Publ. Math. Debrecen
47 / 1-2 (1995), 81–93

Bitopologies and quasi-uniformities on spaces of continuous functions

By SALVADOR ROMAGUERA (Valencia) and MARCOS RUIZ-GÓMEZ (Valencia)

Abstract. We introduce and investigate the notions of 2compact open bitopology and bitopology of quasi-uniform convergence (on 2compacta). In particular, a characterization of 2compact quasi-uniformizable spaces in terms of these bitopologies and a necessary and sufficient condition for (bicomplete) quasi-pseudometrizability of the 2compact open bitopology are presented.

1. Introduction and preliminaries

In this paper the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and positive integer numbers, respectively. If τ is a topology on a set X and $A \subseteq X$, then $\tau \operatorname{cl} A$ ($\tau \operatorname{int} A$) will denote the closure (interior) of A in the topological space (X, τ).

It is well-known that on the space $C(X, Y)$ of continuous functions from a Tychonoff space X into a space Y, the topological properties of X and Y interact with the topological properties of $C(X, Y)$ for the compact open topology, the topology of uniform convergence (on compacta) etc. (See, for example, [4], [8], [9], [10]).

In this paper we introduce and study the notions of 2compact open bitopology and bitopology quasi-uniform convergence (on 2compacta).

We will show that the use of these bitopologies provides several appropriate extensions of classical topological results. For instance, we prove (Theorem 1) that given two bitopological spaces (X, τ_1, τ_2) and (Y, τ_1', τ_2'), their 2compact open bitopology is quasi-uniformizable if and only if

Mathematics Subject Classification: 54E55, 54C35, 54E15, 54D30, 54E35, 54E50.
Key words and phrases: 2compact open bitopology, bitopology of quasi-uniform convergence, quasi-uniformizable space, quasi-metrizable space, hemicompact space, 2compact space.
The first author is supported by DGICYT grant PB89-0611.

(Y, τ_1', τ_2') is quasi-uniformizable. We also characterize (Theorem 2) 2compact bitopological spaces in terms of the 2compact open bitopology and the bitopology of quasi-uniform convergence. The key of this characterization is the bitopological counterpart (see Lemma 2) of a result proved by R. Arens [1] about the coincidence between the compact open topology and the topology of uniform convergence on compacta. Lemma 2 is also used for obtaining (Theorem 4) a necessary and sufficient condition for quasi-pseudometrizability of the 2compact open bitopology in terms of hemicompactness of $(X, \tau_1, \vee\tau_2)$ and quasi-pseudometrizability of (Y, τ_1', τ_2') similar to the metric case. Furthermore, some relevant examples related to these subjects are presented. In particular, the Sorgenfrey line and the Michael line are considered.

A bitopology on a set X is a pair (τ_1, τ_2) such that each τ_i, $i = 1, 2$, is a topology on X. A bitopological space is [5] an ordered triple (X, τ_1, τ_2) such that X is a non-empty set and (τ_1, τ_2) is a bitopology on X. We say that two bitopologies (τ_1, τ_2) and (τ_1', τ_2') on X coincide if $\tau_i = \tau_i'$, $i = 1, 2$. Given a bitopology (τ_1, τ_2) on X, we denote by $\tau_1 \vee \tau_2$ the supremum topology of τ_1 and τ_2.

A bitopological space (X, τ_1, τ_2) is called:

(i) 2Hausdorff [14] if $(X, \tau_1 \vee \tau_2)$ is a Hausdorff space.

(ii) pairwise Hausdorff [5] if, for $x \neq y$, there is a τ_i-neighborhood of x and a disjoint τ_j-neighborhood of y; $i, j = 1, 2$; $i \neq j$.

(iii) pairwise completely regular [7] if for each $x \in X$ and each τ_i-open set U with $x \in U$ there is a τ_i-lower semicontinuous and τ_j-upper semicontinuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) = 0$; $i, j = 1, 2$; $i \neq j$.

(iv) 2compact [14] if $(X, \tau_1 \vee \tau_2)$ is compact.

A quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that: (i) for each $U \in \mathcal{U}$, $\Delta = \{(x, x) : x \in X\} \subseteq U$ and (ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ satisfying $V^2 \subseteq U$ where $V^2 = V \circ V$.

If $\mathcal{U}$ is a quasi-uniformity on X, then $T(\mathcal{U}) = \{A \subseteq X : \text{if } x \in A$ there is $U \in \mathcal{U}$ with $U(x) \subseteq A\}$ is a topology on X, where $U(x) = \{y \in X : (x, y) \in U\}$. On the other hand, for each $U \in \mathcal{U}$ we can define $U^{-1} = \{x, y\} : (x, y) \in U\}$. Then $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X called conjugate of $\mathcal{U}$. The coarsest uniformity finer than both $\mathcal{U}$ and $\mathcal{U}^{-1}$ is denoted by $\mathcal{U} \vee \mathcal{U}^{-1}$.

We say that a quasi-uniformity $\mathcal{U}$ on X is compatible with a bitopology (τ_1, τ_2) on X if $T(\mathcal{U}) = \tau_1$ and $T(\mathcal{U}^{-1}) = \tau_2$. A bitopological space (X, τ_1, τ_2) is said to be quasi-uniformizable if there is a quasi-uniformity $\mathcal{U}$ on X compatible with (τ_1, τ_2).

A quasi-pseudometric on a set X is a non-negative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$ and (ii) $d(x, y) \leq$

$d(x,z)+d(z,y)$. A quasi-pseudometric d is called separating if $d(x,y)+d(y,x)>0$ whenever $x\neq y$ and is called a quasi-metric if $d(x,y)>0$ whenever $x\neq y$.

Each quasi-(pseudo)metric d on X generates a topology $T(d)$ on X which has as a base the family of d-balls $\{B_d(x,r): x\in X, r>0\}$ where $B_d(x,r)=\{y\in X: d(x,y)<r\}$. Since the conjugate of d, d^{-1}, given by $d^{-1}(x,y)=d(y,x)$ is also a quasi-(pseudo)metric, there is another topology $T(d^{-1})$ on X. Thus, a quasi-(pseudo)metric d on X is called compatible with a bitopology (τ_1,τ_2) on X if $T(d)=\tau_1$ and $T(d^{-1})=\tau_2$. A bitopological space (X,τ_1,τ_2) is said to be (separated) quasi-(pseudo)metrizable if there is a (separating)quasi-(pseudo)metric on X compatible with (τ_1,τ_2).

Remark 1. Note that a quasi-pseudometric d on X is separating if and only if $(X,T(d),T(d^{-1}))$ is 2Hausdorff. Similarly, d is quasi-metric if and only if $(X,T(d),T(d^{-1}))$ is pairwise Hausdorff.

If d is a separating quasi-pseudometric on a set X, then $d\vee d^{-1}$, defined by $(d\vee d^{-1})(x,y)=\max\{d(x,y),d^{-1}(x,y)\}$ is a metric on X compatible with $T(d)\vee T(d^{-1})$.

Let $X=\mathbb{R}$ and let $d(x,y)=\max\{y-x,0\}$ for all $x,y\in X$. Then, d is a separating quasi-pseudometric on X and basic $T(d)$-open sets are of the form $]-\infty,a[$, $a\in\mathbb{R}$; basic $T(d^{-1})$-open sets are of the form $]a,+\infty[$, $a\in\mathbb{R}$. The bitopological space $(X,T(d),T(d^{-1}))$ is 2Hausdorff (with $T(d\vee d^{-1})$ the usual topology on $\mathbb{R}$) but not pairwise Hausdorff. In the rest of the paper, we denote by u and l, above topologies $T(d)$ and $T(d^{-1})$, respectively. Note that $([0,1],u,l)$ is a 2compact space.

2. The 2compact open bitopology

Let (X,τ_1,τ_2) and (Y,τ'_1,τ'_2) be two bitopological spaces. A function f from X into Y is called bicontinuous if it is continuous from (X,τ_i) into (Y,τ'_i) for $i=1,2$. In this case we will say that $f:(X,\tau_1,\tau_2)\to(Y,\tau'_1,\tau'_2)$ is bicontinuous.

Now denote by Y^X the set of all continuous functions from $(X,\tau_1\vee\tau_2)$ into $(Y,\tau'_1\vee\tau'_2)$ and by BY^X the subset of Y^X which consists of all bicontinuous functions from (X,τ_1,τ_2) into (Y,τ'_1,τ'_2). In particular, by $BY^{[0,1]}$ we will denote the set of all bicontinuous functions from $([0,1],u,l)$ into (Y,τ'_1,τ'_2) and by $B[0,1]^X$ the set of all bicontinuous functions from (X,τ_1,τ_2) into $([0,1],u,l)$. Similarly we define $BY^{\mathbb{R}}$ and $B\mathbb{R}^X$. (In some results we will assume that (Y,τ'_1,τ'_2) contains a pairwise path, i.e., a bicontinuous function $p:([0,1],u,l)\to(Y,\tau'_1,\tau'_2)$ such that $p(0)\in Y\setminus\tau'_1\operatorname{cl}p(1)$ and $p(1)\in Y\setminus\tau'_2\operatorname{cl}p(0)$).

Let $\mathcal{K} = \{K \subseteq X : K$ is $\tau_1 \vee \tau_2$-compact$\}$. For each $K \in \mathcal{K}$ and each $G_i \in \tau'_i$, $i = 1, 2$; consider the set

$$[K, G_i] = \{f \in Y^X : f(K) \subseteq G_i\}.$$

Obviously, $\{[K, G_i] : K \in \mathcal{K}$ and $G_i \in \tau'_i\}$ is a subbase for a topology T^i_k, $i = 1, 2$, on Y^X. We will say that the bitopology (T^1_k, T^2_k) is the 2compact open bitopology on Y^X. The subset BY^X of Y^X with the restriction of this bitopology is denoted by (BY^X, T^1_k, T^2_k).

Example 1. Let $X = Y = \mathbb{R}$, $\tau_1 = \tau'_1 = u$ and $\tau_2 = \tau'_2 = l$. Then the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = |x|$ for all $x \in \mathbb{R}$, is in Y^X but, clearly, not in BY^X.

Remark 2. $B\mathbb{R}^X$ is dense in $(\mathbb{R}^X, T^i_k)$, $i = 1, 2$. In fact, given $f \in \mathbb{R}^X$ such that $f \in \bigcap\{[K_j, G_j] : j = 1, \ldots, n\}$ with $K_j \in \mathcal{K}$ and G_j u-open, then $G_j =]-\infty, a_j[$, $j = 1, \ldots, n$. Let $a = \min\{a_1, \ldots, a_n\}$ and $g \in B\mathbb{R}^X$ such that $g(x) = a - 1$ for all $x \in X$. Then $g \in \bigcap\{[K_j, G_j] : j = 1, \ldots, n\}$. Similarly, for G_j l-open sets.

The following example shows that, in general, the topologies T^1_k and T^2_k are not comparable.

Example 2. Let $X = Y = \mathbb{R}$, $\tau_1 = \tau'_1 = u$ and $\tau_2 = \tau'_2 = l$. If $f : \mathbb{R} \to \mathbb{R}$ is the identity function on $\mathbb{R}$, then $f \in BY^X \subseteq Y^X$. Put $K = \{1\}$. So, $f \in [K, G] \in T^2_k$ where $G =]0, +\infty[$. Assume W a T^1_k-open set such that $f \in W \subseteq [K, G]$. Then there exist $\tau_1 \vee \tau_2$-compact sets $L_1, \ldots, L_n$, and τ'_1-open sets $H_1, \ldots, H_n$, such that $f \in \bigcap\{[L_j, H_j] : j = 1, \ldots, n\} \subseteq W$. Therefore $L_j \subset H_j$, $j = 1, \ldots, n$. Let $H_j =]-\infty, b_j[$ and $b = \max\{b_j : j = 1, \ldots, n\}$. If $b \leq 0$, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x$ for $x \leq b$ and $g(x) = b$ for $x > b$, satisfies $g \in [L_j, H_j]$, $j = 1, \ldots, n$. However, $g(1) = b \leq 0$, this is $g \notin [K, G]$. If $b > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x - b$ for $x \leq b$ and $g(x) = 0$ for $x > b$. Newly, $g \in [L_j, H_j]$, $j = 1, \ldots, n$, but $g(1) \leq 0$. Thus, we have shown that $T^2_k \not\subset T^1_k$. Similarly, $T^1_k \not\subset T^2_k$.

The easy proof of the following result is omitted.

Proposition 1. *Let (X, τ_1, τ_2) and (Y, τ'_1, τ'_2) be two bitopological spaces. Then:*

(a) (Y^X, T^1_k, T^2_k) is 2Hausdorff if and only if (Y, τ'_1, τ'_2) is 2Hausdorff.

(b) (Y^X, T^1_k, T^2_k) is pairwise Hausdorff if and only if (Y, τ'_1, τ'_2) is pairwise Hausdorff.

In the proof of our next result we will use the fact that a bitopological space is quasi-uniformizable if and only if it is pairwise completely regular [7, Theorem 4.2].

Theorem 1. *Let (X, τ_1, τ_2) and (Y, τ'_1, τ'_2) be two bitopological spaces. Then the following are equivalent.*

(1) (Y, τ'_1, τ'_2) is quasi-uniformizable.

(2) (Y^X, T^1_k, T^2_k) is quasi-uniformizable.

(3) (BY^X, T^1_k, T^2_k) is quasi-uniformizable.

Proof. (1) $\to$ (2). Let $f \in Y^X$, $K \in \mathcal{K}$ and $G \in \tau'_1$ such that $f \in [K, G]$. Then for each $a \in K$ there is $\psi_a \in B[0,1]^Y$ such that $\psi_a(f(a)) = 0$ and $\psi_a(Y \setminus G) = 1$. Take $\varepsilon > 0$ with $\varepsilon < 1$. So, there is a finite subset $K' \subseteq K$ such that $\{\psi_a^{-1}[0, \varepsilon[\,: a \in K'\}$ is a τ'_1-open cover of $f(K)$. Put $\psi = \min\{\psi_a : a \in K'\}$ and $\Phi : Y \to [0,1]$ such that $\Phi(y) = 0$ if $0 \le \psi(y) < \varepsilon$ and $\Phi(y) = (\psi(y) - \varepsilon)/(1 - \varepsilon)$ otherwise. It is easy to see that $\Phi \in B[0,1]^Y$ and that $\Phi(f(a)) = 0$ for all $a \in K$ and $\Phi(Y \setminus G) = 1$. Now define $\varphi : Y^X \to [0,1]$ by

$$\varphi(h) = \sup\{\Phi(h(a)) : a \in K\}.$$

Then φ is a bicontinuous function from (Y^X, T^1_k, T^2_k) into $([0,1], u, l)$. In fact, if $0 < \delta \le 1$ and $h \in \varphi^{-1}[0, \delta[$, there is $\mu > 0$, $\mu < \delta$, with $h(K) \subseteq U$ where $U = \Phi^{-1}[0, \mu[$. So, $h \in [K, U] \in T^1_k$ and, clearly, $[K, U] \subseteq \varphi^{-1}[0, \delta[$. Now, let $0 \le \delta < 1$ and $h \in \varphi^{-1}]\delta, 1]$. In this case there exists $a_0 \in K$ with $\Phi(h(a_0)) > \delta$. Thus, $h \in [a_0, U] \in T^2_k$ where $U = \Phi^{-1}]\delta, 1]$. If $g \in [a_0, U]$ it follows $\varphi(g) > \delta$ and, hence, $[a_0, U] \subseteq \varphi^{-1}]\delta, 1]$. Furthermore, $\varphi(f) = 0$ and $\varphi(Y^X \setminus [K, G]) = 1$. When $G \in \tau'_2$, a similar argument permits us to conclude that (Y^X, T^1_k, T^2_k) is a pairwise completely regular space. Consequently, it is quasi-uniformizable.

(2) $\to$ (3). Obvious.

(3) $\to$ (1). Let $\mathcal{U}$ be a quasi-uniformity on BY^X compatible with (T^1_k, T^2_k). For each $y \in Y$ define $f_y : X \to Y$ by $f_y(x) = y$ for all $x \in X$. Now put for each $U \in \mathcal{U}$, $\hat{U} = \{(y, z) \in Y \times Y : (f_y, f_z) \in U\}$. Then $\{\hat{U} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity on Y compatible with (τ'_1, τ'_2) (we omit the details).

Remark 3. It follows from the preceding theorem that for every space (X, τ_1, τ_2), (BY^X, T^1_k, T^2_k) is quasi-uniformizable whenever $Y = \mathbb{R}$, $\tau'_1 = u$ and $\tau'_2 = l$. Similarly, for Y any set, τ'_1 any T_1 topology on Y and τ'_2 the discrete topology on Y.

3. The bitopology of quasi-uniform convergence (on 2compacta)

Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces such that (Y, τ_1', τ_2') is quasi-uniformizable and let $\mathcal{U}$ be a quasi-uniformity on Y compatible with (τ_1', τ_2'). Then, for each $K \in \mathcal{K}$ and each $U \in \mathcal{U}$ we should consider the set

$$(K, U) = \{(f, g) \in Y^X \times Y^X : (f(x), g(x)) \in U \text{ for all } x \in K\}$$

It is easy to show that $\{(K, U) : K \in \mathcal{K} \text{ and } U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_k$ on Y^X called the quasi-uniformity of quasi-uniform convergence (of $\mathcal{U}$) on 2compacta. The bitopology $(T(\mathcal{U}_k), T(\mathcal{U}_k^{-1}))$, generated by $\mathcal{U}_k$ is said to be the bitopology of quasi-uniform convergence (of $\mathcal{U}$) on 2compacta.

Similarly, for each $U \in \mathcal{U}$ we should consider the set

$$(X, U) = \{(f, g) \in Y^X \times Y^X : (f(x), g(x)) \in U \text{ for all } x \in X\}.$$

Then, $\{(X, U) : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_X$ on Y^X called the quasi-uniformity of quasi-uniform convergence (of $\mathcal{U}$). The bitopology $(T(\mathcal{U}_X), T(\mathcal{U}_X^{-1}))$ generated by $\mathcal{U}_X$ is said to be the bitopology of quasi-uniform convergence (of $\mathcal{U}$).

If $\mathcal{U}$ is a quasi-uniformity on Y compatible with (τ_1', τ_2') and $(\mathcal{U} \vee \mathcal{U}^{-1})_X ((\mathcal{U} \vee \mathcal{U}^{-1})_k)$ denotes the uniformity of uniform convergence (on compacta) of $\mathcal{U} \vee \mathcal{U}^{-1}$ relative to $(X, \tau_1 \vee \tau_2)$ and $(Y, \tau_1' \vee \tau_2')$, then we have:

Lemma 1. *$(\mathcal{U} \vee \mathcal{U}^{-1})_k = \mathcal{U}_k \vee \mathcal{U}_k^{-1}$ and $(\mathcal{U} \vee \mathcal{U}^{-1})_X = \mathcal{U}_X \vee \mathcal{U}_X^{-1}$.*

Proof. Note that if $K \in \mathcal{K}$ and $U \in \mathcal{U}$ then $(K, \mathcal{U} \cap U^{-1}) = (K, U) \cap (K, U^{-1}) = (K, U) \cap (K, U)^{-1}$. Hence, $(\mathcal{U} \vee \mathcal{U}^{-1})_k \subseteq (\mathcal{U}_k \vee \mathcal{U}_k^{-1})$. On the other hand, given $K, L \in \mathcal{K}$ and $U, V \in \mathcal{U}$ we have $(K \cup L, U \cap V^{-1}) \subseteq (K, U) \cap (L, V^{-1})$ and, thus, $(\mathcal{U}_k \vee \mathcal{U}_k^{-1}) \subseteq (\mathcal{U} \vee \mathcal{U}^{-1})_k$. The second equality follows similarly.

A quasi-uniformity $\mathcal{U}$ on a set X is called bicomplete [3] if the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ is complete (see also [6], [13]).

Corollary. *Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces. If (τ_1', τ_2') has a compatible bicomplete quasi-uniformity $\mathcal{U}$ then $\mathcal{U}_X$ is a bicomplete quasi-uniformity.*

Proof. Since $\mathcal{U} \vee \mathcal{U}^{-1}$ is a complete uniformity, it follows from [4, Chapter 7, Theorem 9] that $(\mathcal{U} \vee \mathcal{U}^{-1})_X$ is a complete uniformity on Y^X. By Lemma 1, $\mathcal{U}_X$ is bicomplete.

In [1], ARENS proved the following crucial result. Let $\mathcal{F}$ be a family of continuous functions on a topological space X to a uniform space $(Y,\mathcal{U})$. Then the topology of uniform convergence on compacta is the compact open topology. The next result, which plays an important role in our study, provides an approprite extension of Arens' theorem.

Lemma 2. *Let (X,τ_1,τ_2) be a space and (Y,τ_1',τ_2') a quasi-uniformizable space. Then, for each quasi-uniformity on Y compatible with (τ_1',τ_2') the bitopology of quasi-uniform convergence on 2compacta coincides with the 2compact open bitopology.*

PROOF. Let $\mathcal{U}$ be a quasi-uniformity on Y compatible with (τ_1',τ_2'). We will show that $T(\mathcal{U}_k)=T_k^1$ and $T(\mathcal{U}_k^{-1})=T_k^2$. Let $f\in Y^X$, $K\in\mathcal{K}$ and $G\in\tau_1'$ such that $f\in[K,G]$. For each $a\in K$ there is $U_a\in\mathcal{U}$ with $U_a(f(a))\subseteq G$. Take $V_a\in\mathcal{U}$ such that $V_a^2\subseteq U_a$. Then there exists a finite subset $K'\subseteq K$ such that $f(K)\subseteq\bigcup\{V_a(f(a)):a\in K'\}$. Put $V=\bigcap\{V_a:a\in K'\}$. Thus, $V\in\mathcal{U}$. Now we prove that $(K,V)(f)\subseteq[K,G]$. In fact, given $g\in(K,V)(f)$ we have $(f(x),g(x))\in V$ for all $x\in K$. Since for each $x\in K$ there is $a\in K'$ with $f(x)\in V_a(f(a))$ it follows $(f(a),g(x))\in V_a^2\subseteq U_a$. So, $g(x)\in U_a(f(a))\subseteq G$ for all $x\in K$. This proves that $T_k^1\subseteq T(\mathcal{U}_k)$. Similarly, $T_k^2\subseteq T(\mathcal{U}_k^{-1})$. In order to prove the opposite inclusions let $f\in Y^X$, $K\in\mathcal{K}$ and $U\in\mathcal{U}$. Take $V\in\mathcal{U}$ such that $V^3\subseteq U$. Put $V^*=V\cap V^{-1}$. Then there is a closed entourage $W\in(\mathcal{U}\vee\mathcal{U}^{-1})$ with $W\subseteq V^*$. Now there also exists a finite subset $K'\subseteq K$ such that $f(K)\subseteq\bigcup\{W(f(a)):a\in K'\}$. For each $a\in K'$ consider the $\tau_1\vee\tau_2$-compact set $K_a=K\cap f^{-1}(W(f(a)))$. If for each $a\in K'$ we write $G_a=\tau_1'\operatorname{int}V^2(f(a))$ and $A=\bigcap\{[K_a,G_a]:a\in K'\}$, then $f\in A\in T_k^1$. Furthermore, given $g\in A$ and $x\in K$ there is $a\in K'$ with $f(x)\in W(f(a))$. Thus, $(f(a),f(x))\in V^*$. Therefore $(f(x),g(x))\in V^3\subseteq U$. Consequently, $g\in(K,U)(f)$. This proves that $T(\mathcal{U}_k)\subseteq T_k^1$. Similarly we show that $T(\mathcal{U}_k^{-1})\subseteq T_k^2$. The proof is complete.

As a first application of Lemma 2 we obtain the following characterization of 2compact spaces.

Theorem 2. *A 2Hausdorff quasi-uniformizable space (X,τ_1,τ_2) is 2-compact if and only if for each quasi-uniformizable space (Y,τ_1',τ_2') containing a pairwise path and each quasi-uniformity on Y compatible with (τ_1',τ_2'), the bitopology of quasi-uniform convergence coincides with the 2compact open bitopology.*

PROOF. Let (X,τ_1,τ_2) be a 2Hausdorff 2compact space and $\mathcal{U}$ a quasi-uniformity on Y compatible with (τ_1',τ_2'). Since (X,τ_1,τ_2) is 2compact, the bitopology of quasi-uniform convergence (of $\mathcal{U}$) coincides with the bitopology of quasi-uniform convergence (of $\mathcal{U}$) on 2compacta. By

Lemma 2 the bitopology of quasi-uniform convergence (of $\mathcal{U}$) coincides with the 2compact open bitopology.

Conversely, let F be a proper $\tau_1 \vee \tau_2$-closed subset of X and (Y, τ_1', τ_2') a quasi-uniformizable space. Let $\mathcal{U}$ be a quasi-uniformity on Y compatible with (τ_1', τ_2') and let p be a pairwise path for (Y, τ_1', τ_2'). Put $G = Y \setminus \tau_1' \operatorname{cl} p(1)$. Then there is $U \in \mathcal{U}$ such that $U(p(0)) \subseteq G$. Now define $f : X \to Y$ by $f(x) = p(0)$ for all $x \in X$. So $(X, U)(f)$ is a $T(\mathcal{U}_X)$-neighborhood of f. By hypothesis there exist $\tau_1 \vee \tau_2$-compact sets $K_1, \ldots, K_n$, and τ_1'-open sets $G_1, \ldots, G_n$, such that $f \in \bigcap\{[K_j, G_j] : j = 1, \ldots, n\} \subseteq (X, U)(f)$. We will show that $F \subseteq K$ where $K = \bigcup\{K_j : j = 1, \ldots, n\}$. To this end suppose that there is $x_0 \in F \setminus K$. Then there also exists $\Phi \in [0,1]^X$ satisfying $\Phi(x_0) = 1$ and $\Phi(K) = 0$. Hence, $(p \circ \Phi) \in Y^X$. Furthermore, $(p \circ \Phi)(K) = p(0)$ and thus, $(p \circ \Phi) \in (X, U)(f)$. Consequently, $(f(x_0), (p \circ \Phi)(x_0)) \in U$. Therefore, $(p(0), p(1)) \in U$ and, then, $p(1) \in G$, a contradiction. So, $F \subseteq K$ and thus, F is $\tau_1 \vee \tau_2$-compact. The proof is complete.

Another consequence of Lemmas 1 and 2 is the following surprising fact which is used later on.

Lemma 3. *Let (X, τ_1, τ_2) be a space and (Y, τ_1', τ_2') a quasi-uniformizable space. If the compact open topology relative to $(X, \tau_1 \vee \tau_2)$ and $(Y, \tau_1' \vee \tau_2')$ is denoted by T_k^*, then $T_k^* = T_k^1 \vee T_k^2$.*

Proof. Let $\mathcal{U}$ be a quasi-uniformity on Y compatible with (τ_1', τ_2'). By Arens' theorem, cited above, $T_k^* = T((\mathcal{U} \vee \mathcal{U}^{-1})_k)$ and, by Lemma 1, $T_k^* = T(\mathcal{U}_k \vee \mathcal{U}_k^{-1})$. Since $T(\mathcal{U}_k \vee \mathcal{U}_k^{-1}) = T(\mathcal{U}_k) \vee T(\mathcal{U}_k^{-1})$, it follows from Lemma 2 that $T_k^* = T_k^1 \vee T_k^2$.

We conclude this section by studying the notion of supremum (separated) quasi-(pseudo)metric bitopology.

Suppose that the space (Y, τ_1', τ_2') is (separated) quasi-(pseudo)-metrizable. Then, every bounded (separating) quasi-(pseudo)metric d on Y compatible with (τ_1', τ_2') induces a (separating) quasi-(pseudo)metric $\hat{d}$ on Y^X defined by

$$\hat{d}(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

We will say that $\hat{d}$ is the supremum (separating) quasi-(pseudo)metric induced by d. So, $(T(\hat{d}), T(\hat{d}^{-1}))$ is a bitopology on Y^X called the supremum (separated) quasi-(pseudo)metric bitopology on Y^X.

On the other hand, it is well-known that every quasi-pseudometric d on Y induces a quasi-uniformity $\mathcal{U}(d)$ on Y such that a base of it consists of all sets of the form $\{(y, z) \in Y \times Y : d(y, z) < 2^{-n}\}$ for $n \in \mathbb{N}$.

Theorem 3. *Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces such that (Y, τ_1', τ_2') is (separated) quasi-(pseudo)metrizable. If d is a bounded (separating) quasi-(pseudo)metric on Y compatible with (τ_1', τ_2') and $\mathcal{U}(d)$ is the quasi-uniformity on Y induced by d, then the supremum (separated) quasi-(pseudo)metric bitopology coincides with the bitopology of quasi-uniform convergence of $\mathcal{U}(d)$.*

Proof. Given $\varepsilon > 0$ let $(X, U_\varepsilon) = \{(f, g) \in Y^X \times Y^X : d(f(x), g(x)) < \varepsilon$ for all $x \in X\}$. Then, $(X, U_\varepsilon) \in (\mathcal{U}(d))_X$. If $f \in Y^X$ we have $(X, U_{\varepsilon/2})(f) \subseteq B_{\hat{d}}(f, \varepsilon) \subseteq (X, U_\varepsilon)(f)$. This proves that $T(\hat{d}) = T((\mathcal{U}(d))_X)$. Similarly, $T(\hat{d}^{-1}) = T((\mathcal{U}(d)^{-1})_X)$.

With respect to the above theorem it is interesting to consider the case that the range space is $([0,1], u, l)$. If d is the separating quasi-pseudometric on $[0,1]$ defined by $d(x, y) = \max\{y - x, 0\}$, then the supremum separating quasi-pseudometric on $[0,1]^X$ is given by

$$\hat{d}(f, g) = \sup\{\max(g(x) - f(x), 0) : x \in X\}.$$

Since every 2compact space has a unique compatible quasi-uniformity [13, Theorem 4.5], it follows from the above theorem that $(T(\hat{d}), T(\hat{d}^{-1}))$ is the bitopology of quasi-uniform convergence for all bounded separating quasi-pseudometrics on $[0,1]$ compatible with (u, l). If, in addition, (X, τ_1, τ_2) is 2compact, Theorems 2 and 3 show that the supremum separating quasi-pseudometric bitopology $(T(\hat{d}), T(\hat{d}^{-1}))$ is the 2compact open bitopology.

4. Quasi-pseudometrizability of the 2compact open bitopology

It is well-known (see [9]) that if (X, τ) is a Tychonoff space and (Y, τ') is a space containing a nontrivial path, the compact open topology is (completely) metrizable if and only if (X, τ) is a hemicompact space (and a k-space) and (Y, τ') is (completely) metrizable. In this section we investigate the (bicomplete) quasi-pseudometrizability of the 2compact open bitopology.

Recall that a topological space (X, τ) is said to be hemicompact if it has a sequence (K_n) of compact subsets such that for each compact K there is some $n \in \mathbb{N}$ satisfying $K \subseteq K_n$.

Theorem 4. *Let (X, τ_1, τ_2) be a 2Hausdorff quasi-uniformizable space and (Y, τ_1', τ_2') a bitopological space containing a pairwise path. Then the 2compact open bitopology is separated quasi-pseudometrizable if and only* *if $(X,\tau_1\vee\tau_2)$ is hemicompact and (Y,τ_1',τ_2') is separated quasi-pseudometrizable.*

Proof. Necessary condition. Suppose that there is a separating quasi-pseudometric d on Y^X compatible with (T_k^1,T_k^2). For each $y\in Y$ let $f_y : X\to Y$ defined by $f_y(x)=y$ for all $x\in X$. Then, $f_y\in Y^X$. Now define, for all $y,z\in Y$, $\rho(y,z)=d(f_y,f_z)$. It is easy to see that ρ is a separating quasi-pseudometric on Y compatible with (τ_1',τ_2'). Since $d\vee d^{-1}$ is a metric on Y^X compatible with $T_k^1\vee T_k^2$, it follows from Lemma 3 that the compact open topology T_k^* (relative to $(X,\tau_1\vee\tau_2)$ and $(Y,\tau_1'\vee\tau_2')$) coincides with $T(d\vee d^{-1})$. Therefore, $(X,\tau_1\vee\tau_2)$ is hemicompact.

Sufficient condition. Let (K_n) be a sequence of $\tau_1\vee\tau_2$-compact subsets of X witnessing hemicompactness of $(X,\tau_1\vee\tau_2)$ and let d be a separating quasi-pseudometric on Y compatible with (τ_1',τ_2'). Put, for each $m\in\mathbb{N}$, $U_m=\{(y,z)\in Y\times Y : d(y,z)<1/m\}$. Clearly, $\mathcal{B}=\{(K_n,U_m) : n,m\in\mathbb{N}\}$ is a base for the quasi-uniformity of quasi-uniform convergence (of $\mathcal{U}(d)$) on 2compacta. Then there exists a quasi-pseudometric ρ on Y^X compatible with the bitopology of quasi-uniform convergence (of $\mathcal{U}(d)$) on 2 compacta [2, Theorem 4], [11, 2.3, page 51]. By Lemma 2, ρ is compatible with the 2compact open bitopology (T_k^1,T_k^2). Finally, ρ is separating by Remark 1 and Proposition 1(a).

Example 3. Let (X,τ_1,τ_2) be a 2Hausdorff quasi-uniformizable space such that $(X,\tau_1\vee\tau_2)$ is hemicompact. It follows from Theorem 4 that the space $B\mathbb{R}^X$ of all τ_1-upper semicontinuous and τ_2-lower semicontinuous functions with the 2compact open bitopology is separated quasi-pseudometrizable. In particular, the results is true whenever: (i) $X=\mathbb{R}$, $\tau_1=u$ and $\tau_2=l$; (ii) $X=\mathbb{N}$, τ_1 the cofinite topology on X and τ_2 the discrete topology on X.

A quasi-pseudometric d on a set X is called bicomplete if the pseudometric $d\vee d^{-1}$ is complete. We will say that a bitopological space is bicompletely quasi-(pseudo)metrizable if it has a compatible bicomplete quasi-(pseudo)metric.

Theorem 5 [12]. *A (separated) quasi-(pseudo)metrizable bitopological space (X,τ_1,τ_2) is bicompletely (separated) quasi-(pseudo)-metrizable if and only if the space $(X,\tau_1\vee\tau_2)$ is completely metrizable.*

Theorem 6. *Let (X,τ_1,τ_2) be a 2Hausdorff quasi-uniformizable space and (Y,τ_1',τ_2') a bitopological space containing a pairwise path. Then the 2compact open bitopology is bicompletely separated quasi-pseudometrizable if and only if $(X,\tau_1\vee\tau_2)$ is a hemicompact k-space and (Y,τ_1',τ_2') is bicompletely separated quasi-pseudometrizable.*

Proof. Necessary condition. Let d be a bicomplete separating quasi-pseudometric on Y^X compatible with (T_k^1, T_k^2). By Theorem 4, $(X, \tau_1 \vee \tau_2)$ is hemicompact and (Y, τ_1', τ_2') is separated quasi-pseudometrizable. On the other hand, $d \vee d^{-1}$ is a complete metric on Y^X compatible with the compact open topology T_k^* by Lemma 3. So, $(X, \tau_1 \vee \tau_2)$ is a k-space and $(Y, \tau_1' \vee \tau_2')$ is completely metrizable. Therefore, (Y, τ_1', τ_2') is bicompletely separated quasi-pseudometrizable by Theorem 5.

Sufficient condition. Since $(Y, \tau_1' \vee \tau_2')$ is completely metrizable it follows that the compact open topology T_k^* is completely metrizable. Thus, $T_k^1 \vee T_k^2$ is completely metrizable. On the other hand, Theorem 4 shows that (Y^X, T_k^1, T_k^2) is separated quasi-pseudometrizable. Theorem 5 concludes the proof.

Corollary. *Let (X, τ_1, τ_2) be a 2Hausdorff quasi-uniformizable space and (Y, τ_1', τ_2') a separated quasi-pseudometrizable space containing a pairwise path. Then the compact open topology relative to $(X, \tau_1 \vee \tau_2)$ and $(Y, \tau_1' \vee \tau_2')$ is (completely) metrizable if and only if the 2compact open bitopology is (bicompletely) quasi-pseudometrizable.*

Remark 4. Note that in Theorems 4 and 6 the hypothesis that the range space contains a pairwise path, is only used in the proof of the necessary condition.

In the following examples we will suppose that (X, τ_1, τ_2) is a 2Hausdorff quasi-uniformizable space such that $(X, \tau_1 \vee \tau_2)$ is a hemicompact k-space. (For instance, $X = \mathbb{R},\ \tau_1 = u,\ \tau_2 = l$).

Example 4. Let $Y = \mathbb{R}$ and let d be the bicomplete quasi-metric on Y defined by $d(x, y) = y - x$ if $x \leq y$ and $d(x, y) = 1$ if $x > y$. Then $T(d)$ is the Sorgenfrey line on $\mathbb{R}$ (basic $T(d)$-open sets are of the form $[x, y[$ where $x < y$) and $T(d^{-1})$ is the Sorgenfrey comjugate line on $\mathbb{R}$ (basic $T(d^{-1})$-open sets are of the form $]x, y]$ where $x < y$). It follows from Theorem 6 and Remark 4 that the 2compact open bitopology on Y^X is bicompletely quasi-metrizable.

Example 5. Let $Y = \mathbb{R}$ and let d be the quasi-metric on Y defined by $d(x, y) = \min\{1, |x - y|\}$ if x is rational, $d(x, y) = 1$ if $x \neq y$ and x is irrational and $d(x, x) = 0$. Then, $T(d)$ is the Michael line on $\mathbb{R}$. It follows from Theorem 4 and Remark 4 that the 2compact open bitopology on Y^X is quasi-metrizable.

It is clear that if (Y^X, T_k^1, T_k^2) is (separated) quasi-(pseudo)metrizable, then (BY^X, T_k^1, T_k^2) is (separated) quasi-(pseudo)metrizable. We do not know if the converse is also true. However, we have the following partial solution to this question.

Proposition 2. *Let (X, τ_1, τ_2) be a quasi-uniformizable space such that for all $x, y \in X$, with $x \neq y$, there is a $\tau_1 \vee \tau_2$-neighborhood U of x and a τ_i-neighborhood V of y, $i = 1, 2$, such that $U \cap V = \emptyset$, and let (Y, τ_1', τ_2') be a bitopological space containing a pairwise path. Then the following are equivalent.*

(a) *$(X, \tau_1 \vee \tau_2)$ is hemicompact and (Y, τ_1', τ_2') is separated quasi-pseudometrizable.*

(b) *(Y^X, T_k^1, T_k^2) is separated quasi-pseudo-metrizable.*

(c) *(BY^X, T_k^1, T_k^2) is separated quasi-pseudo-metrizable.*

Proof. (a) → (b). Theorem 4.

(b) → (c). Obvious.

(c) → (a). Let d be a separated quasi-pseudo metric on BY^X compatible with (T_k^1, T_k^2). Similarly to the proof of Theorem 4 there is a separated quasi-pseudo metric ρ on Y compatible with (τ_1', τ_2').

It remains to show that $(X, \tau_1 \vee \tau_2)$ is hemicompact. To this end let p be a pairwise path for (Y, τ_1', τ_2'). Define $f : X \to Y$ by $f(x) = p(0)$ for all $x \in X$. Then $f \in BY^X$. Now let $U_n = \{g \in BY^X : d(f, g) < 1/n\}$ for all $n \in \mathbb{N}$. Since each U_n is a T_k^1-neighborhood of f, there is a sequence (K_n) of $\tau_1 \vee \tau_2$-compact subsets of X and a decreasing sequence (δ_n) of positive real numbers such that $\delta_n \to 0$ and $f \in [K_n, B_\rho(p(0), \delta_n)] \subseteq U_n$. Since $p(1) \in Y \setminus \tau_2' \operatorname{cl} p(0)$ there is δ_m such that $\rho(p(0), p(1)) \geq \delta_m$. Given $K \in \mathcal{K}$ take $A = [K, B_\rho(p(0), \delta_m)]$. Then there is $U_n \subseteq A$. We will show that $K \subseteq K_n$. Assume the contrary. Then there exists an $x \in K \setminus K_n$. From the separation hypothesis it follows that every $\tau_1 \vee \tau_2$-compact subset is τ_1 and τ_2 closed. Hence, there is $\Phi \in B[0,1]^X$ such that $\Phi(x) = 1$ and $\Phi(K_n) = 0$. Therefore, $(p \circ \Phi) \in BY^X$ and $(p \circ \Phi) \in [K_n, B_\rho(p(0), \delta_n)] \subseteq U_n \subseteq A$. However, $(p \circ \Phi)(x) = p(1)$ implies $(p \circ \Phi) \in BY^X \setminus A$, a contradiction. This completes the proof.

Example 6. Let $X = Y = \mathbb{R}$, τ_1 the Sorgenfrey line on $\mathbb{R}$, τ_2 the Sorgenfrey conjugate line on $\mathbb{R}$, $\tau_1' = u$ and $\tau_2' = l$. Since $(X, \tau_1 \vee \tau_2)$ is not hemicompact, it follows from Proposition 2 that the space $B\mathbb{R}^X$ of all τ_1-upper semicontinuous and τ_2-lower semicontinuous functions with the 2compact open bitopology is not quasi-pseudometrizable.

The same conclusion is obtained when τ_1 is the Michael line on $\mathbb{R}$ and τ_2 is the topology $T(d^{-1})$ of Example 5.

References

[1] R. Arens, A topology of spaces of transformations, *Annals of Math.* **47** (1946), 480–495.

[2] P. Fletcher, H. B. Hoyle, III and C. W. Patty, The comparison of topologies, *Duke Math. J.* **36** (1969), 325–331.

[3] P. Fletcher and W. F. Lindgren, Quasi-Uniform Spaces, *Marcel Dekker*, *New York*, 1982.

[4] J. L. Kelley, General Topology, *Van Nostrand*, *New York*, 1955.

[5] J. C. Kelly, Bitopological spaces, *Proc. London Math. Soc.* **13** (1963), 71–89.

[6] H. P. A. Künzi and N. Ferrario, Bicompleteness of the fine quasi-uniformity, *Proc. Cambridge Phil. Soc.* **109** (1991), 167–186.

[7] E. P. Lane, Bitopological spaces and quasi-uniform spaces, *Proc. London Math. Soc.* **17** (1967), 241–256.

[8] R. A. McCoy and I. Ntantu, Completeness properties of function spaces, *Topology Appl.* **22** (1986), 191–206.

[9] R. A. McCoy and I. Ntantu, Topological Properties of Spaces of Continuous Functions, *Lectures Notes in Math.* **1315** Berlin (1988).

[10] A. Okuyama and T. Terada, Function Spaces, In Topics in General Topology (K. Morita and J. Nagata, eds.), *North Holland*, *Amsterdam*, 1989, pp. 411–458.

[11] I. L. Reilly, Quasi-gauges, Quasi-Uniformities and Biotopological Spaces, Ph.D. Thesis, *University of Illinois*, 1970.

[12] S. Romaguera and S. Salbany, On bicomplete quasi-pseudometrizability, *Top. Appl.* **50** (1993), 283–289.

[13] S. Salbany, Bitopological Spaces, Compactifications and Completions, *Math. Monographs of the Univ. of Cape Town* **1** (1974).

[14] S. Salbany, A bitopological view of topology and order, Categorical Topology, Proc. Tolede, Ohio 1983, *Heldermann*, *Berlin*, 1984, pp. 481–504.

SALVADOR ROMAGUERA
ESCUELA DE CAMINOS
DEPARTAMENTO DE MATEMÁTICA APLICADA
UNIVERSIDAD POLITÉCNICA
46071, VALENCIA
SPAIN

MARCOS RUIZ–GÓMEZ
ESCUELA DE CAMINOS
DEPARTAMENTO DE MATEMÁTICA APLICADA
UNIVERSIDAD POLITÉCNICA
46071, VALENCIA
SPAIN

(Received February 25, 1994)