *or $\{p_{ij}\}$ is nonincreasing and satisfies* (1), (2), *and* (10), *and $(\overline{N},p_{ij})$ is factorable, then $(\overline{N},p_{ij})$ and $(C,1,1)$ are aboslutely equivalent of order $k \ge 1$.*

Although Theorem 2 is proved only for factorable matrices, the lemmas needed in the proof are true for more general weighted mean matrices, and will be so stated and proved. We shall also indicate where the restriction of factorability seems to be required.

PROOF of Theorem 2, Part 1.

$$
\begin{aligned}
T_{mn} &= \frac{1}{P_{mn}} \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \sum_{\mu=i}^{m} \sum_{\nu=j}^{n} p_{\mu\nu} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} - \frac{1}{P_{mn}} \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} (P_{m,j-1} + P_{i-1,n} - P_{i-1,j-1}),
\end{aligned}
$$

whence

$$
\begin{aligned}
T_{mn} - T_{m-1,n} &= \sum_{j=0}^{n} a_{mj} - \frac{1}{P_{mn}} \sum_{j=0}^{n} a_{mj} (P_{m,j-1} + P_{m-1,n} - P_{m-1,j-1}) \\
&\quad + \left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}} \right) \sum_{i=0}^{m-1} \sum_{j=0}^{n} a_{ij} (P_{i-1,n} - P_{i-1,j-1}) \\
&\quad + \sum_{i=0}^{m-1} \sum_{j=0}^{n} a_{ij} \left(\frac{P_{m-1,j-1}}{P_{m-1,n}} - \frac{P_{m,j-1}}{P_{mn}} \right) \\
&= \left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}} \right) \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} (P_{i-1,n} - P_{i-1,j-1}) \\
&\quad + \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \left(\frac{P_{m-1,j-1}}{P_{m-1,n}} - \frac{P_{m,j-1}}{P_{mn}} \right) \\
&\quad - \left(\frac{1}{P_{m-j,n}} - \frac{1}{P_{mn}} \right) \sum_{j=0}^{n} a_{mj} (P_{m-1,n} - P_{m-1,j-1}) \\
&\quad - \sum_{j=0}^{n} a_{mj} \left(\frac{P_{m-1,j-1}}{P_{m-1,n}} - \frac{P_{m,j-1}}{P_{mn}} \right) + \sum_{j=0}^{n} a_{mj} \\
&\quad - \frac{1}{P_{mn}} \sum_{j=0}^{n} a_{mj} (P_{m,j-1} + P_{m-1,n} - P_{m-1,j-1}).
\end{aligned}
$$

Note that all of the terms involving $\sum_{j=0}^{n} a_{mj}$ add up to zero. Thus,

$$
\begin{aligned}
T_{mn} - T_{m-1,n} &= \left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}} \right) \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} (P_{i-1,n} - P_{i-1,j-1}) \\
&\quad + \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \left(\frac{P_{m-1,j-1}}{P_{m-1,n}} - \frac{P_{m,j-1}}{P_{mn}} \right).
\end{aligned}
$$

Analogously,

$$T_{m,n-1} - T_{m-1,n-1} = \left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m,n-1}}\right)$$
$$\times \sum_{i=0}^{m}\sum_{j=0}^{n-1} a_{ij}(P_{i-1,n-1} - P_{i-1,j-1}) + \sum_{i=0}^{m}\sum_{j=0}^{n-1} a_{ij}\left(\frac{P_{m-1,j-1}}{P_{m-1,n-1}} - \frac{P_{m,j-1}}{P_{m,n-1}}\right).$$

Combining the last two expressions yields

$$\begin{aligned}
\Delta_{11}T_{m-1,n-1} &= \left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}}\right)\sum_{i=0}^{m} a_{in}(P_{i-1,n} - P_{i-1,n-1}) \\
&\quad + \left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}} - \frac{1}{P_{m-1,n-1}} + \frac{1}{P_{m,n-1}}\right)\sum_{i=0}^{m}\sum_{j=0}^{n-1} a_{ij}(-P_{i-1,j-1}) \\
&\quad + \sum_{i=0}^{m}\sum_{j=0}^{n-1} a_{ij}\Bigg[P_{i-1,n}\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}}\right) \\
&\quad - P_{i-1,n-1}\left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m,n-1}}\right)\Bigg] \\
&\quad + \sum_{i=0}^{m} a_{in}\left(\frac{P_{m-1,n-1}}{P_{m-1,n}} - \frac{P_{m,n-1}}{P_{mn}}\right) \\
&\quad + \sum_{i=0}^{m}\sum_{j=0}^{n-1} a_{ij}\left(\frac{P_{m-1,j-1}}{P_{m-1,n}} - \frac{P_{m,j-1}}{P_{mn}} - \frac{P_{m-1,j-1}}{P_{m-1,n-1}} + \frac{P_{m,j-1}}{P_{m,n-1}}\right) \\
&= \Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}P_{i-1,j-1} \\
&\quad + \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\Bigg[P_{m-1,j-1}\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{m-1,n-1}}\right) \\
&\quad + P_{m,j-1}\left(\frac{1}{P_{m,n-1}} - \frac{1}{P_{mn}}\right)\Bigg] \\
&\quad + \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\Bigg[P_{i-1,n}\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}}\right) \\
&\quad - P_{i-1,n-1}\left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m,n-1}}\right)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&- \Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\sum_{i=0}^{m} a_{in}P_{i-1,n-1} \\
&- \sum_{i=0}^{m} a_{in}\left[P_{m-1,n-1}\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{m-1,n-1}}\right)\right. \\
&\left. + P_{m,n-1}\left(\frac{1}{P_{m,n-1}} - \frac{1}{P_{mn}}\right)\right] \\
&- \sum_{i=0}^{m} a_{in}\left[P_{i-1,n}\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}}\right)\right. \\
&\left. - P_{i-1,n-1}\left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m,n-1}}\right)\right] \\
&+ \left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}}\right)\sum_{i=0}^{m} a_{in}(P_{i-1,n} - P_{i-1,n-1}) \\
&+ \sum_{i=0}^{m} a_{in}\left(\frac{P_{m-1,n-1}}{P_{m-1,n}} - \frac{P_{m,n-1}}{P_{mn}}\right).
\end{aligned}
$$

Note that the terms involving $\sum_{i=0}^{m} a_{in}$ add up again to zero. Thus,

$$
\begin{aligned}
\Delta_{11}T_{m-1,n-1} = {} & \Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}P_{i-1,j-1} \\
& + \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\left[P_{m-1,j-1}\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{m-1,n-1}}\right)\right. \\
& \left. + P_{m,j-1}\left(\frac{1}{P_{m,n-1}} - \frac{1}{P_{mn}}\right)\right] \\
& + \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\left[P_{i-1,n}\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}}\right)\right. \\
& \left. - P_{i-1,n-1}\left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m,n-1}}\right)\right].
\end{aligned}
\tag{14}
$$

If $p_{ij} = p_i q_j$, then $P_{mn} = P_n Q_n$, so that each of the quantities in brackets in (14) adds to zero, so the second two series disappear.

Introduce the quantity

$$t_{mn} := \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} ij a_{ij} \quad (m, n = 0, 1, \dots).$$

It is clear that

$$(m+1)(n+1)t_{mn} - m(n+1)t_{m-1,n} = \sum_{j=0}^{n} mj a_{mj},$$

whence

$$\Delta_{11}(mn\, t_{m-1,n-1}) = mn\, a_{mn}. \tag{15}$$

Substituting (15) into (14) yields

$$\Delta_{11} T_{m-1,n-1} = \Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right) \sum_{i=1}^{m} \sum_{j=1}^{n} P_{i-1,j-1} \frac{\Delta_{11}(ij\, t_{i-1,j-1})}{ij}$$

$$= \Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right) \sum_{i=1}^{m} \left[\sum_{j=1}^{n} P_{i-1,j-1} t_{i-1,j-1} \right.$$

$$- \sum_{j=1}^{n} P_{i-1,j-1} \left(\frac{j+1}{j}\right) t_{i-1,j} - \sum_{j=1}^{n} P_{i-1,j-1} \left(\frac{i+1}{i}\right) t_{i,j-1}$$

$$\left. + \sum_{j=1}^{n} P_{i-1,j-1} \left(\frac{i+1}{i}\right) \left(\frac{j+1}{j}\right) t_{ij} \right]$$

$$= \Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right) \sum_{i=1}^{m} \left[\sum_{j=1}^{n-1} t_{i-1,j} \left(P_{i-1,j} - \left(\frac{j+1}{j}\right) P_{i-1,j-1}\right) \right.$$

$$\left. - P_{i-1,n-1} \left(\frac{n+1}{n}\right) t_{i-1,n} \right]$$

$$\times \left[- \left(\frac{i+1}{i}\right) \left\{ \sum_{j=1}^{n-1} t_{ij} \left(P_{i-1,j} - \left(\frac{j+1}{j}\right) P_{i-1,j-1}\right) \right\} \right.$$

$$\left. + \left(\frac{i+1}{i}\right) \left(\frac{n+1}{n}\right) P_{i-1,n-1} t_{in} \right]$$

$$= \Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right) \sum_{i=1}^{m} \left[\sum_{j=1}^{n-1} \left(t_{i-1,j} - t_{ij}\left(\frac{i+1}{i}\right)\right) \right.$$

$$
\begin{aligned}
&\times \left(P_{i-1,j} - \left(\frac{j+1}{j}\right) P_{i-1,j-1}\right) \\
&- P_{i-1,n-1}\left(\frac{n+1}{n}\right)\left(t_{i-1,n} - \left(\frac{i+1}{i}\right) t_{in}\right)\Bigg] \\
= \Delta_{11}&\left(\frac{1}{P_{m-1,n-1}}\right)\left[\sum_{j=1}^{n-1}\left\{\sum_{i=1}^{m-1} t_{ij}\left[P_{ij} - \left(\frac{j+1}{j}\right) P_{i,j-1} + \right.\right.\right. \\
&- \left(\frac{i+1}{i}\right)\left(P_{i-1,j} - \left(\frac{j+1}{j}\right) P_{i-1,j-1}\right)\Bigg] \\
&- t_{mj}\left(\frac{m+1}{m}\right)\left(P_{m-1,j} - \left(\frac{j+1}{j}\right) P_{m-1,j-1}\right) \\
&- \left(\frac{n+1}{n}\right)\left[\sum_{i=1}^{m-1}\left(P_{i,n-1} - \left(\frac{i+1}{i}\right) P_{i-1,n-1}\right) t_{in}\right] \\
&\left.\left. + \left(\frac{n+1}{n}\right)\left(\frac{m+1}{m}\right) P_{m-1,n-1} t_{mn}\right\}\right] \\
= I_1 &+ I_2 + I_3, \text{ say.}
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
P_{ij} - &\left(\frac{j+1}{j}\right) P_{i,j-1} - \left(\frac{i+1}{i}\right)\left(P_{i-1,j} - \left(\frac{j+1}{j}\right) P_{i-1,j-1}\right) \\
&= P_{ij} - P_{i,j-1} - \frac{P_{i,j-1}}{j} - P_{i-1,j} + P_{i-1,j-1} \\
&\quad + \frac{P_{i-1,j-1}}{j} - \frac{P_{i-1,j}}{i} + \frac{P_{i-1,j-1}}{i} + \frac{P_{i-1,j-1}}{ij} \\
&= \Delta_{11} P_{i-1,j-1} - \frac{1}{j}(P_{i,j-1} - P_{i-1,j-1}) \\
&\quad - \frac{1}{i}(P_{i-1,j} - P_{i-1,j-1}) + \frac{1}{ij} P_{i-1,j-1},
\end{aligned}
$$

we may write $I_1 = \sum_{i=1}^{4} I_{1i}$.

Lemma 3. *If* $\{p_{ij}\}$ *is either nondecreasing and satisfies* (5), *or is nonincreasing, then*

$$
\frac{P_{mn}}{P_{m-1,n}}, \quad \frac{P_{mn}}{P_{m,n-1}} = \mathcal{O}(1).
$$

Proof. Suppose $\{p_{mn}\}$ is nondecreasing and satisfies (5). Clearly,

$$\frac{P_{mn}}{P_{m-1,n}} = \frac{P_{mn}}{P_{mn} - \sum_{j=0}^{n} p_{mj}} = \frac{1}{1 - \left(\sum_{j=0}^{n} p_{mj}\right)/P_{mn}}.$$

But

$$\frac{\sum_{j=0}^{n} p_{mj}}{P_{mn}} \le \frac{(n+1)p_{mn}}{P_{mn}} = \frac{\mathcal{O}(1)}{m+1}.$$

Therefore

$$1 - \frac{\sum_{j=0}^{n} p_{mj}}{P_{mn}} \ge 1 - \frac{\mathcal{O}(1)}{m+1}$$

and

$$0 \le \frac{P_{mn}}{P_{m-1,n}} \le \frac{1}{1 - \mathcal{O}(1)/(m+1)}.$$

Since the expression on the right is the mth term of a convergent sequence in one variable, it is bounded.

Suppose $\{p_{mn}\}$ is nonincreasing. Then

$$\frac{\sum_{j=0}^{n} p_{mj}}{P_{mn}} = \frac{(m+1)\sum_{j=0}^{n} p_{mj}}{(m+1)P_{mn}} \le \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij}}{(m+1)P_{mn}} = \frac{1}{m+1},$$

and the result follows.

The case for $P_{mn}/P_{m,n-1}$ is proved similarly.

Lemma 4. *Let $\Delta_{11}(1/P_{m-1,n-1})$ be of fixed sign. If either $\{p_{mn}\}$ is nondecreasing and satisfies* (5), *or is nonincreasing and satisfies* (10), *then*

$$\frac{P_{mn}P_{m-1,n-1}}{p_{mn}} \Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right) = \mathcal{O}(1).$$

Proof. It is easy to see that

$$\frac{P_{mn}P_{m-1,n-1}}{p_{mn}} \left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m,n-1}} - \frac{1}{P_{m-1,n}} + \frac{1}{P_{mn}}\right)$$
$$= \frac{P_{mn}P_{m-1,n-1}}{p_{mn}} \left(\frac{\sum_{j=0}^{n-1} p_{mj}}{P_{m-1,n-1}P_{m,n-1}} - \frac{\sum_{j=0}^{n} p_{mj}}{P_{m-1,n}P_{mn}}\right)$$
$$= \frac{P_{mn}P_{m-1,n-1}}{p_{mn}} \sum_{j=0}^{n-1} p_{mj} \left(\frac{1}{P_{m-1,n-1}P_{m,n-1}} - \frac{1}{P_{m-1,n}P_{mn}}\right)$$
$$- \frac{P_{m-1,n-1}}{P_{m-1,n}}.$$

The last term is clearly $\mathcal{O}(1)$. The first term is equal to

$$\frac{\sum_{j=0}^{n-1} p_{mj}}{p_{mn}P_{m-1,n}P_{m,n-1}}(P_{m-1,n}P_{mn} - P_{m-1,n-1}P_{m,n-1})$$

$$= \frac{\sum_{j=0}^{n-1} p_{mj}}{p_{mn}P_{m-1,n}P_{m,n-1}}\left[P_{m-1,n}\left(P_{m,n-1} + \sum_{i=0}^{m} p_{in}\right) - P_{m-1,n-1}P_{m,n-1}\right]$$

$$= \frac{\sum_{j=0}^{n-1} p_{mj}}{p_{mn}P_{m-1,n}P_{m,n-1}}\left[P_{m,n-1}(P_{m-1,n} - P_{m-1,n-1}) + P_{m-1,n}\sum_{i=0}^{m} p_{in}\right]$$

$$= \frac{\sum_{j=0}^{n-1} p_{mj}}{p_{mn}P_{m-1,n}P_{m,n-1}}\left[P_{m,n-1}\sum_{i=0}^{m-1} p_{in} + P_{m-1,n}\sum_{i=0}^{m} p_{in}\right]$$

$$= \frac{\sum_{j=0}^{n-1} p_{mj}\sum_{i=0}^{m-1} p_{in}}{p_{mn}P_{m-1,n}} + \frac{\sum_{j=0}^{n-1} p_{mj}\sum_{i=0}^{m} p_{in}}{p_{mn}P_{m,n-1}} = J_1 + J_2, \text{say.}$$

If $\{p_{mn}\}$ is nondecreasing and satisfies (5), then, by using Lemma 3,

$$J_1 \le \frac{(np_{mn})m\,p_{m-1,n}}{p_{mn}P_{m-1,n}} = \mathcal{O}(1)\frac{nm\,p_{mn}}{P_{mn}} = \mathcal{O}(1)$$

and

$$J_2 \le \frac{n\,p_{mn}(m+1)p_{mn}}{p_{mn}P_{m,n-1}} = \mathcal{O}(1)\frac{mn\,p_{mn}}{P_{mn}} = \mathcal{O}(1).$$

If $\{p_{mn}\}$ is nonincreasing and satisfies (10), then

$$J_1 \le \frac{\left(\sum_{i=0}^{m}\sum_{j=0}^{n-1} p_{ij}\right)\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n} p_{ij}\right)}{(m+1)p_{mn}P_{m-1,n}(n+1)} \le \frac{P_{mn}}{(m+1)(n+1)p_{mn}} = \mathcal{O}(1)$$

and

$$J_2 \le \frac{\left(\sum_{i=0}^{m}\sum_{j=0}^{n-1} p_{ij}\right)\left(\sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij}\right)}{(m+1)p_{mn}P_{m,n-1}(n+1)} = \frac{P_{mn}}{(m+1)(n+1)p_{mn}} = \mathcal{O}(1).$$

To complete the proof of Theorem 2, Part 1 it is sufficient to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)I_t\right|^k < \infty \quad \text{for} \quad t = 1, 2, 3.$$

Step 1. By Lemma 4,

$$
\begin{aligned}
K_{11} := & \sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}|I_{11}|^k \\
&= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|\left[\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}|t_{ij}|^k p_{ij}\right] \\
&\qquad \times\left[\frac{1}{P_{m-1,n-1}}\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}p_{ij}\right]^{k-1} \\
&= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}|t_{ij}|^k p_{ij} \\
&= \mathcal{O}(1)\sum_{i=1}^{r}\sum_{j=1}^{s}|t_{ij}|^k p_{ij}\sum_{m=i+1}^{r+1}\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|.
\end{aligned}
$$

Lemma 5. *If $\Delta_{11}(1/P_{m-1,n-1})$ is of fixed sign, then*

$$\sum_{m=i+1}^{r+1}\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| = \frac{\mathcal{O}(1)}{P_{ij}}.$$

Proof. In fact, a simple argument gives

$$
\begin{aligned}
&\sum_{m=i+1}^{r+1}\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| \\
&= \left|\sum_{m=i+1}^{r+1}\sum_{n=j+1}^{s+1}\left(\frac{1}{P_{m-1,n-1}}-\frac{1}{P_{m-1,n}}-\frac{1}{P_{m,n-1}}+\frac{1}{P_{mn}}\right)\right| \\
&= \left|\sum_{m=i+1}^{r+1}\left(\frac{1}{P_{m-1,j}}-\frac{1}{P_{m-1,s+1}}-\frac{1}{P_{m,j}}+\frac{1}{P_{m,s+1}}\right)\right| \\
&= \left|\frac{1}{P_{ij}}-\frac{1}{P_{r+1,j}}-\frac{1}{P_{i,s+1}}+\frac{1}{P_{r+1,s+1}}\right| \\
&< \max\left\{\frac{1}{P_{ij}}+\frac{1}{P_{r+1,s+1}},\frac{1}{P_{r+1,j}}+\frac{1}{P_{i,s+1}}\right\} \le \frac{2}{P_{ij}}.
\end{aligned}
$$

Therefore, by Lemma 5 and (5),

$$K_{11} = \mathcal{O}(1) \sum_{i=1}^{r} \sum_{j=1}^{s} |t_{ij}|^k \frac{p_{ij}}{P_{ij}} = \mathcal{O}(1) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{|t_{ij}|^k}{ij} = \mathcal{O}(1).$$

Using Lemma 4,

$$\begin{aligned} K_{12} &:= \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left(\frac{P_{mn}}{p_{mn}}\right)^{k-1} |I_{12}|^k \\ &= \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| \left[\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \frac{|t_{ij}|^k}{j} \sum_{\nu=0}^{j-1} p_{i\nu}\right] \\ &\quad \times \left[\frac{1}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \sum_{\nu=0}^{j-1} \frac{p_{i\nu}}{j}\right]^{k-1}. \end{aligned}$$

Set

$$A_{12} := \frac{1}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \sum_{\nu=0}^{j-1} \frac{p_{i\nu}}{j}.$$

If $\{p_{ij}\}$ is nondecreasing, $\sum_{\nu=0}^{j-1} p_{i\nu} \le j p_{ij}$ and

$$A_{12} \le \frac{1}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} p_{ij} \le 1.$$

If $\{p_{ij}\}$ is nonincreasing then, by using (1),

$$\begin{aligned} A_{12} &= \frac{1}{P_{m-1,n-1}} \sum_{j=1}^{n-1} \frac{1}{j} \sum_{i=1}^{m-1} \sum_{\nu=0}^{j-1} p_{i\nu} \le \frac{1}{P_{m-1,n-1}} \sum_{j=1}^{n-1} \frac{1}{j} P_{m-1,j-1} \\ &= \frac{\mathcal{O}(1)}{P_{m-1,n-1}} \sum_{j=1}^{n-1} (m-1) p_{m-1,j-1} = \frac{\mathcal{O}(1)(m-1)}{P_{m-1,n-1}} \sum_{j=1}^{n-1} p_{m-1,j-1} \\ &= \frac{\mathcal{O}(1)}{P_{m-1,n-1}} \sum_{i=0}^{m-1} \sum_{j=1}^{n-1} p_{i,j-1} = \mathcal{O}(1). \end{aligned}$$

By Lemma 5,

$$K_{12} = \mathcal{O}(1) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{|t_{ij}|^k}{j} \sum_{\nu=0}^{j-1} p_{i\nu} \sum_{m=i+1}^{r+1} \sum_{n=j+1}^{s+1} \left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|$$

$$= \mathcal{O}(1) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{|t_{ij}|^k}{jP_{ij}} \sum_{\nu=0}^{j-1} p_{i\nu}.$$

Lemma 6. *If $\{p_{ij}\}$ is either nondecreasing and satisfies* (5) *or is nonincreasing, then*

$$\text{(a)} \quad \frac{1}{P_{ij}} \sum_{\nu=0}^{j-1} p_{i\nu} = \frac{\mathcal{O}(1)}{i}, \qquad \text{(b)} \quad \frac{1}{P_{ij}} \sum_{\mu=0}^{i-1} p_{\mu j} = \frac{\mathcal{O}(1)}{j}.$$

Proof. If $\{p_{jk}\}$ is nondecreasing, then by (5),

$$\frac{q}{P_{ij}} \sum_{\nu=0}^{j-1} p_{i\nu} \le \frac{jp_{ij}}{P_{ij}} = \frac{\mathcal{O}(1)}{i}.$$

If $\{p_{ij}\}$ is nonincreasing, then

$$\frac{1}{P_{ij}} \sum_{\nu=0}^{j-1} p_{i\nu} = \frac{1}{iP_{ij}} i \sum_{\nu=0}^{j-1} p_{i\nu} \le \frac{1}{iP_{ij}} \sum_{j=0}^{i} \sum_{\nu=0}^{j-1} p_{i\nu} = 1.$$

Part (b) is proved similarly.

By Lemma 6, $K_{12} = \mathcal{O}(1)$.

Using Lemma 4,

$$K_{13} := \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left(\frac{P_{mn}}{p_{mn}} \right)^{k-1} |I_{13}|^k$$
$$= \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left| \Delta_{11} \left(\frac{1}{P_{m-1,n-1}} \right) \right| \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \frac{|t_i j|^k}{i} \sum_{\mu=0}^{i-1} p_{\mu j}$$
$$\times \left[\frac{1}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \frac{1}{i} \sum_{\mu=0}^{i-1} p_{\mu j} \right]^{k-1}.$$

If $\{p_{ij}\}$ is nondecreasing, then $\sum_{\mu=0}^{i-1} p_{\mu j} \le ip_{ij}$ and the quantity in brackets is ≤ 1. If $\{p_{ij}\}$ is nonincreasing, then the quantity in brackets is dominated by

$$\frac{1}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \frac{1}{i} P_{i-1,n-1} = \frac{\mathcal{O}(1)}{P_{m-1,n-1}} \sum_{i=1}^{m-1} (n-1) p_{i-1,n-1}$$
$$= \frac{\mathcal{O}(1)}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \sum_{j=0}^{n-1} p_{i-1,j} = \mathcal{O}(1).$$

Using Lemmas 5 and 6,

$$K_{13} = \mathcal{O}(1) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{|t_{ij}|^k}{i} \sum_{\mu=0}^{i-1} p_{\mu j} \sum_{m=i+1}^{r+1} \sum_{n=j+1}^{s} \left| \Delta_{11} \left(\frac{1}{P_{m-1,n-1}} \right) \right|$$
$$= \mathcal{O}(1) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{|t_{ij}|^k}{i} \frac{1}{P_{ij}} \sum_{\mu=0}^{i-1} p_{\mu j} = \mathcal{O}(1) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{|t_{ij}|^k}{ij} = \mathcal{O}(1).$$

Using Lemma 4,

$$K_{14} := \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left(\frac{P_{mn}}{p_{mn}} \right)^{k-1} |I_{14}|^k$$
$$= \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left| \Delta_{11} \left(\frac{1}{Pm-1, n-1} \right) \right| \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \frac{|t_{ij}|^k}{ij} P_{i-1,j-1}$$
$$\times \left[\frac{1}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \frac{P_{i-1,j-1}}{ij} \right]^{k-1} .$$

From (10) the quantity in brackets is

$$\frac{\mathcal{O}(1)}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} p_{ij} = \mathcal{O}(1).$$

Using Lemma 5,

$$K_{14} = \mathcal{O}(1) \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \frac{|t_{ij}|^k}{ij} P_{i-1,j-1} \sum_{m=i+1}^{r+1} \sum_{n=j+1}^{s+1} \left| \Delta_{11} \left(\frac{1}{P_{m-1,n-1}} \right) \right|$$
$$= \mathcal{O}(1).$$

Step 2. We may write $I_2 = I_{21} + I_{22}$. Using Lemma 4,

$$
\begin{aligned}
K_{21} &:= \sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}|I_{21}|^k \\
&= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|\left(\frac{m+1}{m}\right)^k \\
&\qquad\times\sum_{j=1}^{n-1}|t_{mj}|^k(P_{m-1,j}-P_{m-1,j-1})\left[\frac{1}{P_{m-1,n-1}}\sum_{j=1}^{n-1}\sum_{\mu=0}^{m-1}p_{\mu j}\right]^{k-1} \\
&= \mathcal{O}(1)\sum_{j=1}^{n-1}\sum_{m=2}^{r+1}|t_{mj}|^k\sum_{\mu=0}^{m-1}p_{\mu j}\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|.
\end{aligned}
$$

Lemma 7. *Let $\{p_{mn}\}$ be such that $\Delta_{11}(1/P_{m-1,n-1})$ is of fixed sign. If either $\{p_{mn}\}$ is nondecreasing and satisfies* (5), *or $\{p_{mn}\}$ is nonincreasing, then*

$$\text{(a)}\qquad \sum_{i=0}^{m-1}p_{ij}\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| = \frac{\mathcal{O}(1)}{jm}$$

and

$$\text{(b)}\qquad \sum_{j=0}^{n-1}p_{ij}\sum_{m=i+1}^{r+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| = \frac{\mathcal{O}(1)}{in}.$$

Proof. Clearly

$$
\begin{aligned}
\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| &= \left|\sum_{n=j+1}^{s+1}\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| \\
&= \left|\frac{1}{P_{m-1,j}}-\frac{1}{P_{m-1,s+1}}-\frac{1}{P_{mj}}+\frac{1}{P_{m,s+1}}\right| \\
&\le \max\left\{\frac{P_{mj}-P_{m-1,j}}{P_{mj}P_{m-1,j}},\frac{P_{m,s+1}-P_{m-1,s+1}}{P_{m,s+1}P_{m-1,s+1}}\right\}.
\end{aligned}
$$

If $\{p_{mn}\}$ is nondecreasing, then, from (5),

$$\frac{1}{P_{mj}P_{m-1,j}}\sum_{i=0}^{m-1}p_{ij}\sum_{\nu=0}^{j}p_{m\nu} \le \frac{mp_{n-1,j}(j+1)p_{mj}}{p_{mj}P_{m-1,j}} = \frac{\mathcal{O}(1)}{jm},$$

and

$$\frac{1}{P_{m,s+1}P_{m-1,s+1}}\sum_{i=0}^{m-1}p_{ij}\sum_{\nu=0}^{s+1}p_{m\nu}$$
$$\leq \frac{mp_{n-1,j}(s+2)p_{m,s+1}}{P_{m,s+1}P_{m,s+1}} = \frac{\mathcal{O}(1)}{(s+1)m} = \frac{\mathcal{O}(1)}{jm},$$

since $j \leq s+1$.

If $\{p_{mn}\}$ is nonincreasing, then

$$\frac{1}{P_{mj}P_{m-1,j}}\sum_{i=0}^{m-1}p_{ij}\sum_{\nu=0}^{j}p_{m\nu} = \frac{1}{jmP_{mj}P_{m-1,j}}j\sum_{i=0}^{m-1}p_{ij}m\sum_{\nu=0}^{j}p_{m\nu}$$
$$\leq \frac{1}{jmP_{mj}P_{m-1,j}}\left(\sum_{i=0}^{m-1}\sum_{\nu=0}^{j}p_{i\nu}\right)\left(\sum_{i=0}^{m-1}\sum_{\nu=0}^{j}p_{i\nu}\right) \leq \frac{1}{jm}$$

and

$$\frac{1}{P_{m,s+1}P_{m-1,s+1}}\sum_{i=0}^{m-1}p_{ij}\sum_{\nu=0}^{s+1}p_{m\nu}$$
$$= \frac{1}{jmP_{m,s+1}P_{m-1,s+1}}j\sum_{i=0}^{m-1}p_{ij}m\sum_{\nu=0}^{s+1}p_{m\nu}$$
$$\leq \frac{1}{jmP_{m,s+1}P_{m-1,s+1}}\sum_{i=0}^{m-1}\sum_{\nu=0}^{j}p_{i\nu}\sum_{j=0}^{m-1}\sum_{\nu=0}^{s+1}p_{i\nu} \leq \frac{1}{jm}.$$

Part (b) is proved similarly.

Using Lemma 7, $K_{21} = \mathcal{O}(1)$.

Using Lemma 4, (10), and Lemma 6,

$$K_{22} := \sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}|I_{22}|^k$$
$$= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|\left(\frac{m+1}{m}\right)^k\sum_{j=1}^{n-1}\frac{1}{j}P_{m-1,j-1}|t_{mj}|^k$$
$$\times\left[\frac{1}{P_{m-1,n-1}}\sum_{j=1}^{n-1}\frac{1}{j}P_{m-1,j-1}\right]^{k-1}$$

$$= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|\sum_{j=1}^{n-1}\frac{1}{j}P_{m-1,j-1}|t_{mj}|^k$$

$$= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{j=1}^{s}\frac{|t_{mj}|^k}{j}P_{m-1,j-1}\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|.$$

Lemma 8. *Let $\{p_{mn}\}$ be such that $\Delta_{11}(1/P_{m-1,n-1})$ is of fixed sign. If either $\{p_{mn}\}$ is nondecreasing, or $\{p_{mn}\}$ is nonincreasing and satisfies* (10), *then*

$$\text{(a)} \qquad P_{m-1,j-1}\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| = \frac{\mathcal{O}(1)}{m}$$

and

$$\text{(b)} \qquad P_{i-1,n-1}\sum_{m=i+1}^{r+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| = \frac{\mathcal{O}(1)}{n}.$$

Proof. Let

$$M_1 := P_{m-1,j-1}\sum_{n=j+1}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|$$
$$= \mathcal{O}(1)P_{m-1,j-1}\left[jm\sum_{i=0}^{m-1}p_{ij}\right]^{-1}.$$

If $\{p_{mn}\}$ is nondecreasing, then

$$j\sum_{i=0}^{m-1}p_{ij} \geq \sum_{i=0}^{m-1}\sum_{v=0}^{j-1}p_{iv} = P_{m-1,j-1},$$

and $M_1 = \mathcal{O}(1)m^{-1}$.

If $\{p_{mn}\}$ is nonincreasing, then

$$j\sum_{i=0}^{m-1}p_{ij} \geq jmp_{mj},$$

and, from (10), $M_1 = \mathcal{O}(1)m^{-1}$.

Using Lemma 8, $K_{22} = \mathcal{O}(1)$.

Step 3. We may write

$$I_3 = -\left(\frac{n+1}{n}\right)\sum_{i=1}^{m-1}(P_{i,n-1}-P_{i-1,n-1})t_{in} + \left(\frac{n+1}{n}\right)\sum_{i=1}^{m-1}P_{i-1,n-1}\frac{t_{in}}{i}$$

$$+\left(\frac{n+1}{n}\right)\left(\frac{m+1}{m}\right)P_{m-1,n-1}t_{mn}$$

$$= I_{31}+I_{32}+I_{33}, \text{ say.}$$

Using Lemmas 4 and 7,

$$\begin{aligned}
K_{31} &:= \sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}|I_{31}|^k \\
&= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|\left(\frac{n+1}{n}\right)^k \\
&\quad\times\sum_{i=1}^{m-1}(P_{i,n-1}-P_{i-1,n-1})|t_{in}|^k \\
&\quad\times\left[\frac{1}{P_{m-1,n-1}}\sum_{i=1}^{m-1}\sum_{j=0}^{n-1}p_{ij}\right]^{k-1} \\
&= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|\sum_{i=1}^{m-1}\sum_{j=0}^{n-1}p_{ij}|t_{in}|^k \\
&= \mathcal{O}(1)\sum_{i=1}^{r}\sum_{n=2}^{s+1}|t_{in}|^k\sum_{j=0}^{n-1}p_{ij}\sum_{m=i+1}^{r+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right| \\
&= \mathcal{O}(1)\sum_{i=1}^{r}\sum_{n=2}^{s+1}\frac{|t_{in}|^k}{in} = \mathcal{O}(1).
\end{aligned}$$

Using Lemma 4, (10), Lemmas 6 and 8,

$$\begin{aligned}
K_{32} &:= \sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}|I_{32}|^k \\
&= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left|\Delta_{11}\left(\frac{1}{P_{m-1,n-1}}\right)\right|\left(\frac{n+1}{n}\right)^k\sum_{i=1}^{m-1}P_{i-1,n-1}\frac{|t_{in}|^k}{i}
\end{aligned}$$

$$\times \left[\frac{1}{P_{m-1,n-1}} \sum_{i=1}^{m-1} \frac{P_{i-1,n-1}}{i} \right]^{k-1}$$

$$= \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left| \Delta_{11} \left(\frac{1}{P_{m-1,n-1}} \right) \right| \sum_{i=1}^{m-1} P_{i-1,n-1} \frac{|t_{in}|^k}{i}$$

$$= \mathcal{O}(1) \sum_{i=1}^{r} \sum_{n=2}^{s+1} \frac{|t_{in}|^k}{i} P_{i-1,n-1} \sum_{m=i+1}^{r+1} \left| \Delta_{11} \left(\frac{1}{P_{m-1,n-1}} \right) \right|$$

$$= \mathcal{O}(1) \sum_{i=1}^{r} \sum_{n=2}^{s+1} \frac{|t_{in}|^k}{in} = \mathcal{O}(1).$$

Using Lemmas 4 and 5, and (5),

$$K_{33} := \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left(\frac{P_{mn}}{p_{mn}} \right)^{k-1} |I_{33}|^k$$

$$= \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left| \Delta_{11} \left(\frac{1}{P_{m-1,n-1}} \right) \right|$$

$$\times \left[\left(\frac{n+1}{n} \right) \left(\frac{m+1}{m} \right) P_{m-1,n-1} |t_{mn}| \right]^k / P_{m-1,n-1}^{k-1}$$

$$= \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left| \Delta_{11} \left(\frac{1}{P_{m-1,n-1}} \right) \right| P_{m-1,n-1} |t_{mn}|^k$$

$$= \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \frac{p_{mn}}{P_{mn}} |t_{mn}|^k = \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \frac{|t_{mn}|^k}{mn} = \mathcal{O}(1).$$

Problem 1. We have now shown that $(\overline{N},p_{jk})$ is k-absolutely stronger than $(C,1,1)$, under the added assumption that $\{p_{jk}\}$ is factorable. Before proving the converse, we shall point out the difficulties that arise without that assumption.

From (14), if $\{p_{jk}\}$ is not factorable, substituting (15) into the last two series and performing the standard summation by parts yields

$$\sum_{j=1}^{n-1} \left[\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{m-1,n-1}} \right) \left(P_{m-1,j} - \left(\frac{j+1}{j} \right) P_{m-1,j-1} \right) \right.$$

$$\left. + \left(\frac{1}{P_{m,n-1}} - \frac{1}{P_{mn}} \right) \left(P_{mj} - \left(\frac{j+1}{j} \right) P_{m,j-1} \right) \right]$$

$$\times \left[-\sum_{i=1}^{m-1} \frac{t_{ij}}{i} - \left(\frac{m+1}{m}\right) t_{mj} \right]$$

$$- \left(\frac{n+1}{n}\right) \left[\frac{P_{m-1,n-1}}{P_{m-1,n}} - \frac{P_{m,n-1}}{P_{mn}}\right] \left[-\sum_{i=1}^{m-1} \frac{t_{in}}{i} + \left(\frac{m+1}{m}\right) t_{mn} \right]$$

$$+ \sum_{i=1}^{m-1} \left[\left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{mn}}\right) \left(P_{in} - \left(\frac{i+1}{i}\right) P_{i-1,n}\right) \right. \tag{16}$$

$$\left. - \left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m,n-1}}\right) \left(P_{i,n-1} - \left(\frac{i+1}{i}\right) P_{i-1,n-1}\right) \right]$$

$$\times \left[-\sum_{j=1}^{n-1} \frac{t_{ij}}{j} - \left(\frac{n+1}{n}\right) t_{in} \right]$$

$$+ \left(\frac{m+1}{m}\right) \left[\frac{P_{m-1,n}}{P_{mn}} - \frac{P_{m-1,n-1}}{P_{m,n-1}}\right] \left[-\sum_{j=1}^{n-1} \frac{t_{mj}}{j} - \left(\frac{n+1}{n}\right) t_{mn} \right].$$

The coefficients of t_{mn} add to zero, but it does not appear to be possible to obtain bounds on the remaining terms. For example, if one defines

$$K_{41} := \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}$$

$$\times \left| \sum_{j=1}^{n-1} \left(\frac{1}{P_{m-1,n}} - \frac{1}{P_{m-1,n-1}}\right) (P_{m-1,j} - P_{m-1,j-1}) \sum_{i=1}^{m-1} \frac{t_{ij}}{i} \right|^k$$

$$= \mathcal{O}(1) \sum_{m=2}^{r+1} \sum_{n=2}^{s+1} \left(\frac{P_{mn}}{p_{mn}}\right)^{k-1}$$

$$\times \sum_{j=1}^{n-1} \left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m-1,n}}\right) \left(\sum_{\mu=0}^{m-1} p_{\mu j}\right) \sum_{i=1}^{m-1} \frac{|t_{ij}|^k}{i}$$

$$\times \left[\sum_{j=1}^{n-1} \left(\frac{1}{P_{m-1,n-1}} - \frac{1}{P_{m-1,n}}\right) \left(\sum_{\mu=0}^{m-1} p_{\mu j}\right) \sum_{i=1}^{m-1} \frac{1}{i} \right]^{k-1},$$

then by Lemma 6,

$$\begin{aligned}
&\sum_{j=1}^{n-1}\left(\frac{1}{P_{m-1,n-1}}-\frac{1}{P_{m-1,n}}\right)\left(\sum_{\mu=0}^{m-1}p_{\mu j}\right)\sum_{i=1}^{m-1}\frac{1}{i}\\
&\quad=\mathcal{O}(1)\log m\left(\frac{1}{P_{m-1,n-1}}-\frac{1}{P_{m-1,n}}\right)\sum_{j=1}^{n-1}\sum_{i=0}^{m-1}p_{ij}\\
&\quad=\frac{\mathcal{O}(1)P_{m-1,n-1}\log m}{P_{m-1,n-1}P_{m-1,n}}\sum_{i=0}^{m-1}p_{in}\\
&\quad=\frac{\mathcal{O}(1)\log m}{n}.
\end{aligned}$$

Thus, from (10) and Lemma 6,

$$\begin{aligned}
K &:= \mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}\left(\frac{P_{mn}\log m}{np_{mn}}\right)^{k-1}\sum_{j=1}^{n-1}\left(\frac{1}{P_{m-1,n-1}}-\frac{1}{P_{m-1,n}}\right)\\
&\quad\times\left(\sum_{\mu=0}^{m-1}p_{\mu j}\right)\sum_{i=1}^{m-1}\frac{|t_{ij}|^k}{i}\\
&=\mathcal{O}(1)\sum_{m=2}^{r+1}\sum_{n=2}^{s+1}(m\log m)^{k-1}\sum_{j=1}^{n-1}\left(\frac{1}{P_{m-1,n-1}}-\frac{1}{P_{m-1,n}}\right)\\
&\quad\times\left(\sum_{\mu=0}^{m-1}p_{\mu j}\right)\sum_{i=1}^{m-1}\frac{|t_{ij}|^k}{i}\\
&=\mathcal{O}(1)\sum_{i=1}^{r}\sum_{j=1}^{s}\frac{|t_{ij}|^k}{i}\sum_{m=i+1}^{r+1}\sum_{n=j+1}^{s+1}(m\log m)^{k-1}\\
&\quad\times\left(\frac{1}{P_{m-1,n-1}}-\frac{1}{P_{m-1,n}}\right)\sum_{\mu=0}^{m-1}p_{\mu j}\\
&=\mathcal{O}(1)\sum_{i=1}^{r}\sum_{j=1}^{s}\frac{|t_{ij}|^k}{i}\sum_{m=i+1}^{r+1}(m\log m)^{k-1}\\
&\quad\times\sum_{\mu=0}^{m-1}p_{\mu j}\left(\frac{1}{P_{m-1,j}}-\frac{1}{P_{m-1,s+1}}\right)
\end{aligned}$$

$$= \mathcal{O}(1) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{|t_{ij}|^k}{ij} \sum_{m=i+1}^{r+1} (m \log m)^{k-1},$$

and K need not be finite.

Even combining terms of (16), or rewriting them, still does not remove the presence of a logarithmic term.

PROOF of Theorem 2, Part 2. To prove that $(C,1,1)$ is k-absolutely stronger than $(\overline{N},p_{jk})$, we shall first begin with the definition of T_{mn}. By taking successive first differences with respect to each variable, it readily follows that

$$p_{mn} s_{mn} = \Delta_{11}(P_{m-1,n-1} T_{m-1,n-1}),$$

which we shall rewrite in the form

$$\begin{aligned} s_{mn} &= \frac{1}{p_{mn}} \big[P_{mn} \Delta_{11} T_{m-1,n-1} + P_{mn}(T_{m,n-1} + T_{m-1,n} - T_{m-1,n-1}) \\ &\qquad - P_{m,n-1} T_{m,n-1} - P_{m-1,n} T_{m-1,n} + P_{m-1,n-1} T_{m-1,n-1} \big] \\ &= \frac{1}{p_{mn}} \big[P_{mn} \Delta_{11} T_{m-1,n-1} + (P_{mn} - P_{m,n-1}) T_{m,n-1} \\ &\qquad + (P_{mn} - P_{m-1,n}) T_{m-1,n} - (P_{mn} - P_{m-1,n-1}) T_{m-1,n-1} \big]. \end{aligned} \tag{17}$$

But

$$P_{mn} - P_{m,n-1} = \left(\sum_{i=0}^{m} \sum_{j=0}^{n} - \sum_{i=0}^{m} \sum_{j=0}^{n-1} \right) p_{ij} = \sum_{i=0}^{m} p_{in}, \tag{18}$$

$$P_{mn} - P_{m-1,n} = \sum_{j=0}^{n} p_{mj}, \tag{19}$$

and

$$P_{mn} - P_{m-1,n-1} = \sum_{j=0}^{n} p_{mj} + \sum_{i=0}^{m-1} p_{in}. \tag{20}$$

Substituting (18)–(20) into (17) yields

$$\begin{aligned} s_{mn} = \frac{1}{p_{mn}} \Bigg[& P_{mn} \Delta_{11} T_{m-1,n-1} + T_{m,n-1} \sum_{i=0}^{m} p_{in} + T_{m-1,n} \sum_{j=0}^{n} p_{mj} \\ & - T_{m-1,n-1} \left(\sum_{i=0}^{m} p_{in} + \sum_{j=0}^{n-1} p_{mj} \right) \Bigg] \end{aligned}$$

$$= \frac{P_{mn}}{p_{mn}} \Delta_{11} T_{m-1,n-1} - \frac{1}{p_{mn}} \sum_{i=0}^{m} p_{in} \Delta_{10} T_{m-1,n-1} - \frac{1}{p_{mn}} \sum_{j=0}^{n} p_{mj} \Delta_{01} T_{m-1,n-1} + T_{m-1,n-1}.$$

Thus,

$$s_{mn} - s_{m,n-1} = \left(\frac{P_{mn}}{p_{mn}} \Delta_{11} T_{m-1,n-1} - \frac{P_{m,n-1}}{p_{m,n-1}} \Delta_{11} T_{m-1,n-2} \right) + \frac{1}{p_{m,n-1}} \sum_{i=0}^{m} p_{i,n-1} \Delta_{11} T_{m-1,n-2} + \left(\frac{1}{p_{m,n-1}} \sum_{i=0}^{m} p_{i,n-1} - \frac{1}{p_{mn}} \sum_{i=0}^{m} p_{in} \right) \Delta_{10} T_{m-1,n-1} - \frac{1}{p_{mn}} \sum_{j=0}^{n} p_{mj} \Delta_{01} T_{m-1,n-1} + \frac{1}{p_{m,n-1}} \sum_{j=0}^{n-1} p_{mj} \Delta_{01} T_{m-1,n-2} - \Delta_{01} T_{m-1,n-2}. \tag{21}$$

Replacing m by $m-1$ in (21), we have

$$s_{m-1,n} - s_{m-1,n-1} = \left(\frac{P_{m-1n}}{p_{m-1,n}} \Delta_{11} T_{m-2,n-1} - \frac{P_{m-1,n-1}}{p_{m-1,n-1}} \Delta_{11} T_{m-2,n-2} \right) + \frac{1}{p_{m-1,n-1}} \sum_{i=0}^{m-1} p_{i,n-1} \Delta_{11} T_{m-2,n-2} + \left(\frac{1}{p_{m-1,n-1}} \sum_{i=0}^{m-1} p_{i,n-1} - \frac{1}{p_{m-1,n}} \sum_{i=0}^{m-1} p_{in} \right) \Delta_{10} T_{m-2,n-1} - \frac{1}{p_{m-1,n}} \sum_{j=0}^{n} p_{m-1,j} \Delta_{01} T_{m-2,n-1} + \frac{1}{p_{m-1,n-1}} \sum_{j=0}^{n-1} p_{m-1,j} \Delta_{01} T_{m-2,n-2} - \Delta_{01} T_{m-2,n-2}. \tag{22}$$

Subtracting (22) from (21) yields

$$
\begin{aligned}
(23)\quad a_{mn} = {} & \Delta_{11}\left(\frac{P_{m-1,n-1}}{p_{m-1,n-1}}\Delta_{11}T_{m-2,n-2}\right)\\
& + \frac{1}{p_{m,n-1}}\sum_{i=0}^{m} p_{i,n-1}\Delta_{11}T_{m-1,n-2}\\
& - \frac{1}{p_{m-1,n-1}}\sum_{i=0}^{m-1} p_{i,n-1}\Delta_{11}T_{m-2,n-2}\\
& + \left(\frac{1}{p_{m,n-1}}\sum_{i=0}^{m-1} p_{i,n-1} - \frac{1}{p_{mn}}\sum_{i=0}^{m-1} p_{in}\right)\Delta_{10}T_{m-1,n-1}\\
& - \left(\frac{1}{p_{m-1,n-1}}\sum_{i=0}^{m-2} p_{i,n-1} - \frac{1}{p_{m-1,n}}\sum_{i=0}^{m-2} p_{in}\right)\Delta_{10}T_{m-2,n-1}\\
& - \frac{1}{p_{m-1,n-1}}\sum_{j=0}^{n-1} p_{m-1,j}\Delta_{11}T_{m-2,n-2}\\
& + \left(-\frac{1}{p_{m-1,n-1}}\sum_{j=0}^{n-1} p_{m-1,j} + \frac{1}{p_{m,n-1}}\sum_{j=0}^{n-1} p_{mj}\right)\Delta_{01}T_{m-1,n-2}\\
& + \frac{1}{p_{m-1,n}}\sum_{j=0}^{m-1} p_{m-1,j}\Delta_{11}T_{m-2,n-1}\\
& + \left(\frac{1}{p_{m-1,n}}\sum_{j=0}^{m-1} p_{m-1,j} - \frac{1}{p_{mn}}\sum_{j=0}^{n} p_{mj}\right)\Delta_{01}T_{m-1,n-1}\\
& + \Delta_{11}T_{m-2,n-2}.
\end{aligned}
$$

If $\{p_{ij}\}$ is factorable, then the coefficients of $\Delta_{10}T_{m-1,n-1}$, $\Delta_{10}T_{m-2,n-1}$, $\Delta_{01}T_{m-1,n-2}$, and $\Delta_{01}\ T_{m-1,n-1}$ are zero. Also,

$$T_{mn} = \frac{1}{P_m Q_n}\sum_{i=0}^{m} p_i \sum_{j=0}^{n} q_j s_{ij}. \tag{24}$$

Using (24) with $m=0$, we obtain

$$T_{0n} = \frac{1}{Q_n}\sum_{j=0}^{n} q_j s_{0j} \ ,$$

$$Q_n T_{0n} - Q_{n-1}T_{0,n-1} = q_n s_{0n}. \tag{25}$$

Using (24) with $m=1$ yields

$$T_{1n} = \frac{1}{P_1 Q_n}\sum_{i=0}^{1}\sum_{j=0}^{n} p_i q_j s_{ij} \ ,$$

$$P_1(Q_n T_{1n} - Q_{n-1}T_{1,n-1}) = \sum_{i=0}^{1} p_i q_n s_{in} \ ,$$

$$\frac{P_1(Q_n T_{1n} - Q_{n-1}T_{1,n-1})}{q_n} = p_0 s_{0n} + p_1 s_{1n} = (p_0+p_1)s_{0n} + p_1\sum_{k=0}^{n} a_{1k}.$$

Using (25) gives

$$\begin{aligned} p_1\sum_{k=0}^{n} a_{1k} &= \frac{P_1(Q_n T_{1n} - Q_{n-1}T_{1,n-1})}{q_n} - \frac{P_1(Q_n T_{0n} - Q_{n-1}T_{0,n-1})}{q_n} \\ &= P_1\left[\frac{Q_n}{q_n}(T_{1n}-T_{1,n-1}) + T_{1,n-1} - \frac{Q_n}{q_n}(T_{0n}-T_{0,n-1}) - T_{0,n-1}\right] \\ &= P_1\left[\frac{Q_n}{q_n}\Delta_{11}T_{0,n-1} - \Delta_{10}T_{0,n-1}\right]. \end{aligned}$$

Thus,

$$\begin{aligned} a_{1n} &= \frac{P_1}{p_1}\left[\frac{Q_n}{q_n}\Delta_{11}T_{0,n-1} - \Delta_{10}T_{0,n-1} - \frac{Q_{n-1}}{q_{n-1}}\Delta_{11}T_{0,n-2} + \Delta_{10}T_{0,n-2}\right] \\ &= \frac{P_1}{p_1}\left[\frac{Q_n}{q_n}\Delta_{11}T_{0,n-1} - \frac{Q_{n-1}}{q_{n-1}}\Delta_{11}T_{0,n-2} + \Delta_{11}T_{0,n-2}\right]. \end{aligned} \tag{26}$$

Similarly,

$$a_{m1} = \frac{Q_1}{q_1}\left[\frac{P_m}{p_m}\Delta_{11}T_{m-1,0} - \frac{P_{m-1}}{p_{m-1}}\Delta_{11}T_{m-2,0} + \Delta_{11}T_{m-2,0}\right]. \tag{27}$$

In a similar manner it can be shown that

$$a_{11} = \frac{P_1 Q_1}{p_1 q_1} \Delta_{11} T_{00}, \tag{28}$$

and equation (23) becomes

$$\begin{aligned} a_{mn} = \Delta_{11} &\left(\frac{P_{m-1} Q_{n-1}}{p_{m-1} q_{n-1}} \Delta_{11} T_{m-2,n-2} \right) \\ &+ \frac{Q_n}{q_n} \Delta_{11} T_{m-2,n-1} - \frac{Q_{n-1}}{q_{n-1}} \Delta_{11} T_{m-2,n-2} \\ &+ \frac{P_m}{p_m} \Delta_{11} T_{m-1,n-2} - \frac{P_{m-1}}{p_{m-1}} \Delta_{11} T_{m-2,n-2} + \Delta_{11} T_{m-1,n-1}. \end{aligned} \tag{29}$$

We now substitute (26) - (29) into

$$\begin{aligned} t_{mn} &= \frac{1}{(m+1)(n+1)} \sum_{i=1}^{m} \sum_{j=1}^{n} ij a_{ij} \\ &= \frac{1}{(m+1)(n+1)} \left[a_{11} + \sum_{i=2}^{m} i a_{i1} + \sum_{j=2}^{n} j a_{1j} + \sum_{i=2}^{m} \sum_{j=2}^{n} ij a_{ij} \right] \\ &= \frac{1}{(m+1)(n+1)} \Bigg[\frac{P_1 Q_1}{p_1 q_1} \Delta_{11} T_{00} \\ &\quad + \frac{Q_1}{q_1} \sum_{i=2}^{m} i \left(\frac{P_i}{p_i} \Delta_{11} T_{i-1,0} - \frac{P_{i-1}}{p_{i-1}} \Delta_{11} T_{i-2,0} + \Delta_{11} T_{i-2,0} \right) \\ &\quad + \frac{P_1}{p_1} \sum_{j=2}^{n} j \left(\frac{Q_j}{q_j} \Delta_{11} T_{0,j-1} - \frac{Q_{j-1}}{q_{j-1}} \Delta_{11} T_{0,j-2} + \Delta_{11} T_{0,j-2} \right) \\ &\quad + \sum_{i=2}^{m} \sum_{j=2}^{n} ij \left(\Delta_{11} \left(\frac{P_{i-1} Q_{j-1}}{p_{i-1} q_{j-1}} \Delta_{11} T_{i-2,j-2} \right) + \frac{Q_j}{q_j} \Delta_{11} T_{i-2,j-1} \right. \\ &\quad - \frac{Q_{j-1}}{q_{j-1}} \Delta_{11} T_{i-2,j-2} - \frac{P_i}{p_i} \Delta_{11} T_{i-1,j-2} - \frac{P_{i-1}}{p_{i-1}} \Delta_{11} T_{i-2,j-2} \\ &\quad \left. + \Delta_{11} T_{i-1,j-1} \right) \Bigg] \\ &= L_1 + L_2 + L_3 + L_4, \text{ say.} \end{aligned} \tag{30}$$

For single summability, a series $\sum a_n$ is said to be absolutely summable (C, α) with index $k \geq 1$, or summable $|C, \alpha|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}|^k < \infty, \tag{31}$$

where σ_n^α denotes the nth Cesàro mean of order α of the sequence $s_n = \sum_{k=0}^n a_k$. If $t_n^\alpha := n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$, then, by a result of Kogbetliantz [4], (31) is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n}|t_n^\alpha|^k < \infty.$$

Since, for double summability, the mnth terms of the (C,α,β) means are factorable, the result of Kogbetliantz applies. In particular, if $\alpha = \beta = 1$, to complete the proof, it is sufficient to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+1)(n+1)}|L_i|^k < \infty, \quad i = 1,2,3,4.$$

From (30),

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+1)(n+1)}|L_1|^k &= \mathcal{O}(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+1)^{-k-1}(n+1)^{-k-1}\\ &= \mathcal{O}(1).\end{aligned}$$

Write $L_2 = L_{21} + L_{22}$, where

$$\begin{aligned}L_{21} &:= \frac{Q_1}{q_1(m+1)(n+1)}\left[\sum_{i=2}^{m} i\frac{P_i}{p_i}\Delta_{11}T_{i-1,0} - \sum_{i=2}^{m} i\frac{P_{i-1}}{p_{i-1}}\Delta_{11}T_{i-2,0}\right]\\ &= \frac{\mathcal{O}(1)}{(m+1)(n+1)}\left[\frac{mP_m}{p_m}\Delta_{11}T_{m-1,0} - \frac{2P_1}{p_1}\Delta_{11}T_{00} - \sum_{i=2}^{m-1}\frac{P_i}{p_i}\Delta_{11}T_{i-1,0}\right]\\ &= L_{211} + L_{212} + L_{213}, \text{ say.}\end{aligned}$$

Using (10),

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+1)(n+1)}|L_{211}|^k\\ &\qquad = \mathcal{O}(1)\sum_{n=1}^{\infty}(n+1)^{-k-1} \times \sum_{m=1}^{\infty}\frac{1}{m+1}\left(\frac{P_m}{p_m}\right)^k |\Delta_{11}T_{m-1,0}|^k\\ &\qquad = \mathcal{O}(1)\sum_{m=1}^{\infty}\left(\frac{P_m}{p_m}\right)^{k-1}|\Delta_{11}T_{m-1,0}|^k = \mathcal{O}(1)\end{aligned}$$

and similarly,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{212}|^k$$
$$=\mathcal{O}(1)\sum_{m=1}^{\infty}(m+1)^{-k-1}\times\sum_{n=1}^{\infty}(n+1)^{-k-1}=\mathcal{O}(1).$$

Using Hölder's inequality and (10),

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{213}|^k$$
$$=\mathcal{O}(1)\sum_{n=1}^{\infty}(n+1)^{-k-1}\sum_{m=1}^{\infty}(m+1)^{-k-1}\left|\sum_{i=2}^{m-1}\frac{P_i}{p_i}\Delta_{11}T_{i-1,0}\right|^k$$
$$=\mathcal{O}(1)\sum_{m=1}^{\infty}(m+1)^{-2}\left[\sum_{i=2}^{m-1}\left(\frac{P_i}{p_i}\right)^k|\Delta_{11}T_{i-1,0}|^k\right]\left[\frac{1}{m+1}\sum_{i=2}^{m-1}1\right]^{k-1}$$
$$=\mathcal{O}(1)\sum_{i=2}^{\infty}\left(\frac{P_i}{p_i}\right)^k|\Delta_{11}T_{i-1,0}|^k\sum_{m=i+1}^{\infty}(m+1)^{-2}$$
$$=\mathcal{O}(1)\sum_{i=2}^{\infty}\left(\frac{P_i}{p_i}\right)^{k-1}|\Delta_{11}T_{i-1,0}|^k=\mathcal{O}(1)$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{22}|^k$$
$$=\mathcal{O}(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}\left|\frac{1}{(m+1)(n+1)}\sum_{i=2}^{m}i\Delta_{11}T_{i-2,0}\right|^k$$
$$=\mathcal{O}(1)\sum_{n=1}^{\infty}(n+1)^{-k-1}\sum_{m=1}^{\infty}(m+1)^{-2}$$
$$\times\left[\sum_{i=2}^{m}i^k|\Delta_{11}T_{i-2,0}|^k\right]\left[\frac{1}{m+1}\sum_{i=2}^{m}1\right]^{k-1}$$
$$=\mathcal{O}(1)\sum_{i=2}^{\infty}i^k|\Delta_{11}T_{i-2,0}|^k\sum_{m=i+1}^{\infty}(m+1)^{-2}$$

$$= \mathcal{O}(1)\sum_{i=2}^{\infty} i^{k-1}|\Delta_{11}T_{i-2,0}|^k = \mathcal{O}(1)\sum_{i=2}^{\infty}\left(\frac{P_i}{p_i}\right)^{k-1}|\Delta_{11}T_{i-2,0}|^k$$

$$= \mathcal{O}(1).$$

In a similar manner,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_3|^k = \mathcal{O}(1).$$

Write

$$L_4 = \sum_{i=1}^{6} L_{4i},$$

where

$$L_{41} = \frac{1}{(m+1)(n+1)}\sum_{i=2}^{m} i\left[\sum_{j=2}^{n} j\Delta_{10}\frac{P_{i-1}Q_{j-1}}{p_{i-1}q_{j-1}}\Delta_{11}T_{i-2,j-2} - \sum_{j=2}^{n} j\Delta_{10}\frac{P_{i-1}Q_j}{p_{i-1}q_j}\Delta_{11}T_{i-2,j-1}\right]$$

$$= \frac{1}{(m+1)(n+1)}\sum_{i=2}^{m} i\left[2\Delta_{10}\frac{P_{i-1}Q_1}{p_{i-1}q_1}\Delta_{11}T_{i-2,0} - n\Delta_{10}\frac{P_{i-1}Q_n}{p_{i-1}q_n}\Delta_{11}T_{i-2,n-1} + \sum_{j=2}^{n-1}\Delta_{10}\frac{P_{i-1}Q_j}{p_{i-1},q_j}\Delta_{11}T_{i-2,j-1}\right]$$

$$= \frac{1}{(m+1)(n+1)}\left[2\left(\sum_{i=2}^{m} i\frac{P_{i-1}Q_1}{p_{i-1}q_1}\Delta_{11}T_{i-2,0} - \sum_{i=2}^{m} i\frac{P_iQ_1}{p_iq_1}\Delta_{11}T_{i-1,0}\right) - n\left(\sum_{i=2}^{m} i\frac{P_{i-1}Q_n}{p_{i-1}q_n}\Delta_{11}T_{i-2,n-1} - \sum_{i=2}^{m} i\frac{P_iQ_n}{p_iq_n}\Delta_{11}T_{i-1,n-1}\right) + \sum_{j=2}^{n-1}\left(\sum_{i=2}^{m} i\frac{P_{i-1}Q_j}{p_{i-1}q_j}\Delta_{11}T_{i-2,j-1} - \sum_{i=2}^{m} i\frac{P_iQ_j}{p_iq_j}\Delta_{11}T_{i-1,j-1}\right)\right]$$

$$= L_{411} + L_{412} + L_{413}, \text{ say.}$$

We have

$$L_{411} = \frac{2}{(m+1)(n+1)}\Big[2\frac{P_1Q_1}{p_1q_1}\Delta_{11}T_{00} - m\frac{P_mQ_1}{p_mq_1}\Delta_{11}T_{m-1,0} + \sum_{i=2}^{m-1}\frac{P_iQ_1}{p_iq_1}\Delta_{11}T_{i-1,0}\Big].$$

Using the same arguments as applied to L_{211}, L_{212}, and L_{213}, it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{411}|^k = \mathcal{O}(1).$$

Similarly,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{412}|^k = \mathcal{O}(1).$$

Continuing in a similar fashion as above yields

$$\begin{aligned} L_{413} &= \frac{1}{(m+1)(n+1)}\sum_{j=2}^{n-1}\Big[2\frac{P_1Q_j}{p_1q_j}\Delta_{11}T_{0,j-1} - m\frac{P_mQ_j}{p_mq_j}\Delta_{11}T_{m-1,j-1} + \sum_{i=2}^{m-1}\frac{P_iQ_j}{p_iq_j}\Delta_{11}T_{i-1,j-1}\Big] \\ &= L_{4131} + L_{4132} + L_{4133}, \text{ say,} \end{aligned}$$

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}&\frac{1}{(m+1)(n+1)}|L_{4131}|^k \\ &= \mathcal{O}(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+1)^{-k-1}(n+1)^{-k-1}\left|\sum_{j=2}^{n-1}\Delta_{11}T_{0,j-1}\right|^k \\ &= \mathcal{O}(1)\sum_{m=1}^{\infty}(m+1)^{-k-1}\sum_{n=1}^{\infty}(n+1)^{-2}\left[\sum_{j=2}^{n-1}|\Delta_{11}T_{0,j-1}|^k\right] \\ &\quad\times\left[\frac{1}{n+1}\sum_{j=2}^{n-1}1\right]^{k-1} \\ &= \mathcal{O}(1)\sum_{j=2}^{\infty}|\Delta_{11}T_{0,j-1}|^k\sum_{n=j+1}^{\infty}(n+1)^{-2} = \mathcal{O}(1), \end{aligned}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{4132}|^k=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}$$

$$\times\left|\frac{m}{(m+1)(n+1)}\sum_{j=2}^{n-1}\frac{P_mQ_j}{p_mq_j}\Delta_{11}T_{m-1,j-1}\right|^k$$

$$=\mathcal{O}(1)\sum_{m=1}^{\infty}m^{-1}\sum_{n=1}^{\infty}n^{-2}\left[\sum_{j=2}^{n-1}\left(\frac{P_mQ_j}{p_mq_j}\right)^k|\Delta_{11}T_{m-1,j-1}|^k\right]$$

$$\times\left[\frac{1}{n+1}\sum_{j=2}^{n-1}1\right]^{k-1}$$

$$=\mathcal{O}(1)\sum_{m=1}^{\infty}\left(\frac{P_m}{p_m}\right)^{k-1}\sum_{j=2}^{\infty}\left(\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{m-1,j-1}|^k\sum_{n=j+1}^{\infty}n^{-2}$$

$$=\mathcal{O}(1)\sum_{m=1}^{\infty}\sum_{j=2}^{\infty}\left(\frac{P_mQ_j}{p_mq_j}\right)^{k-1}|\Delta_{11}T_{m-1,j-1}|^k=\mathcal{O}(1),$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{4133}|^k$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+1)^{-k-1}(n+1)^{-k-1}\left|\sum_{j=2}^{n-1}\sum_{i=2}^{m-1}\frac{P_iQ_j}{p_iq_j}\Delta_{11}T_{i-1,j-1}\right|^k$$

$$=\mathcal{O}(1)\sum_{m=1}^{\infty}m^{-2}\sum_{n=1}^{\infty}n^{-2}\left[\sum_{j=2}^{n-1}\sum_{i=2}^{m-1}\left(\frac{P_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\right]$$

$$\times\left[\frac{1}{(m+1)(n+1)}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}1\right]^{k-1}$$

$$=\mathcal{O}(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(\frac{P_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\sum_{m=i+1}^{\infty}m^{-2}\sum_{n=j+1}^{\infty}n^{-2}$$

$$=\mathcal{O}(1),$$

$$L_{42}+L_{43}=\frac{1}{(m+1)(n+1)}\left[\sum_{i=2}^{m}\sum_{j=2}^{n}ij\frac{Q_j}{q_j}\Delta_{11}T_{i-2,j-1}\right.$$
$$\left.-\sum_{i=2}^{m}\sum_{j=2}^{n}ij\frac{Q_{j-1}}{q_{j-1}}\Delta_{11}T_{i-2,j-2}\right]$$
$$=\frac{1}{(m+1)(n+1)}\sum_{i=2}^{m}i\left[n\frac{Q_n}{q_n}\Delta_{11}T_{i-2,n-1}-2\frac{Q_1}{q_1}\Delta_{11}T_{i-2,0}\right.$$
$$\left.-\sum_{j=2}^{n-1}\frac{Q_j}{q_j}\Delta_{11}T_{i-2,j-1}\right]$$
$$=L_{421}+L_{422}+L_{423},\ \text{say},$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{421}|^k$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+1)^{-k-1}(n+1)^{-k-1}\left|n\frac{Q_n}{q_n}\sum_{i=2}^{m}i\Delta_{11}T_{i-2,n-1}\right|^k$$
$$=\mathcal{O}(1)\sum_{m=1}^{\infty}m^{-2}\sum_{n=1}^{\infty}n^{-1}\left(\frac{Q_n}{q_n}\right)^k\left[\sum_{i=2}^{m}i^k|\Delta_{11}T_{i-2,n-1}|^k\right]$$
$$\times\left[\frac{1}{m+1}\sum_{i=2}^{m}1\right]^{k-1}$$
$$=\mathcal{O}(1)\sum_{n=1}^{\infty}\left(\frac{Q_n}{q_n}\right)^{k-1}\sum_{i=2}^{\infty}i^k|\Delta_{11}T_{i-2,n-1}|^k\sum_{m=i+1}^{\infty}m^{-2}$$
$$=\mathcal{O}(1)\sum_{n=1}^{\infty}\left(\frac{Q_n}{q_n}\right)^{k-1}\sum_{i=2}^{\infty}i^{k-1}|\Delta_{11}T_{i-2,n-1}|^k$$
$$=\mathcal{O}(1)\sum_{i=2}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_nP_i}{q_np_j}\right)^{k-1}|\Delta_{11}T_{i-2,n-1}|^k=\mathcal{O}(1).$$

Using the same argument as that for L_{22}, it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{422}|^k=\mathcal{O}(1),$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{423}|^k = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+1)^{-k-1}(n+1)^{-k-1}$$

$$\times \Big|\sum_{i=2}^{m} i \sum_{j=2}^{n-1}\frac{Q_j}{q_j}\Delta_{11}T_{i-2,j-1}\Big|^k$$

$$\leq \sum_{m=1}^{\infty} m^{-2}\sum_{n=2}^{\infty} n^{-2}\left[\sum_{i=2}^{m}\sum_{j=2}^{n-1}\left(i\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{i-2,j-1}|^k\right]$$

$$\times\left[\frac{1}{(m+1)(n+1)}\sum_{i=2}^{m}\sum_{j=2}^{n-1}1\right]^{k-1}$$

$$= \mathcal{O}(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(i\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{i-2,j-1}|^k\sum_{m=i+1}^{\infty} m^{-2}\sum_{n=j+1}^{\infty} n^{-2}$$

$$= \mathcal{O}(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(\frac{P_iQ_j}{p_iq_j}\right)^{k-1}|\Delta_{11}T_{i-2,j-1}|^k = \mathcal{O}(1).$$

The argument for $L_{44}+L_{45}$ is similar.
Finally, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+1)(n+1)}|L_{46}|^k = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+1)^{-k-1}(n+1)^{-k-1}$$

$$\times\left|\sum_{i=2}^{m}\sum_{j=2}^{n} ij\Delta_{11}T_{i-1,j-1}\right|^k$$

$$\leq \sum_{m=1}^{\infty} m^{-2}\sum_{n=2}^{\infty} n^{-2}\left[\sum_{i=2}^{m}\sum_{j=2}^{n}(ij)^k|\Delta_{11}T_{i-1,j-1}|^k\right]$$

$$\times\left[\frac{1}{(m+1)(n+1)}\sum_{i=2}^{m}\sum_{j=2}^{n}1\right]^{k-1}$$

$$= \mathcal{O}(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(ij)^k|\Delta_{11}T_{i-1,j-1}|^k\sum_{m=i+1}^{\infty} m^{-2}\sum_{n=j+1}^{\infty} n^{-2}$$

$$= \mathcal{O}(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(ij)^{k-1}|\Delta_{11}T_{i-1,j-1}|^k$$

$$= \mathcal{O}(1) \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \left(\frac{P_{i-1} Q_{j-1}}{p_{i-1} q_{j-1}} \right)^{k-1} |\Delta_{11} T_{i-1,j-1}|^k = \mathcal{O}(1).$$

Problem 2. If $\{p_{ij}\}$ is not factorable then, from (23), one has terms involving one-sided differences Δ_{10} and Δ_{01}. It is not possible to estimate the size of these terms, since every estimation must involve the first complete difference Δ_{11}.

For other papers treating absolute k-inclusion of weighted mean matrices, the reader may wish to consult [6]–[9].

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(Received December 10, 1993; revised January 19, 1995)