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Stability properties of functional equations in several variables

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1. Introduction

Let S be a semigroup and we consider the functional equation

$$F(x,y)+F(xy,z)=F(x,yz)+F(y,z) \tag{1}$$

where $F:S\times S\to\mathbb{C}$ is a function and (1) is supposed to hold for all x,y in S. It is easy to check that (1) holds for any F of the form

$$F(x,y)=f(xy)-f(x)-f(y) \tag{2}$$

where $f:S\to\mathbb{C}$ is any function. The converse of this statement for symmetric F on any Abelian group S has been proved in [7]. The proof depends heavily on the commutative structure of S. The general solution of (1) has also been found on several classes of commutative semigroups, see e.g. [2]. Now we prove that any bounded solution F of (1) has a representation of the form (2) with a bounded f, if S is an amenable semigroup. Concerning amenable groups and semigroups the reader should refer to [5], [6]. Further we study the stability of (1). Again, let S be a semigroup, $F:S\times S\to\mathbb{C}$ a function, and suppose, that the three-place function

$$(x,y,z)\to F(x,y)+F(xy,z)-F(x,yz)-F(y,z)$$

is bounded on $S\times S\times S$. In the classical cases of Hyers–Ulam stability this implies that $F-K$ satisfies (1) with some bounded K. Here we prove that this is the case.

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In the second part we study the functional equation

$$(3)\quad F(xy,z)+F(xy^{-1},z)-2F(y,z)=F(x,yz)+F(x,yz^{-1})-2F(x,y)$$

with $F : S\times S\to\mathbb{C}$, where S is a group. Equation (3) has been arisen in [9], where the question concerning (3) was, whether any solution F of (3) can be represented in the form

$$(4)\qquad F(x,y)=f(xy)+f(xy^{-1})-2f(x)-2f(y).$$

(Obviously, any F of the form (4) satisfies (3) if S is commutative.) This question has been answered in [3] in the negative, by presenting a counterexample. Nevertheless, the problem of the general solution of (3) remains open. In [4] it has been proved that in the case $S=\mathbb{R}$ any twice differentiable solution of (3) has the form (4) (with twice differentiable f). Here we show, that any bounded solution of (3) has the form (4) with bounded f, if the group S is amenable. Further, we show that equation (3) has the similar remarkable stability property, like equation (1): if the function $F : S\times S\to\mathbb{C}$ has the property, that the function

$$(x,y,z)\to F(xy,z)+F(xy^{-1},z)-2F(y,z)-F(x,yz)-F(x,yz^{-1})+2F(x,y)$$

is bounded, then $F-K$ is a solution of (3) with a bounded function K, supposing S is Abelian.

2. The functional equation (1)

Concerning (1) we first prove the following theorem:

Theorem 2.1. *Let S be a right amenable semigroup and let $F : S\times S\to\mathbb{C}$ be a bounded function satisfying* (1)*. Then there exists a unique bounded function $f : S\to\mathbb{C}$ with*

$$F(x,y)=f(xy)-f(x)-f(y)$$

for all x,y in S.

Proof. Let M denote any right invariant mean defined on the set of all bounded complex valued functions on S and we write M_x to indicate that M is applied on the argument as a function of x. Now apply M on both sides of (1) as functions of x, for any fixed y,z in S. We obtain

$$M_x[F(x,y)]+M_x[F(xy,z)]=M_x[F(x,yz)]+M_x[F(y,z)],$$

that is, by the right invariance of M and by $M(1)=1$,

$$M_x[F(x,y)]+M_x[F(x,z)]=M_x[F(x,yz)]+F(y,z),$$

hence we may choose

$$f(y) = M_x[F(x,y)]$$

for all y in S. As F is bounded, it follows that f is bounded. For the uniqueness of f it is easy to see that the difference of two f's is a complex homomorphism of S, which cannot be bounded unless it is zero. Hence our theorem is proved.

Now our stability theorem follows for (1).

Theorem 2.2. *Let S be a right amenable semigroup and let $F : S \times S \to \mathbb{C}$ be a function, for which the function*

$$(x,y,z) \to F(x,y) + F(xy,z) - F(x,yz) - F(y,z)$$

is bounded. Then there exists a function $\Phi : S \times S \to \mathbb{C}$ satisfying (1)*, for which $F - \Phi$ is bounded.*

Proof. We define the function $\Phi : S \times S \to \mathbb{C}$ by the formula

$$\Phi(y,z) = M_x[F(x,y) + F(xy,z) - F(x,yz)]$$

for all y, z in S, where M denotes an arbitrary right invariant mean defined on all bounded complex valued functions on S. Then we have for all y, z

$$\begin{gathered}\Phi(y,z) + \Phi(yz,u) - \Phi(y,zu) - \Phi(z,u) = \\ = M_x[F(x,y) + F(xy,z) - F(x,yz) + F(x,yz) + \\ + F(xyz,u) - F(x,yzu) - F(x,y) - F(xy,zu) + F(x,yzu) - \\ - F(x,z) - F(xz,u) + F(x,zu)] = \\ = M_x[\{F(xy,z) + F(xyz,u) - F(xy,zu)\} - \\ - \{F(x,z) + F(xz,u) - F(x,zu)\}] = 0,\end{gathered}$$

by the right invariance of M. On the other hand, it follows

$$F(y,z) - \Phi(y,z) = M_x[F(y,z) + F(x,yz) - F(x,y) - F(xy,z)],$$

which is bounded, by the properties of M and F. Hence our theorem is proved.

3. The functional equation (3)

Concerning (3) we first prove the following theorem:

Theorem 3.1. *Let S be an amenable group and let $F : S \times S \to \mathbb{C}$ be a bounded function satisfying* (3)*. Then there exists a unique bounded function $f : S \to \mathbb{C}$ with*

$$F(x,y) = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)$$

for all x, y in S.

Proof. Let M denote any invariant mean defined on the set of all bounded complex valued functions on S. If we apply M on both sides of (3) as functions of x, for any fixed y, z in S, then we obtain

$$\begin{aligned} &M_x[F(xy,z)] + M_x[F(xy^{-1},z)] - 2M_x[F(y,z)] = \\ &= M_x[F(x,yz)] + M_x[F(x,yz^{-1})] - 2M_x[F(x,y)], \end{aligned}$$

that is, by the properties of M

$$\begin{aligned} &M_x[F(x,z)] + M_x[F(x,z)] - 2F(y,z) = \\ &= M_x[F(x,yz)] + M_x[F(x,yz^{-1})] - 2M_x[F(x,y)], \end{aligned}$$

which shows that our statement holds if we choose

$$f(y) = -\frac{1}{2} M_x[F(x,y)]$$

for all y in S. The boundedness of f follows from the properties of M. The uniqueness of f follows similarly as in Theorem 2.1.

For the stability theorem for equation (3) we need also commutativity on S.

Theorem 3.2. *Let S be an Abelian group and let $F : S \times S \to \mathbb{C}$ be a function, for which the function*

$$\begin{aligned} (x,y,z) \to F(x+y,z) + F(x-y,z) - 2F(y,z) - \\ -F(x,y+z) - F(x,y-z) + 2F(x,y) \end{aligned}$$

is bounded. Then there exists a function $\Phi : S \times S \to \mathbb{C}$ satisfying (3)*, for which $F - \Phi$ is bounded.*

Proof. We define the function $\Phi : S \times S \to \mathbb{C}$ by the formula

$$\Phi(y,z) = -\frac{1}{2} M_x[-F(x+y,z) - F(x-y,z) + F(x,y+z) + F(x,y-z) - 2F(x,y)]$$

for any y, z in S, where M is an invariant mean on S. Now we can compute as follows

$$\Phi(y+z,u)+\Phi(y-z,u)-2\Phi(z,u)-\Phi(y,z+u)-\Phi(y,z-u)+2\Phi(y,z)=$$
$$=-\frac{1}{2}M_x[-F(x+y+z,u)-F(x-y-z,u)+F(x,y+z+u)+$$
$$+F(x,y+z-u)-2F(x,y+z)-F(x+y-z,u)-F(x-y+z,u)+$$
$$+F(x,y-z+u)+F(x,y-z-u)-2F(x,y-z)+2F(x+z,u)+$$
$$+2F(x-z,u)-2F(x,z+u)-2F(x,z-u)+4F(x,z)+$$
$$+F(x+y,z+u)+F(x-y,z+u)-F(x,y+z+u)-F(x,y-z-u)+$$
$$+2F(x,y)+F(x+y,z-u)+F(x-y,z-u)-F(x,y+z-u)-$$
$$-F(x,y+u-z)+2F(x,y)-2F(x+y,z)-2F(x-y,z)+$$
$$+2F(x,y+z)+2F(x,y-z)-4F(x,y)]=$$
$$=-\frac{1}{2}M_x[\{-F(x+y+z,u)--F(x+y-z,u)+F(x+y,z+u)+$$
$$+F(x+y,z-u)-2F(x+y,z)\}+\{F(x+z,u)+F(x-z,u)-$$
$$-F(x,z+u)-F(x,z-u)+2F(x,z)\}]-\frac{1}{2}M_x[\{-F(x-y+z,u)-$$
$$-F(x-y-z,u)+F(x-y,z+u)+F(x-y,z-u)-2F(x-y,z)\}+$$
$$+\{F(x+z,u)+F(x-z,u)-F(x,z+u)-F(x,z-u)+2F(x,z)\}]=0$$

by the invariance of M. On the other hand,

$$F(y,z)-\Phi(y,z)=\frac{1}{2}M_x[2F(y,z)-F(x+y,z)-F(x-y,z)+$$
$$+F(x,y+z)+F(x,y-z)-2F(x,y)]$$

which is bounded, by the properties of M and F. Hence our theorem is proved.

We note, that actually we haven't used the commutativity of S, only the following property of F:

$$F(xyz,uvw)=F(xzy,uwv)$$

and the fact, that there exists a right invariant mean on S. A similar condition to this one has been used for functions of one variable in [1], [8], [10]. Hence we can prove the following corollary.

Corollary 3.3. *Let S be an amenable group and let $F : S\times S\to\mathbb{C}$ be a function, which satisfies*

$$F(xyz,uvw)=F(xzy,uwv)$$

for all x, y, z, u, v, w in S, and for which the function

$$(x, y, z) \to F(xy, z)+F(xy^{-1}, z)-2F(y, z)-F(x, yz)-F(x, yz^{-1})+2F(x, y)$$

is bounded. Then there exists a function $\Phi : S \times S \to \mathbb{C}$ satisfying (3), *for which $F - \Phi$ is bounded.*

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