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Nullstellensatz theorems and radical classes

By N. R. McCONNELL (Queensland) and TIMOTHY STOKES (Tasmania)

Let $\mathcal{U}$ be a universal class of associative rings. It is well-known that the class of nil rings in $\mathcal{U}$ is a radical class. For a field K, finitely generated commutative nil K-algebras are significant in algebraic geometry because of their relationship to the Hilbert Nullstellensatz, which states that for all $I \triangleleft K[x_1, x_2, \dots, x_n]$, the nil radical of $K[x_1, x_2, \dots, x_n]/I$ is $\mathcal{IV}(I)/I$ where $\mathcal{IV}(I)$ is the ideal of all polynomials which vanish over any algebraic closure L of K whenever all polynomials in I vanish over L. We show that such a situation is typical for any universal class $\mathcal{U}$ of multioperator groups: whenever there is some kind of a Nullstellensatz (literally, "zero place theorem") for some R in the variety generated by $\mathcal{U}$, there is a corresponding induced radical class, and the radical and semisimple objects may be described in terms of the Nullstellensatz for R. This leads to information about the possible kinds of Nullstellensatz theorems which can arise. A number of examples for groups and rings are considered.

Throughout, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the rings of rational, real and complex numbers respectively.

1. Multi-operator groups

Multi-operator groups of signature Ω are henceforth referred to as Ω-*groups*, as in [2]. We refer the reader to [2] for the definition and proofs of basis properties of Ω-groups. We shall use additive notation for the group operation on all Ω-groups, whether or not the operation is commutative, consistent with the notational convention used in [2].

Let M be an Ω-group, I a normal subgroup of $\langle M, +\rangle$. I is an *ideal* of M if for each $n \in \mathbb{N}$ and each n-ary operator $\varrho \in \Omega$,

$$-\varrho(a_1, a_2, \dots, a_{j-1}, i + a_j, a_{j+1}, \dots, a_n)+ \\ + \varrho(a_1, a_2, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n) \in I,$$

for all $a_k \in M$, $i \in I$, $j = 1, 2, \ldots, n$. Notation: $I \lhd M$. This definition reduces to the usual definitions of normal subgroup, ideal and R-submodule (where R is an associative ring) in the universal classes of groups, rings and R-modules respectively.

Let I be a normal subgroup of the Ω-group M. From ([2], p.101), I is an ideal of M if and only if the quotient group M/I is a Ω-group, with the mapping ϱ_n defined by

$$\varrho_n(a_1 + I, a_2 + I, \ldots, a_n + I) = \varrho_n(a_1, a_2, \ldots, a_n) + I$$

for all $a_1, a_2, \ldots, a_n \in M$.

The set of ideals of the Ω-group M is a complete lattice where meet is intersection, and join is additive normal subgroup product, as is shown in [2]. Hence the join of a set of ideals $\{I_\lambda : \lambda \in \Lambda\}$ of M is denoted by $\sum\{I_\lambda : \lambda \in \Lambda\}$; the join of two ideals $I, J \lhd M$ will be denoted by $I + J$.

For $S \subseteq M$, the *ideal generated by* S is the smallest ideal of M containing S, and is denoted by $(S)_M$. If $S = \{s_1, s_2, \ldots, s_r\} \subseteq M$, then $(S)_M$ will often be denoted by $(s_1, s_2, \ldots, s_r)_M$.

Let $\mathcal{W}$ be a variety, with $\mathcal{G}$ a class of free Ω-groups in $\mathcal{W}$. Let $Q(\mathcal{G})$ denote the class of all homomorphic images in $\mathcal{W}$ of the elements of $\mathcal{G}$. If $\mathcal{G}$ is all free Ω-groups in the variety $\mathcal{W}$, then $Q(\mathcal{G})$ is the universal class $\mathcal{W}$. If $\mathcal{W}$ is the variety of commutative rings and $\mathcal{G}$ is all finitely generated free commutative rings, then $Q(\mathcal{G})$ is the universal class of all finitely generated commutative rings; similar remarks apply for abelian groups, K-vector spaces and commutative K-algebras for any field K.

2. Closure operations and radical classes

We now introduce the concept of a *radical operation*, which provides a characterisation of the notion of a radical class that is particularly useful for our purposes. Recall that a closure operation on some set $\mathcal{P}$ of subsets of a set S is a function $C : \mathcal{P} \to \mathcal{P}$, satisfying, for all $X, Y \in \mathcal{P}$:

(i) $X \subseteq C(X)$;

(ii) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$; and

(iii) $C(C(X)) = C(X)$.

Definition 2.1. A radical operation on a class of Ω-groups $\mathcal{U}$ is a collection $\mathcal{C} = \{C_R : R \in \mathcal{U}\}$ where each C_R is a closure operation on the set of ideals of R, such that for all $R, S \in \mathcal{U}$, if $I, K \lhd R$ with $K \subseteq I$, $J \lhd S$, and $I/K \cong C_S(J)/J$, then $I \subseteq C_R(K)$.

For Ω-groups, a radical class can be defined as follows (c.f. Definition 1.3.1 of [1]):

Definition 2.2. A non-empty subclass $\mathcal{R}$ of a universal class $\mathcal{U}$ of Ω-groups is a radical class if

(a) $A \in \mathcal{R}, A \cong B$ imply $B \in \mathcal{R}$,

(b) $I \triangleleft A \in \mathcal{R}$ implies $A/I \in \mathcal{R}$,

(c) $\mathcal{R}(A) = \sum\{J : J \triangleleft A, J \in \mathcal{R}\} \in \mathcal{R}$ for any $A \in \mathcal{U}$ and

(d) $\mathcal{R}(A/\mathcal{R}(A)) = \{0\}$ for any $A \in \mathcal{U}$.

$A \in \mathcal{U}$ is said to be $\mathcal{R}$-*radical* if $\mathcal{R}(A) = A$ (equivalently, if $A \in \mathcal{R}$) and $\mathcal{R}$-*semisimple* if $\mathcal{R}(A) = \{0\}$.

Theorem 2.3. *Let $\mathcal{U}$ be a universal class of Ω-groups.*

(i) Let $\mathcal{R}$ be a radical class on $\mathcal{U}$. For all $R \in \mathcal{R}$ and $I \triangleleft R$, define

$$C_R(I) = \sum\{J : J \triangleleft R, I \subseteq J, J/I \in \mathcal{R}\}.$$

Then $\mathcal{C} = \{C_R : R \in \mathcal{U}\}$ is a radical operation on $\mathcal{U}$.

(ii) Let $\mathcal{C} = \{C_R : R \in \mathcal{U}\}$ be a radical operation on $\mathcal{U}$. Then

$$\mathcal{R} = \{A : \text{ there exist } R \in \mathcal{U} \text{ and } M \triangleleft R \text{ such that } A \cong C_R(M)/M\}$$

is a radical class on $\mathcal{U}$, and $\mathcal{R}(A) = C_A(\{0\})$ for all $A \in \mathcal{U}$.

Proof. (*i*) If $\mathcal{R}$ is a radical class, let I, K be ideals of $R \in \mathcal{U}$, J an ideal of $S \in \mathcal{U}$ with $K \subseteq I$ and $I/K \cong C_S(J)/J$. Then $C_S(J)/J \in \mathcal{R}$ (from the definition of C_S and by (c) of Definition 2.2), so $I/K \in \mathcal{R}$ too. But then by the definition of C_R, $I \subseteq C_R(K)$, so $\mathcal{C}$ is a radical operation.

(*ii*) (a) of Definition 2.2 is immediate from the definition of a radical operation. Let $\mathcal{R}$ be as described in the theorem. If $I \triangleleft A \in \mathcal{R}$, let $A \cong C_R(J)/J$ for some $J \triangleleft R$, $R \in \mathcal{U}$, and let $I \cong M/J$. Then $A/I \cong (C_R(J)/J)/(M/J) \cong C_R(J)/M$. Now $J \subseteq M$, so $C_R(J) \subseteq C_R(M)$; also $M \subseteq C_R(J)$, so $C_R(M) \subseteq C_R(C_R(J)) = C_R(J)$, and so $C_R(J) = C_R(M)$. Thus $A/I \cong C_R(M)/M \in \mathcal{R}$ and (b) holds.

For $\mathcal{R}$ as above, let $\mathcal{R}(A)$ be defined as in (c) for $A \in \mathcal{U}$, and consider $J \in \mathcal{R}$. Then there exist R, K with $K \triangleleft R$ and $J \cong C_R(K)/K$. Hence $J/\{0\} \cong C_R(K)/K$, so $J \subseteq C_A(\{0\})$ and so $\mathcal{R}(A) \subseteq C_A(\{0\})$. But $C_A(\{0\}) \cong C_A(\{0\})/\{0\} \in \mathcal{R}$, so $C_A(\{0\}) \subseteq \mathcal{R}(A)$. That is, $\mathcal{R}(A) = C_A(\{0\}) \in \mathcal{R}$ and (c) holds.

Finally, let $L \triangleleft A/\mathcal{R}(A) = A/C_A(\{0\})$, and let $L \cong M/C_A(\{0\})$. If $L \in \mathcal{R}$, then $M/C_A(\{0\}) \cong C_R(J)/J$ for some $J \triangleleft R \in \mathcal{U}$. Thus $M \subseteq$

$C_A(C_A(\{0\}))$, so $L = \{0\}$ and $\mathcal{R}(A/\mathcal{R}(A)) = \{0\}$. Hence (d) holds and $\mathcal{R}$ is a radical class. □

Hence there is a one-to-one correspondence between radical operations on $\mathcal{U}$ and radical classes in $\mathcal{U}$.

Proposition 2.4. *Let $\mathcal{G}$ be a class of free Ω–groups in the variety $\mathcal{W}$ such that $\mathcal{U} = Q(\mathcal{G})$ is a universal class. Let $\mathcal{C} = \{C_F : F \in \mathcal{G}\}$ be a radical operation on $\mathcal{G}$. For all $F \in \mathcal{G}$, $J \triangleleft F$ and $R \cong F/J \in \mathcal{U}$ via $\psi : F/J \to R$, define $C_R(I) \triangleleft R$ by setting $C_R(I) = \psi(C_F(I')/J)$, where $I' \triangleleft F$ satisfies $I \cong I'/J$. Then $C_{\mathcal{U}} = \{C_R : R \in \mathcal{U}\}$ is a radical operation on $\mathcal{U}$.*

Proof. First we show that C_R is well-defined for each $R \in Q(\mathcal{G})$, that is, that it is independent of the representation of R as a quotient of a free Ω-group. Let F, $G \in \mathcal{G}$ and $J \triangleleft F$, $K \triangleleft G$ be such that there exist isomorphisms $\psi : F/J \to R$, $\phi : G/K \to R$. Let $I \triangleleft R$, with $I_F \triangleleft F$ and $I_G \triangleleft G$ such that $\psi(I_F/J) = I = \phi(I_G/K)$. Define $C_R(I) = \phi(C_F(I_F)/J)$.

Now there exists $N \triangleleft G$ with $I_G \subseteq N$ and $C_R(I) = \phi(N/K)$. Hence $C_F(I_F/J) \cong N/K$, so $C_F(I_F)/I_F \cong (C_F(I_F)/J)/(I_F/J) \cong (N/K)/(I_G/K) \cong N/I_G$, so by Definition 2.1, $N \subseteq C_G(I_G)$. Similarly we may define C'_R in terms of G, and there exists $M \triangleleft F$ with $I_F \subseteq M$ such that $C'_R(I) = \psi(M/J)$. Then $C_G(I_G)/K \cong M/J$, and we may obtain $M \subseteq C_F(I_F)$ as above. So $C_R(I) \cong N/K \subseteq C_G(I_G)/K \cong M/J \subseteq C_F(I_F)/J \cong C_R(I)$, so that in fact $N/K = C_G(I_G)/K$ and $M/J = C_F(I_F)/J$, and so $C_R(I) = C'_R(I)$ as required. Hence C_R is indeed well-defined.

Clearly C_R is a closure operation for all $R \in \mathcal{U}$. Suppose $R, S \in \mathcal{U}$, $I, K \triangleleft R$, $K \subseteq I$, $J \triangleleft S$ and $I/K \cong C_S(J)/J$, and let $S \cong F/N$, $R \cong G/M$ for some $F, G \in \mathcal{G}$. Let $I \cong I'/M$, $K \cong K'/M$, $J \cong J'/N$. Then $I'/K' \cong (I'/M)/(K'/M) \cong I/K \cong C_S(J)/J \cong (C_G(J')/N)/(J'/N) \cong C_G(J')/J'$, so by assumption $I' \subseteq C_F(K')$; that is, $I \cong I'/M \subseteq C_F(K')/M \cong C_R(K)$. Hence $\mathcal{C}_{\mathcal{U}}$ is a radical operation. □

Hence if $\mathcal{U} = Q(\mathcal{G})$ for some collection of free Ω-groups $\mathcal{G}$ in some variety $\mathcal{W}$, then any radical operation on $Q(\mathcal{G})$ is completely determined by its effect on the free Ω-groups in $Q(\mathcal{G})$.

Let $\mathcal{U}$ be a universal class. Denote by $\mathcal{R}^{\mathcal{C}}$ the radical class in $\mathcal{U}$ associated with the radical operation $\mathcal{C}$ defined on $\mathcal{U}$. If $\mathcal{U} = Q(\mathcal{G})$ where $\mathcal{G} \subseteq \mathcal{W}$ is a collection of free Ω-groups, and $\mathcal{C}$ is a radical operation on

$\mathcal{G}$, let $\mathcal{R}^{\mathcal{C}}$ denote the radical class on $Q(\mathcal{G})$ associated with the induced radical operation on all of $Q(\mathcal{G})$ as in Proposition 2.4.

Proposition 2.5. *Let $\mathcal{C} = \{C_R : R \in \mathcal{U}\}$ be a radical operation on $\mathcal{U}$, $I \lhd M \in \mathcal{U}$. Then $\mathcal{R}^{\mathcal{C}}(M/I) = C_M(I)/I$.*

Proof.

$$\begin{aligned}\mathcal{R}^{\mathcal{C}}(M/I) &= \sum\{J/I : J/I \lhd M/I,\ J/I \in \mathcal{R}^{\mathcal{C}}\}\\ &= \sum\{J : J \lhd M,\ I \subseteq J,\ J/I \in \mathcal{R}^{\mathcal{C}}\}/I\\ &= C_M(I)/I. \qquad \square\end{aligned}$$

Lemma 2.6. *Let $\mathcal{G}$ be a class of free Ω-groups in $\mathcal{W}$ such that $Q(\mathcal{G})$ is a universal class, with $\mathcal{C} = \{C_F : F \in \mathcal{G}\}$ a family of closure operations on $\mathcal{G}$. The following conditions are equivalent.*

(i) *$\mathcal{C}$ is a radical operation on $\mathcal{G}$.*

(ii) *Let $F, G \in \mathcal{G}$. For $I, K \lhd F$ with $C_F(K) \subseteq I$ and $J \lhd G$, if $I/C_F(K) \cong C_G(J)/J$, then $I = C_F(K)$.*

Proof. Suppose (ii) holds. We show that, whenever $I, K \lhd F$ with $K \subseteq I$, $J \lhd G$, if $I/K \cong C_G(J)/J$, then $I \subseteq C_F(K)$. Suppose rather that I properly contains $C_F(K)$. Then $C_F(K) \lhd I$ and $(I/K)/(C_F(K)/K) \cong I/C_F(K)$, and so $\{0\} \neq I/C_F(K) \cong (C_G(J)/J)/(L/J) \cong C_G(J)/L$, for some ideal L of G satisfying $J \subseteq L \subseteq C_G(J)$, whence $C_F(J) \subseteq C_G(L) \subseteq C_G(C_G(J))$, so that $C_G(J) = C_G(L)$. Thus $I/C_F(K) \cong C_G(L)/L \neq \{0\}$, which is impossible by assumption, so $I \subseteq C_F(K)$, and $\mathcal{C}$ is a radical operation, that is, (i) holds. The converse is clear. $\square$

3. Ideals and zero sets over Ω-groups

Let $\mathcal{W}$ be a variety of Ω-groups, $M \in \mathcal{W}$. Let F_X be a free Ω-group in $\mathcal{W}$ on the generators $X = \{x_i : i \in \mathcal{J}\}$, $\mathcal{J}$ an index set. Let M^X be the set of functions $X \to M$ viewed as the $|X|$-fold Cartesian product of copies of M; (a_i) denotes a typical element of M^X.

F_X may be be viewed as an algebra of (not necessarily distinct) polynomial functions acting on elements of M^X by substitution. For any $f \in F_X$, we shall use the notation $f(a)$ to denote the result of substituting each a_i in $a = (a_i) \in M^X$ for the corresponding $x_i \in X$ occurring in f. We shall at times use the notation $a_i \in M$ as an abbreviation for $a_1, a_2, \ldots a_k \in M$ for some specified k; similarly for $b_j \in M$, and so on.

Definition 3.1. The *zero set* associated with $H \subseteq F_X$ is the set

$$\mathcal{V}_M^X(H) = \{a : a \in M^X, f(a) = 0 \text{ for all } f \in H\}.$$

The *ideal* associated with $S \subseteq M^X$ is the set

$$\mathcal{I}_M^X(S) = \{f : f \in F_X, f(a) = 0 \text{ for all } a \in S\}.$$

Algebraic geometry provides the canonical example for these concepts. One difference is that the notion of polynomial used here requires all "constant" terms to be zero. In order to minimise confusion between its two quite distinct meanings, we use the term "zero set" in preference to "variety".

Proposition 3.2.

$$\mathcal{I}_M^X(S) \lhd F_X.$$

Proof. Let $f_1, f_2 \in \mathcal{I}_M^X(S)$. Then for all $s \in S$, $(f_1 - f_2)(s) = f_1(s) - f_2(s) = 0 - 0 = 0$, so $f_1 - f_2 \in \mathcal{I}_M^X(S)$, and so $\mathcal{I}_M^X(S)$ is a subgroup of F_X. If $g \in F_X$, then for all $s \in S$,

$$(g + f_1 - g)(s) = g(s) + f_1(s) - g(s) = g(s) + 0 - g(s) = 0,$$

so $g + f_1 - g \in \mathcal{I}_M^X(S)$. Hence $\mathcal{I}_M^X(S)$ is normal.

If $\varrho \in \Omega$ has arity n, $f \in \mathcal{I}_M^X(S)$ and $g_1, g_2, \ldots, g_n \in F_X$, then for all $s \in S$,

$$\begin{aligned}&\big[-\varrho(g_1, g_2, \ldots, f + g_i, \ldots, g_n) + \varrho(g_1, g_2, \ldots, g_i, \ldots, g_n)\big](s)\\ &\quad= -\varrho(g_1(s), \ldots, f(s) + g_i(s), \ldots, g_n(s)) + \varrho(g_1(s), \ldots, g_i(s), \ldots, g_n(s))\\ &\quad= -\varrho(g_1(s), \ldots, 0 + g_i(s), \ldots, g_n(s)) + \varrho(g_1(s), \ldots, g_i(s), \ldots, g_n(s))\\ &\quad= 0,\end{aligned}$$

so

$$-\varrho(g_1, g_2, \ldots, f + g_i, \ldots, g_n) + \varrho(g_1, g_2, \ldots, g_i, \ldots, g_n) \in \mathcal{I}_M^X(S)$$

which is therefore an ideal of F_X. □

Proposition 3.3. *Let H_1, $H_2 \subseteq F_X$ and S_1, $S_2 \subseteq M^X$.*

(i) *If $H_1 \subseteq H_2$, then $\mathcal{V}_M^X(H_1) \supseteq \mathcal{V}_M^X(H_2)$.*

(ii) *If $S_1 \subseteq S_2$, then $\mathcal{I}_M^X(S_1) \supseteq \mathcal{I}_M^X(S_2)$.*

(iii) *$S_1 \subseteq \mathcal{V}_M^X(\mathcal{I}_M^X(S_1))$.*

(iv) *$H_1 \subseteq \mathcal{I}_M^X(\mathcal{V}_M^X(H_1))$.*

The proofs of (i) to (iv) are straightforward. Then establish the existence of a *Galois correspondence* (see [5]) between the lattice of zero sets in M^X and the lattice of ideals of F_X of the form $\mathcal{I}_M^X(S)$, both of which are therefore complete lattices. Hence we have the following

Proposition 3.4. *The function $\mathcal{IV}_M^X : J \mapsto \mathcal{I}_M^X(\mathcal{V}_M^X(J))$, $J \lhd F_X$, is a closure operation on the ideals of F_X.*

In the next section we show that $\mathcal{IV}_M^X$ is in fact a radical closure operation on $\mathcal{U}$ providing there is a Nullstellensatz for M is a certain sense; in such a case the radical and semisimple objects may be described equationally.

4. Equationally defined radical classes arising from Nullstellensatz theorems

For the remainder of this article, we assume that $\mathcal{W}$ is a variety of Ω-groups and $\mathcal{G}$ is a collection of free Ω-groups in $\mathcal{W}$ containing all finitely generated free Ω-groups in $\mathcal{W}$, such that $\mathcal{U} = Q(\mathcal{G})$ is a universal class.

Let $M \in \mathcal{W}$. Let $X_n = \{x_1, x_2, \dots, x_n\}$; then $F_{X_n} \in \mathcal{U}$ for all n by assumption. Let $\mathcal{N} = \{x_1, x_2 \dots\}$. Viewing F_{X_n} as a subset of $F_{X_{n+1}}$ and of $F_\mathcal{N}$ in the obvious way, we see that $F_\mathcal{N} = \bigcup_{i\in\mathbb{N}} F_{X_i}$. For $f \in F_\mathcal{N}$ and $a, a_1, a_2, \dots \in M$, let $f(a, a_1, a_2, \dots)$ be denoted by $f(a, a_i)$.

Definition 4.1. Suppose $\mathcal{F} \subseteq F_\mathcal{N}$, and $M \in \mathcal{U}$ with $S \subseteq M$. Define

$$\mathcal{F}_M(S) = \{a : a \in M, \text{ there exist } f \in \mathcal{F} \text{ and } a_i \in M \\ \text{such that } f(a, a_i) \in (S)_M\}.$$

Lemma 4.2. *Let $\mathcal{F} \subseteq F_\mathcal{N}$ be such that $\mathcal{F}_M(\{0\}) = \{0\}$. Let $R = F_X/\mathcal{IV}_M^X(H)$. Then for all $H \subseteq F_X$, $\mathcal{F}_R(\{0\}) = \{0\}$.*

Proof. Suppose that $\mathcal{F}_M(\{0\}) = \{0\}$, that is, that for some $f \in F_\mathcal{N}$ and for all $a, a_i \in M$, it is the case that $a = 0$ whenever $f(a, a_i)$=0. Suppose $h \in \mathcal{F}_R(\{0\})$; thus there exists $f \in \mathcal{F}$ and $h_i \in R$ such that $f(h, h_i) = 0$. But $h = g + \mathcal{IV}_M^X(H)$ and $h_i = g_i + \mathcal{IV}_M^X(H)$ for some $g, g_i \in F_X$. Hence $f(g + \mathcal{IV}_M^X(H),\ g_i + \mathcal{IV}_M^X(H)) = 0 + \mathcal{IV}_M^X(H)$, that is, $f(g, g_i) \in \mathcal{IV}_M^X(H)$. Hence, for all $(a_j) \in \mathcal{V}_M^X(H)$, $f(g(a_j), g_i(a_j)) = 0$, and so $g(a_j) = 0$, whence $g \in \mathcal{IV}_M^X(H)$. That is, $h = g + \mathcal{IV}_M^X(H) = 0 \in R$. □

Definition 4.3. M possesses a Nullstellensatz in $\mathcal{U}$ with family $\mathcal{F} \subseteq F_{\mathcal{N}}$ if, for each $F_X \in \mathcal{G}$, $\mathcal{F}_{F_X}(H) = \mathcal{IV}_M^X(H)$ for all $H \subseteq F_X$.

Let $\mathcal{G}$ be the class of all free finitely generated commutative $\mathbb{Q}$-algebras, each of which is a copy of $\mathbb{Q}_0[x_1.x_2, \ldots, x_n]$, the ring of rational polynomials in n variables without constant term, $n \in \mathbb{N}$. Now the Hilbert Nullstellensatz may easily be restricted to polynomials having zero constant term (that is, to the elements of $\mathcal{G}$). It then states that $\mathbb{C}$ possesses a Nullstellensatz in $\mathcal{G}$, with family $\mathcal{F} = \{x_1^n : n \in \mathbb{N}\}$.

Let $\mathcal{G}$ be as above, $H \subseteq F_{X_n} = \mathbb{Q}_0[x_1, x_2, \ldots, x_n]$. The Real Nullstellensatz of [6] may likewise be restricted to polynomials having zero constant term, and it then says that $f \in \mathcal{IV}_{\mathbb{R}}^X(H)$ if and only if there exist nonnegative integers c_i, m and t such that $x_1^{2m} + c_2x_2^2 + \cdots + c_tx_t^2 \in (H)_{F_{X_n}}$. Hence $\mathbb{R}$ possesses a Nullstellensatz in $\mathcal{G}$ with family $\mathcal{F} = \{x_1^{2m} + c_2x_2^2 + \cdots + c_tx_t^2 : c_i, m, t \in \mathbb{N}\}$.

We assume for the rest of the section that M possesses a Nullstellensatz in $\mathcal{U}$ with family $\mathcal{F} \subseteq F_{\mathcal{N}}$.

Lemma 4.4.

$$\mathcal{F}_M(\{0\}) = \{0\}.$$

Proof. Let $f \in F$. Then $f \in \bigcup_{i \in \mathbb{N}} F_{X_i}$, so for some $n > 0$, $f \in F_{X_n} \in \mathcal{U}$. Now M possesses a Nullstellensatz with family $\mathcal{F}$, so $\mathcal{F}_{F_{X_n}}(\{f\}) = \mathcal{IV}_M^{X_n}(\{f\})$. But $f(x_1, x_2, \ldots, x_n) \in (f)_{F_{X_n}}$, so by definition, $x_1 \in \mathcal{F}_{F_{X_n}}(\{f\}) = \mathcal{IV}_M^{X_n}(\{f\})$. Thus if $f(a, a_i) = 0$ for some $a, a_i \in M$, then $a = 0$. Now if $a \in \mathcal{F}_M(\{0\})$, then there exists $f \in \mathcal{F}$, $a_i \in M$ such that $f(a, a_i) = 0$, whence $a = 0$. □

Theorem 4.5. *$\mathcal{C} = \{\mathcal{IV}_M^X : F_X \in \mathcal{G}\}$ is a radical operation on $\mathcal{G}$.*

Proof. We begin by noting that $\mathcal{IV}_M^X$ is a closure operation on the ideals of F_X for each $F_X \in \mathcal{G}$ by Proposition 3.4.

Let F_X, $F_Y \in \mathcal{G}$. Let $I \lhd F_X$, $L \lhd F_Y$. Suppose $I/C_{F_X}(K) \cong C_{F_Y}(L)/L \neq \{0\}$. Now $\mathcal{F}_{F_X/C_{F_X}(K)}(\{0\}) = \{0\}$, so for all $f \in F$, for all $a, a_i \in F_X/C_{F_X}(K)$, $f(a, a_i) = 0$ implies $a = 0$ by Lemmas 4.2 and 4.4. Hence the same is true for all $a, a_i \in I/C_{F_X}(K) \lhd F_X/C_{F_X}(K)$, that is, $\mathcal{F}_{I/C_{F_X}(K)}(\{0\}) = \{0\}$.

Let $g \in C_{F_Y}(L) - L$. Then there exist $f \in F$ and $g_i \in C_{F_Y}(L)$ such that $f(g, g_i) \in L$. Hence $f(g + L, g_i + L) = 0 + L$, yet $g + L \neq 0 + L$ since $g \notin L$, so $\mathcal{F}_{C_{F_Y}(L)/L}(\{0\}) \neq \{0\}$, contradicting the assumed isomorphism. Hence $C_{F_Y}(L)/L = \{0\}$ and $\mathcal{C}$ is indeed a radial operation. □

Thus, by Proposition 2.4, all of $\mathcal{U} = Q(\mathcal{G})$ inherits the radical operation $\mathcal{C} = \{C_R : R \in \mathcal{U}\}$ from $\mathcal{G}$ in a natural way.

Proposition 4.6. *For all $R \in \mathcal{U}$, $C_R(I) = \mathcal{F}_R(I)$ for all $I \triangleleft R$.*

Proof. Let $R = F_X/J$, $I \triangleleft R$, with $I = I'/J$ for some $I' \triangleleft F_X$. Now for all $g \in F_X$, the following statements are equivalent:

(i) $b + J \in \mathcal{F}_R(I)$;
(ii) there exist $g_i \in F_X$ such that $f(b + J, g_i + J) \in I = I'/J$;
(iii) there exist $g_i \in F_X$ such that $f(b, g_i) + J \in I'/J$;
(iv) there exist $g_i \in F_X$ such that $f(b, g_i) \in I'$;
(v) $b \in \mathcal{F}_{F_X}(I')$;
(vi) $b + J \in \mathcal{F}_{F_X}(I')/J$.

Hence $\mathcal{F}_R(I) = (\mathcal{F}_{F_X}(I')/J = \mathcal{I}\mathcal{V}_M^X(I')/J = C_{F_X}(I')/J = C_R(I)$. □

Corollary 4.7. *$R \in \mathcal{U}$ is $\mathcal{R}^{\mathcal{C}}$-semisimple if and only if $\mathcal{F}_R(\{0\}) = \{0\}$, and is $\mathcal{R}^{\mathcal{C}}$-radical if and only if $\mathcal{F}_R(\{0\}) = R$.*

Proof. By Theorem 2.3 (ii), $\mathcal{R}^{\mathcal{C}}(R) = C_R(\{0\}) = \mathcal{F}_R(\{0\})$. □

Thus if $M \in \mathcal{W}$ possesses a Nullstellensatz in $\mathcal{U}$ with family $\mathcal{F}$, inducing a radical operation $\mathcal{C}$ on $\mathcal{U}$, then $R \in \mathcal{U}$ is $\mathcal{R}^{\mathcal{C}}$-semisimple if for all $a, a_i \in R$ and $f \in \mathcal{F}$, $f(a, a_i) = 0$ implies $a = 0$. Hence M fails to be semisimple if for some non-zero $a \in M$ there exists $f \in F$ and $a_i \in M$ for which $f(a, a_i) = 0$. On the other hand, M is radical if for *every* $a \in M$ there exists $f \in F$ and $a_i \in M$ for which $f(a, a_i) = 0$.

Example 4.8. The Nil Radical:

Let $\mathcal{G}$ be the class of finitely generated free commutative $\mathbb{Q}$-algebras and $\mathcal{U} = Q(\mathcal{G})$ the universal class of finitely generated commutative $\mathbb{Q}$-algebras. Then by the restricted Hilbert Nullstellensatz and Corollary 4.7, the subclass of $\mathcal{U}$ consisting of all R such that for all $r \in R$ there exists $n>0$ such that $r^n = 0$ is a radical class (the nil radical on $\mathcal{U}$); the semisimple class consists of all $R \in \mathcal{U}$ fo which, for all $r \in R$, if $r^n = 0$ then $r = 0$. Hence the associated Nullstellensatz family is $\{x_1^n : n \in \mathbb{N}\}$. Of course, it turns out that the class of nil associative rings is a radical class although the semisimple rings are not so easily described.

If $\mathcal{U}$ is the class of finitely generated commutative rings, then the Nullstellensatz family for $\mathbb{C}$ becomes $\mathcal{F} = \{mx_1^n : m, n \in \mathbb{N}\}$. The resulting radical class is the *Veldsman radical* on $\mathcal{U}$, and similar remarks apply as for the nil radical.

Example 4.9. The Real Radical:

Let $\mathcal{G}$ and $\mathcal{U}$ be the same as at the beginning of the previous example. The Real Nullstellensatz for $\mathcal{G}$ has family

$$\mathcal{F} = \{x_1^{2m} + c_2x_2^2 + \cdots + c_tx_t^2 : c_i, m, t \in \mathbb{N}\}.$$

The induced radical and semisimple classes are then easily described by means of Corollary 4.7. If $\mathcal{U}$ is instead the class of finitely generated commutative rings, then the associated family is

$$\mathcal{F} = \{c_1x_1^{2m} + c_2x_2^2 + \cdots + c_tx_t^2 : c_i, m, t \in \mathbb{N}\}.$$

Example 4.10. Fields in General:

Let k be a field and let K be a subfield of the algebraic closure of k such that $k \subseteq K$. Let $\mathcal{U}$ be the universal class of finitely generated commutative k-algebras. Let $k_m = k[x_1, x_2, \ldots, x_m]$, the polynomial ring in m commuting variables over k (and *not* simply the free commutative k-algebra on m generators). Let $P_m = \{f \in k_m : f$ is homogeneous and $x_1 \in \mathcal{IV}_K(f)\}$ (where $\mathcal{I}_K$ and $\mathcal{V}_K$ have their obvious meanings in terms of k_m). Let $\mathcal{P} = \bigcup_{m\geq 0} P_m$. Note that $\mathcal{P}$ contains no polynomials with nonzero constant term, so $\mathcal{P} \subseteq F_{\mathcal{N}}$. The main theorem in [3] states that for all $I \lhd k_m$, $\mathcal{IV}_K(I) = \mathcal{P}(I)$. By restricting to polynomials with zero constant term as above, it follows easily that K possesses a Nullstellensatz in $\mathcal{U}$ with family $\mathcal{P}$. Hence there is an induced radical class in $\mathcal{U}$ consisting of all R such that for all $a \in R$ there exists $f \in \mathcal{P}$ and $a_i \in R$ such that $f(a, a_i) = 0$. The previous two examples provide explicit Nullstellensatz families for their respective fields. In general, pinning down such an explicitly defined family is not possible. (See [3].)

Example 4.11. A Ring with Involution:

Let $\mathcal{U}$ be the universal class of finitely generated commutative $\mathbb{Q}$-algebras with involution. Define $\mathbb{C}(i) = \{\alpha + \beta i : \alpha, \beta \in \mathbb{C}\}$ to be the commutative $\mathbb{C}$-algebra with identity satisfying $i^2 = -1$, and with involution defined by $\bar{1} = 1$ and $\bar{i} = -i$. Let $M = \langle \mathbb{C}(i), +, \times, ^{-} \rangle$. By a result in [7], M possesses a Nullstellensatz with family $\mathcal{F} = \{x_1^n : n \in \mathbb{N}\}$.

Example 4.12. The Additive Rationals:

Let $M = \langle \mathbb{Q}, + \rangle$, the additive group of rationals, and let $\mathcal{U} = Q(\mathcal{G})$ be the universal class of finitely generated abelian groups, $\mathcal{G}$ the class of all free groups in $\mathcal{U}$. Each subgroup of each F_{X_n} is finitely generated, so we assume without loss of generality that $H \subseteq F_{X_n}$ is finite. By a simple argument involving row reduction, $f \in \mathcal{IV}_M^X(H)$ if and only if there exist $\alpha_i \in \mathbb{Q}$ and $h_i \in H$ such that $f = \sum_i \alpha_i h_i$. Multiplying through by the lowest common denominator m of the α_i gives $mf = \sum_i \beta_i h_i$ where the

β_i are integers. Hence any $f \in \mathcal{IV}_M^X(H)$ satisfies $mf \in (H)_{F_{X_n}}$. The converse is obvious, and so

$$\begin{aligned}&\mathcal{IV}_M^X(H)\\&\quad = \{f : f \in F_{X_n},\text{ and there exists } m > 0 \text{ such that } mf \in (H)_{F_{X_n}}\}.\end{aligned}$$

Hence M has a Nullstellensatz with family $\mathcal{F} = \{mx_1 : m \in \mathbb{N}\}$.

By Theorem 4.5, $R \in \mathcal{U}$ is radical if and only if, for all $r \in R$, there exists $m > 0$ such that $mr = 0$; this is the class of *torsion groups* in $\mathcal{U}$. R is semisimple if and only if for all $r \in R$, if $mr = 0$ for any $m > 0$, then $r = 0$; this is the class of *torsion-free groups* in $\mathcal{U}$.

5. A special case

In this section, we limit consideration to cases where each element of $\mathcal{G}$ has a countable number of generators and so is embeddable in $\mathcal{F}_{\mathcal{N}}$.

We note that the subset $\mathcal{IV}_M^X(\{0\})$ of $F_X \in \mathcal{G}$ is closed under substitution, in the sense that if $g \in \mathcal{IV}_M^X(\{0\})$, then $g(p_i) \in \mathcal{IV}_M^X(\{0\})$ for all $p_i \in F_X$.

Definition 5.1. M is semantically minimal *in* $\mathcal{U}$ *if, for all* $F_X \in T$ *and* $H \subseteq F_X$, $\mathcal{IV}_M^X(H) = (H)_{F_X} + \mathcal{IV}_M^X(\{0\})$.

Now for any $M \in \mathcal{U}$ and $H \subseteq F_X$, $\mathcal{IV}_M^X(\{0\}) \subseteq \mathcal{IV}_M^X(H)$, and $H \subseteq \mathcal{IV}_M^X(H) \triangleleft F_X$, so $(H)_{F_X} \subseteq \mathcal{IV}_M^X(H)$. Hence M is semantically minimal if $\mathcal{IV}_M^X(H)$ is as small as possible for all $H \subseteq F_X$.

If $\mathcal{U}$ is a variety and M is semantically minimal and generates $\mathcal{U}$ then $\mathcal{IV}_M^X(\{0\}) = \{0\}$, so $\mathcal{IV}_M^X(H) = (H)_{F_X}$. It is then trivially the case that M possesses a Nullstellensatz in $\mathcal{U}$, with family $F = \{x_1\}$. We may generalise this.

Theorem 5.2. *If M is semantically minimal in $\mathcal{U}$, then M possesses a Nullstellensatz in $\mathcal{U}$ with family*

$$\mathcal{F} = \{x_1 + g(x_1, x_2, \dots) : g \in \mathcal{IV}_M^X(\{0\}), F_X \in \mathcal{G}\}.$$

The induced radical class consists of all $R \in \mathcal{U}$ such that, for all $a \in R$, there exist $g \in \mathcal{IV}_M^X(\{0\})$ and $b_i \in R$ such that $a = g(b_i)$. The induced semisimple class is the intersection of $\mathcal{U}$ with the subvariety of $\mathcal{W}$ defined by the equations $g(x_i) = 0$ for all $g \in \mathcal{IV}_M^X(\{0\})$.

Proof. Let M, $\mathcal{F}$ be as in the theorem statement. Then for all X and all $H \subseteq F_X$, $\mathcal{IV}_M^X(H) = (H)_{F_X} + \mathcal{IV}_M^X(\{0\})$. If $h \in \mathcal{F}_{F_{X_n}}(H)$, then there exists $f \in \mathcal{F}$ and $p_i \in F_X$ such that $f(h, p_i) \in (H)_{F_X}$, that is, $h + g(h, p_i) \in (H)_{F_X} \subseteq \mathcal{IV}_M^X(H)$ where $g \in \mathcal{IV}_M^X(\{0\})$, so $g(h, p_i) \in \mathcal{IV}_M^X(\{0\}) \subseteq \mathcal{IV}_M^X(H)$ since $\mathcal{IV}_M^X(\{0\})$ is closed under substitution. Hence $f \in \mathcal{IV}_M^X(H)$.

Conversely, suppose $h \in \mathcal{IV}_M^X(H)$. Then $h \in (H)_{F_X} + \mathcal{IV}_M^X(\{0\})$, so that there exists $g \in \mathcal{IV}_M^X(\{0\})$ such that $h + g \in (H)_{F_X}$. Let $q_i(x_1, x_2, \dots) = x_{i+1}$, $i = 1, 2, \dots$. Let $g'(x_1, x_2, \dots) = g(q_1, q_2, \dots) = g(x_2, x_3, \dots)$. Then $g' \in \mathcal{IV}_M^X(\{0\})$ since $\mathcal{IV}_M^X(\{0\})$ is closed under substitution. Moreover, letting $f(x_1, x_2, \dots) = x_1 + g'(x_1, x_2, \dots) \in \mathcal{F}$ and $p_i(x_1, x_2, \dots) = x_i$, $i = 1, 2, \dots$, we have that $f(h, p_1, p_2, \dots) = h + g'(h, p_1, p_2, \dots) = h + g(x_1, x_2, \dots) \in (H)_{F_X}$. Hence certainly $h \in \mathcal{F}_{F_{X_n}}(H)$.

The descriptions of the radical and semisimple classes induced by the Nullstellensatz for M follow almost immediately from Theorem 4.5. □

Example 5.3. The idempotent radical:

Let R be the zero ring on the rationals. We view R as a $\mathbb{Q}$-algebra by setting αa equal to the usual product of α and a as rationals, for all $\alpha \in \mathbb{Q}$, $a \in R$. Let $F_{X_n} = \mathbb{Q}_0[x_1, x_2, \dots, x_n]$, a typical free ring in the class of finitely generated commutative $\mathbb{Q}$-algebras $\mathcal{U}$ of which R is a member. Let $\mathcal{U} = Q(\bigcup\{F_{X_i} = i \in \mathbb{N}\})$, the universal class of finitely generated $\mathbb{Q}$-algebras. Then $\mathcal{IV}_R^{X_n}(\{0\})$ comprises all polynomials which are sums of products of polynomials, that is, $\sum_j p_j q_j$, where all p_j, $q_j \in F_{X_n}$.

We note that each F_{X_n} is a Noetherian ring, so every ideal has the form $(H)_{F_{X_n}}$ for some finite set $H \subseteq F_{X_n}$, so we assume without loss of generality that H is finite. For any $f \in F_{X_n}$, let f' denote the element of the coset $f + F_{X_n}/\mathcal{IV}_R^{X_n}$ obtained from f by eliminating all non-linear terms in the canonical representation of f. For $H \subseteq F_{X_n}$, let $H' = \{f' : f \in H\}$. Then $\mathcal{V}_R^{X_n}(H) = \mathcal{V}_R^{X_n}(H')$.

Now $\mathcal{V}_R^{X_n}(H')$ is the set of solutions of the linear system obtained by setting each element of H' to zero. If $f \in \mathcal{IV}_R^{X_n}(H) = \mathcal{IV}_R^{X_n}(H')$ then f' is a linear combination of the elements of H', as in Example 4.12. Hence $f' = \sum_i \alpha_i h_i' = \sum_i \alpha_i h_i - \sum_j p_j q_j$ for some products of polynomials $p_j q_j$, where the $h_i \in H$. But $f' = f - \sum_k r_k s_k$ for some products of polynomials $r_k s_k$, so $f = f' + \sum_k r_k s_k = \sum_i \alpha_i h_i - \sum_j p_j q_j + \sum_k r_k s_k \in \mathcal{IV}_R^{X_n} + (H)_{F_{X_n}}$. Conversely, if $f \in \mathcal{IV}_R^{X_n} + (H)_{F_{X_n}}$, then it is obvious that $f \in \mathcal{IV}_R^{X_n}(H)$.

Hence $\mathcal{IV}_R^{X_n}(H) = \mathcal{IV}_R^{X_n}(\{0\}) + (H)_{F_{X_n}}$, so R is semantically minimal. By Theorem 5.2 M therefore has a Nullstellensatz family $\mathcal{F} =$

$\{x_1 - \sum_i p_i q_i : p_i q_i \in F_{X_n}\} \subseteq F_{\mathcal{N}}$, with the induced radical $\mathbb{Q}$-algebras exactly those M such that, for all $a \in M$, there exist rational polynomials p_i, q_i without constant term and $a_i \in M$ such that $a - \sum_i p_i(a_i)q_i(a_i) = 0$; equivalently, $a \in M^2$. Thus M is radical if and only if $M = M^2$, so the induced radical class is nothing but the *idempotent radical* on $\mathcal{U}$. On the other hand, it is easily seen that M is semisimple if and only if it is a zero ring.

Example 5.4. Finite Fields:

Let $\mathcal{U}$ be the universal class of all finitely generated commutative rings. Then for all $n \in \mathbb{N}$, $F_{X_n} = \mathbb{Z}_0[x_1, x_2 \ldots, x_n]$. Let p be a prime, $m > 0$. Let $D_n^{p^m} = \{x_1^{p^m} - x_1, x_2^{p^m} - x_2, \ldots, x_n^{p^m} - x_n, px_1, px_2, \ldots, px_n\} \subseteq F_{X_n}$. Let $B_n^{p^m} = F_{X_n}/(D_n^{p^m})$. It is not hard to show that $(D_n^{p^m})$ sonsists of all elements of the form $\sum_i (f_i^{p^m} - f_i) + \sum_j pg_j$.

Let f' be the image of $f \in F_{X_n}$ under the canonical homomorphism $F_{X_n} \to B_n^{p^m}$; similarly for sets. Again for $H \subseteq F_{X_n}$, let $H' = \{f', f \in H\}$. Let K be the algebraic closure of $GF(p^m)$, the finite field of prime power order p^m.

Now for any $(a_1, a_2, \ldots, a_n) \in \mathcal{V}_K^{X_n}(D_n^{p^m})$, we have $a_i^{p^m} - a_i = 0$ for each i. But K is a field, so $a_i(a_i^{p^m - 1} - 1) = 0$, and so $a_i = 0$ or a_i is a $(p^m - 1)$-th root of 1. But because the multiplicative group of a finite field is a cyclic, each of the $p^m - 1$ non-zero elements of $GF(p^m)$ satisfy this, so all possible such roots in K lie in $GF(p^m)$. Hence $(a_1, a_2, \ldots, a_n) \in GF(p^m)^n \subseteq K^n$. But $\mathcal{V}_K^{X_n}(H \cup D_n^{p^m}) = \mathcal{V}_K^{X_n}(H) \cap \mathcal{V}_K^{X_n}(D_n^{p^m}) = \mathcal{V}_{GF(p^m)}^{X_n}(H) = \mathcal{V}_{GF(p^m)}^{X_n}(H \cup D_n^{p^m})$, so that $\mathcal{IV}_{GF(p^m)}^{X_n}(H) = \mathcal{IV}_{GF(p^m)}^{X_n}(H \cup D_n^{p^m}) = \mathcal{IV}_K^{X_n}(H \cup D_n^{p^m})$. If $f \in \mathcal{IV}_{GF(p^m)}^{X_n}(H)$ then $f \in \mathcal{IV}_K^{X_n}(H \cup D_n^{p^m})$ so there exists $\varrho > 0$ such that $f^\varrho \in (H \cup D_n^{p^m})_{F_{X_n}}$ by the Hilbert Nullstellensatz, and so $f^{p^{ms}} \in (H \cup D_n^{p^m})_{F_{X_n}}$ where s is chosen such that $p^{ms} > \varrho$. Hence $f \in (H \cup D_n^{p^m})_{F_{X_n}}$ (since the quotient ring $F_{X_n}/(H \cup D_n^{p^m}) \cong B_n^{p^m}/(H')_{B_n^{p^m}}$ satisfies $g^{(p^m)^r} = g$ for all $r > 0$), so $f' \in (H')_{B_n^{p^m}} \triangleleft B_n^{p^m}$, since $(H')_{B_n^{p^m}} = (H)'_{B_n^{p^m}}$. Hence $f \in (H)_{F_{X_n}} + (D_n^{p^m})_{F_{X_n}}$.

But $(D_n^{p^m})_{F_{X_n}} \subseteq \mathcal{IV}_{GF(p^m)}^{X_n}(\{0\})$, so $\mathcal{IV}_{GF(p^m)}^{X_n}(H) \subseteq (H)_{F_{X_n}} + (D_n^{p^m})_{F_{X_n}} \subseteq (H)_{F_{X_n}} + \mathcal{IV}_{GF(p^m)}^n \subseteq \mathcal{IV}_{GF(p^m)}^{X_n}(H)$, and so the three sets are equal. Hence $GF(p^m)$ is semantically minimal. Letting $H = \{0\}$, we obtain $(D_n^{p^m})_{F_{X_n}} = \mathcal{IV}_{GF(p^m)}^n$, so by Theorem 5.2, $GF(p^m)$

has a Nullstellensatz, with family

$$\mathcal{F} = \Big\{x_1 - \sum_i (g_i^{p^m} - g_i) - \sum_i ph_j : g_i, h_j \in F_{X_n}\Big\}.$$

By Theorem 5.2, the induced radical class consists of all finitely generated commutative rings $R \in \mathcal{U}$ such that, for all $r \in R$, there exist $r_k \in R$ for which $r = \sum_i [g_i(r_k)^{p^m} - g_i(r_k)] - p\sum_j h_j(r_k)$, for some g_i, $h_j \in \mathbb{Z}_0[x_1, x_2, \ldots, x_m]$ for some m. Equivalently, R is such that for all $r \in R$, there exist $a_i, b \in R$ such that $r = \sum_i (a_i^{p^m} - a_i) + pb$. The semisimple class is likewise easily seen to be all $R \in \mathcal{U}$ satisfying the identities $x^{p^m} = x$, $px = 0$; when $p = 2$ and $m = 1$, this is the class of finitely generated *Boolean rings.*

Example 5.5. The Abelian group $\mathbb{Z}_p$:

Let $M = \langle \mathbb{Z}_p, + \rangle, \mathcal{U}$ and $\mathcal{G}$ as in Example 4.12. Again, every subgroup of F_{X_n} is finitely generated, so let $H \subseteq F_{X_n}$ be finite, $f \in F_{X_n}$. Let f' denote the element of F_{X_n} obtained by reducing the coefficients of f modulo p; as usual, let $H' = \{f' : f \in H\}$.

Now $\mathcal{IV}_M^{X_n}(H) = \mathcal{IV}_M^{X_n}(H')$, and $f \in \mathcal{IV}_M^{X_n}(H)$ if and only if $f' \in \mathcal{IV}_M^{X_n}(H')$. But because $\mathbb{Z}_p$ is a field, by the same linear algebra argument as was used in Example 5.3, $f' \in \mathcal{IV}_M^{X_n}(H')$ if and only if $f' \in \sum_i \alpha_i h_i'$, $0 \leq \alpha_i \leq p-1$, $h_i' \in H$. But each $h_i' = h_i + \sum_j p\alpha_{ij}x_j$ for some $\alpha_{ij} \in \mathbb{Z}$. Likewise $f' = f + \sum_j p\beta_j x_j$, so $f = \sum_i \alpha_i h_i' - \sum_j p\beta_j x_j = \sum_i \alpha_i(h_i + \sum_j p\alpha_{ij}x_j) - \sum_j p\beta_j x_j = \sum_i \alpha_i h_i + pg$ for some $g \in F_{X_n}$. Conversely, any f of this form is evidently an element of $\mathcal{IV}_M^{X_n}(H)$, so $\mathcal{IV}_M^{X_n}(H) = \{f + pg : f \in (H)_{F_{X_n}}, g \in F_{X_n}\} = (H)_{F_{X_n}} + \{pg : g \in F_{X_n}\}$. Now $J = \{pg : g \in F_{X_n}\} \triangleleft F_{X_n}$, with $J \subseteq \mathcal{IV}_M^{X_n}$. Because $(H)_{F_{X_n}} + \mathcal{IV}_M^{X_n} \subseteq \mathcal{IV}_M^{X_n}(H)$, it follows that $J = \mathcal{IV}_M^{X_n}$. Hence M is semantically minimal, and so Theorem 5.2 may be applied to show that M has a Nullstellensatz with family

$$\mathcal{F} = \{x_1 + pg(x_1, x_2, \ldots) : g \in F_{X_n}\}.$$

Simplifying a little, we see that M induces a radial class consisting of all $R \in \mathcal{U}$ such that for all $r \in R$, there exist $b \in R$ such that $a = pb$. This is the class of *p-divisible* groups in $\mathcal{U}$. The corresponding semisimple class is all R such that $pb = 0$ for all $r \in R$, the class of *p-torsion* groups in $\mathcal{U}$.

6. The associating property for $\mathcal{F}$

Let $\mathcal{Y}$ be a universal class. In [4], we define $\mathcal{H} \subseteq F_{\mathcal{N}}$ to be $\mathcal{Y}$-*associating* if, for all $M \in \mathcal{Y}$, $f, g \in \mathcal{F}$, $r, a_i \in M$ and $b_j \in I$ where I is some ideal of M such that $\mathcal{H}_I(\{0\}) = I$, if $g(h(r, a_i), b_j) = 0$, then there exists $h \in \mathcal{F}$ and $c_k \in M$ such that $h(r, c_k) = 0$. It was shown in [4] that if $\mathcal{H}$ is $\mathcal{Y}$-associating, then $\mathcal{RH} = \{M : M \in \mathcal{U},\ M = \mathcal{H}_M(\{0\})\}$ is a radical class in $\mathcal{Y}$, and the following result was proved.

Proposition 6.1. *Let $F \subseteq F_{\mathcal{N}}$. If $\mathcal{H}_M(\{0\}) \triangleleft M$ for all $M \in \mathcal{Y}$, and if $\mathcal{H}_{\mathcal{H}_M(\{0\})}(\{0\}) = \mathcal{H}_M(\{0\})$ for all $M \in \mathcal{Y}$, then $\mathcal{RH}$ is a radical class if and only if $\mathcal{H}$ is $\mathcal{Y}$-associating.*

From this, we obtain the following

Proposition 6.2. *Suppose M possesses a Nullstellensatz in $\mathcal{U}$ with family $\mathcal{F}$. Then $\mathcal{F}$ is $\mathcal{U}$-associating.*

Proof. Let $\mathcal{C}$ be the radical operation on $\mathcal{U}$ induced by the Nullstellensatz for M. Then $\mathcal{R}^{\mathcal{C}} = \mathcal{RF}$ by Theorem 4.5. From Proposition 4.6 and Theorem 2.3 (ii), $\mathcal{F}_R(\{0\}) = \mathcal{R}^{\mathcal{C}}(R) \triangleleft R$. Finally, $\mathcal{F}_{\mathcal{F}_R(\{0\})}(\{0\}) = \mathcal{R}^{\mathcal{C}}(\mathcal{R}^{\mathcal{C}}(R)) = \mathcal{R}^{\mathcal{C}}(R) = \mathcal{F}_R(\{0\})$. Hence by Proposition 6.1, $\mathcal{F}$ is $\mathcal{U}$-associating. □

All the examples of Nullstellensatz families $\mathcal{F} \subseteq F_{\mathcal{N}}$ given in previous sections are in fact *strongly associating* as defined in [4]: $\mathcal{F}$ is strongly associating if for every $f, g \in \mathcal{F}$, there exists $h \in \mathcal{F}$ and $p_k \in F_{\mathcal{N}}$ such that $f(g(x, y_i), z_j) = h(x, p_k(x, y_i, z_j))$. This property is expressed purely in terms of the free algebra $F_{\mathcal{N}}$ and is thus independent of $\mathcal{U}$. If a Nullstellensatz family $\mathcal{F}$ is strongly associating then the induced radical class may be extended to the whole of the variety $\mathcal{W}$ containing $\mathcal{U}$. Thus, for instance, the nil, real and idempotent radicals are actually radical classes of commutative $\mathbb{Q}$-algebras (the nil and idempotent radical classes in fact being radical classes of associative rings). It is easily shown that the Nullstellensatz family arising from a semantically minimal M as in Theorem 5.2 must always be strongly associating.

It is a question of some interest to determine which radical classes arise in some way from a Nullstellensatz theorem. Thus the Jacobson radical for all rings — even all commutative rings — may not arise from a Nullstellensatz theorem, even though the defining family $\mathcal{H} = \{x_1 + x_2 + x_1x_2\}$ is (strongly) associating. However, within the universal class of all finitely generated commutative $\mathbb{Q}$-algebras (where the Jacobson radical equals the nil radical), it does so arise.

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N. R. MCCONNELL
DEPARTMENT OF MATHEMATICS AND COMPUTING
UNIVERSITY OF CENTRAL QUEENSLAND
ROCKHAMPTON M.C.
QUEENSLAND, 4702
AUSTRALIA

TIMOTHY STOKES
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TASMANIA
HOBART, TASMANIA, 7001
AUSTRALIA

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