The influence of minimal p-subgroups
on the structure of finite groups

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Abstract. Let H be a subgroup of a finite group G. H is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G. A finite group G is said to be a (G)-group if every minimal subgroup of G is S-quasinormal in G and for $2 \mid |G|$, either every Sylow 2-subgroup of G is an abelian group or every cyclic subgroup of G of order 4 is S-quasinormal in G. In this paper we prove that a (G)-group is supersolvable and investigate the structure of a minimal non-(G)-group further.

1. Introduction

A number of authors have examined the structure of a finite group G, under the assumption that all minimal subgroups of G are well-situated in the group. ITO [1] has shown that a group G of odd order is nilpotent provided that all minimal subgroups of G lie in the center of the group. A sharpened form of Ito's result is the following statement ([2, p. 435]): If, for an odd prime p, every subgroup of G of order p lies in the center of G, then G is p-nilpotent. If all elements of G of order 2 and 4 lie in the center of G, then G is 2-nilpotent. Along the same lines, BUCKLEY [3] has shown that if all minimal subgroups of an odd order group are normal, then the group is supersolvable. An example in [4] shows that if all minimal subgroups of an even order group are normal, then the group is not necessarily supersolvable. The object of this paper is to generalize Buckley's result.

Throughout, all groups mentioned are assumed to be finite groups. The terminology and notations employed agree with standard usage, as in HUPPERT [2].

2. Basic definitions and preliminary results

Let H be a subgroup of a group G; H is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G. We say that the group G is a (G)-group if every minimal subgroup of G is S-quasinormal in G and for $2 \mid |G|$, either every Sylow 2-subgroup of G is an abelian group or every cyclic subgroup of G of order 4 is S-quasinormal in G.

A group G is called a Sylow tower group of supersolvable type if $p_1 < p_2 < \cdots < p_s$ are all the distinct prime divisors of the order of G and there exist $P_1, P_2, \cdots, P_s$ such that P_i is a Sylow p_i-subgroup of G and $P_k \cdots, P_s \lhd G$ for $k = 1, 2, \cdots, s$.

For the sake of convenience, we list here some of the results used in the proofs of this paper.

Lemma 2.1 [5]. *Let H and K be subgroups of a group G. If H is S-quasinormal in G and $H \leq K$, then H is S-quasinormal in K.*

It follows from Lemma 2.1 that if G is a (G)-group, then every subgroup of G is a (G)-group.

Lemma 2.2 [5, Lemma I5.2 and Lemma I5.5]. *Let x be a p-element of a solvable group G and $\langle x \rangle$ be S-quasinormal in G, then $x \in O_p(G)$ and $G_{p'} \leq N_G(\langle x \rangle)$ for every p'-Hall subgroup $G_{p'}$ of G.*

Lemma 2.3. *Let G be a solvable group, and let $P \lhd G$ ($P \in \mathrm{Syl}_p G$). If x is a q-element of G($q \neq p$ are primes), then $\langle x \rangle$ is S-quasinormal in G if and only if $\langle x \rangle P/P$ is S-quasinormal in G/P and $\langle x \rangle$ is S-quasinormal in $P\langle x \rangle$.*

PROOF. If $\langle x \rangle$ is S-quasinormal in G, it is clear that $\langle x \rangle P/P$ is S-quasinormal in G/P and that $\langle x \rangle$ is S-quasinormal in $P\langle x \rangle$. Assume $\overline{\langle x \rangle} = \langle x \rangle P/P$ is S-quasinormal in $G/P = \bar{G}$ and $\langle x \rangle$ is S-quasinormal in $P\langle x \rangle$. Let $R \in \mathrm{Syl}_r G$ (r is a prime and $r \neq p$), then $\bar{R} = RP/P \in \mathrm{Syl}_r \bar{G}$. If $r \neq q$, by Lemma 2.2 $\bar{R} \leq N_{\bar{G}}(\overline{\langle x \rangle})$. Since $N_G(\langle x \rangle)P/P = N_{\bar{G}}(\overline{\langle x \rangle})$, $RP \leq N_G(\langle x \rangle)P$. Noticing that $\langle x \rangle$ is S-quasinormal in $P\langle x \rangle$ we get $P \leq N_{P\langle x \rangle}(\langle x \rangle) \leq N_G(\langle x \rangle)$ by using Lemma 2.2. Hence $R \leq RP \leq N_G(\langle x \rangle)P \leq N_G(\langle x \rangle)$, and so $R\langle x \rangle = \langle x \rangle R$. If r=q, it follows from $\overline{\langle x \rangle} \bar{R} = \bar{R} \overline{\langle x \rangle}$ that $\overline{\langle x \rangle} \bar{R} = \bar{R}$, and so $\langle x \rangle P \leq PR$. By Sylow's Theorem there exists $g \in PR$ such that $\langle x \rangle^g \leq R$. Since $\langle x \rangle$ is S-quasinormal in $P\langle x \rangle$ we get $\langle x \rangle \leq O_q(P\langle x \rangle)$ by using Lemma 2.2. Noticing that $\langle x \rangle$ is a Sylow q-subgroup of $P\langle x \rangle$, $P\langle x \rangle = P \times \langle x \rangle$. So we can assume $g \in R$. From

$\langle x \rangle^g \leq R$ it follows that $\langle x \rangle \leq R$. Hence $R\langle x \rangle = R = \langle x \rangle R$. The proof of the Lemma is complete.

Theorem 2.4 [6]. *Let G be a minimal non-supersolvable group (every proper subgroup of G is supersolvable but G itself is not supersolvable), then there exists a unique Sylow p-subgroup P of G for some prime p such that $P \lhd G$ and P satisfies the following conditions:*

(1) *P has exponent p if $p \neq 2$ and exponent at most 4 if $p = 2$,*
(2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ and $P/\Phi(P)$ is not a cyclic group.*
(3) *$\Phi(P) \leq Z(P)$, the center of P.*

3. Main results

Theorem 3.1. *Let N be a normal subgroup of a group G such that*

(1) *G/N is supersolvable,*
(2) *Every minimal subgroup of N is S-quasinormal in G and for $2 \mid |N|$ either every Sylow 2-subgroup of N is an Abelian group or every cyclic subgroup of N of order 4 is S-quasinormal in G.*

Then G is supersolvable.

PROOF. We suppose the theorem false and choose for G a counterexample of smallest order. Let L be a proper subgroup of G, then $L/L \cap N \simeq LN/N (\leq G/N)$ is supersolvable. Since L and $L \cap N$ satisfy the hypotheses of the theorem L is supersolvable, and therefore G is a minimal non-supersolvable group. By Theorem 2.4 there exists a normal Sylow p-subgroup P of G such that

(i) If $p > 2$ $\exp P = p$; if $p = 2$ $\exp P = 2$ or 4.

(ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ and $P/\Phi(P)$ is not a cyclic group.

By the Schur-Zassenhaus Theorem there exists a p'-Hall subgroup H in G. Then $G/P \simeq H$ is supersolvable. Hence $G/P \cap N$ is supersolvable. Let $M = P \cap N$, then $M \lhd G$ and $M \neq 1$. We consider the subgroup $M\Phi(P)$; since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, $M\Phi(P) = \Phi(P)$ or P. If $M\Phi(P) = \Phi(P)$, then $M \leq \Phi(P) \leq \Phi(G)$. It follows that $G/\Phi(G)$ is supersolvable. By Huppert's Theorem ([2, Theorem 9.5, p. 719]) G is supersolvable, a contradiction. If $M\Phi(P) = P$, then $M = P (\leq N)$. Hence every minimal subgroup of P is S-quasinormal in G and for $p = 2$

either P is an Abelian group or every cyclic subgroup of P of order 4 is S-quasinormal in G. If $p \neq 2$, then $\exp P = p$. Let $x \in P \setminus \Phi(P)$, then $\langle x \rangle$ is S-quasinormal in G. By Lemma 2.2 $H \leq N_G(\langle x \rangle)$, and so $H\Phi(P)/\Phi(P) \leq N_{G/\Phi(P)}(\langle x \rangle \Phi(P))$. Noticing that $P/\Phi(P)$ is an Abelian group we have $\langle x \rangle \Phi(P)/\Phi(P) \lhd G/\Phi(P)$, in contradiction to (2) of Theorem 2.4. Hence $p = 2$. If $P' \neq 1$, let $x \in P \setminus \Phi(P)$, then $|x| = 2$ or 4 and $\langle x \rangle$ is S-quasinormal in G. Similarly to the above proof $\langle x \rangle \Phi(P)/\Phi(P) \lhd G/\Phi(P)$, a contradiction. So $p = 2$ and $P' = 1$. Let $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle \times \langle b_1 \rangle \times \langle b_2 \rangle \times \cdots \times \langle b_m \rangle$ where $|a_i| = 4$, $|b_j| = 2$; $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$. If $m \neq 0$, then $\langle b_1 \rangle$ is S-quasinormal in G, and we have $\langle b_1 \rangle \Phi(P)/\Phi(P) \lhd G/\Phi(P)$, a contradiction. Hence $m = 0$, and $P = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle$. Now $\langle a_i^2 \rangle$ is S-quasinormal in G and P is an Abelian group $\langle a_i^2 \rangle \lhd G$, and so $\langle a_i^2 \rangle \leq Z(G)$. Since $\Phi(P) = \langle a_1^2 \rangle \times \langle a_2^2 \rangle \times \cdots \times \langle a_n^2 \rangle$ we have $\Phi(P) \leq Z(G)$. Let $g \in H$, then $a_1^g = d(d \in P)$, and $(a_1^2)^g = d^2$. On the other hand, $a_1^2 \in Z(G)$ implies $(a_1^2)^g = (a_1^2)$. Hence $(a_1^2) = d^2$, and it follows that $d = a_1 c$, $c \in \Phi(P)$, so $\langle a_1 \rangle \Phi(P)/\Phi(P) \lhd G/\Phi(P)$, in contradiction to (2) of Theorem 2.4. The proof of the Theorem is complete.

Corollary 3.2. *If G is a (G)-group, then G is supersolvable.*

Theorem 3.3. *Assume that every maximal subgroup of a group G is a (G)-group but G itself is not a (G)-group. Then*

(I) $|G| = p^m q^n$ *where p and q are unequal primes, and m and n are positive integers,*

(II) *there is a unique Sylow p-subgroup P and a Sylow q-subgroup Q is cyclic. Hence $G = QP$ and $P \lhd G$. Furthermore, $Q \ntrianglelefteq G$ and*

(1) *If $p < q$ and $p \neq 2$, then $P = \Omega_1(P)$,*

(2) *If $p = 2$, then $P = \Omega_2(P)$.*

PROOF. By Theorem 3.1 every maximal subgroup of G is supersolvable, and so G is either a Sylow tower group of supersolvable type or a minimal non-nilpotent group by using [6, Hilfsatz C.].

(I) Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $\alpha_j > 0$ and the p_j are distinct primes, and $p_j < p_{j+1}$. Since p-groups are always (G)-groups, we can assume that $s \geq 3$. If G is a minimal non-nilpotent group, then $s = 2$ ([2, 5.2 Satz, p. 281]), a contradiction. Hence G is a Sylow tower group of supersolvable type. Let $P_j \in \mathrm{Syl}_{p_j} G$, then $P_s \lhd G$, and every subgroup

of G of order p_s is contained in P_s. For every subgroup $\langle x \rangle$ of G of order p_s, since $P_j P_s$ is a proper subgroup of $G \langle P_1, P_2, \cdots, P_{s-1} \rangle \leq N_G(\langle x \rangle)$ we get by using Lemma 2.2. Hence $P_j \langle x \rangle = \langle x \rangle P_j$ $(j = 1, 2, \cdots, s-1)$. It is clear that $\langle x \rangle P_s = P_s = P_s \langle x \rangle$. So $\langle x \rangle$ is S-quasinormal in G. Let H be a $\{p_s\}'$-Hall subgroup of G, then $G = HP_s$, and $G/P_s \simeq H$ is a (G) -group. Let $\langle y \rangle$ be a subgroup of G of order $p_j (j = 1, 2, \cdots, s-1)$, then $\langle y \rangle$ is S-quasinormal in $\langle y \rangle P_s$ since $\langle y \rangle P_s$ is a proper subgroup of G, and $\langle y \rangle P_s / P_s$ is S-quasinormal in G/P_s since G/P_s is a (G) -group. By Lemma 2.3 $\langle y \rangle$ is S-quasinormal in G. By using the same reasoning we have that if $p_1 = 2$ but $P_1' \neq 1$ then every cyclic subgroup of G of order 4 is S-quasinormal in G. So G is a (G)-group, in contradiction to the hypothesis. Hence (I) holds.

(II) Since G is either a Sylow tower group of supersolvable type or a minimal non-nilpotent group, there exists a Sylow subgroup, say P $(P \in \mathrm{Syl}_p G)$, such that $P \lhd G$. Let $Q \in \mathrm{Syl}_q G$, if $Q \lhd G$, then $G = P \times Q$ is a (G) -group, so $Q \not\lhd G$, and

(i) Q is a cyclic group.

In fact, if Q is not a cyclic group, then for every element $y \in Q$ $P\langle y \rangle$ is a proper subgroup of G, and therefore $P\langle y \rangle$ is a (G)-group. Hence for every subgroup $\langle x \rangle$ of P of order p (therefore for every subgroup of G of order p) $\langle x \rangle$ is S-quasinormal in $P\langle y \rangle$. By Lemma 2.2 $\langle y \rangle \leq N_G(\langle x \rangle)$, and so $Q \leq N_G(\langle x \rangle)$. It follows that $Q\langle x \rangle = \langle x \rangle Q$. If $p = 2$ and $P' \neq 1$, then for every subgroup $\langle x \rangle$ of G of order 4 we have also that $\langle x \rangle Q = Q\langle x \rangle$. It is clear that $\langle x \rangle P = P = P\langle x \rangle$. Noticing that $G/P \simeq Q$ is a (G)-group and that for every cyclic q-group $\langle y \rangle$ of $G\langle y \rangle P$ is a proper subgroup of G since Q is not a cyclic group. We have that every subgroup of G of order q is S-quasinormal in G and if $q = 2$, and $Q' \neq 1$ then every cyclic subgroup of order 4 is S-quasinormal in G by using Lemma 2.3. So G is a (G)-group, a contradiction, hence Q is a cyclic group.

(ii) If $q > p$ and $p \neq 2$, then $P = \Omega_1(P)$.

In fact, if $\Omega_1(P) \neq P$, then $\Omega_1(P)Q$ is a proper subgroup of G. Since $\Omega_1(P)Q$ is a (G)-group, $\Omega_1(P)Q$ is supersolvable by using Theorem 3.1. So $Q \lhd \Omega_1(P)Q$, and $\Omega_1(P)Q = \Omega_1(P) \times Q$. By

Theorem 7.26 of [7] $G = P \times Q$, a contradiction. Hence $P = \Omega_1(P)$.

Similarly to the proof of (ii) we have

(iii) If $p = 2$, then $P = \Omega_2(P)$.

From (i), (ii) and (iii) it follows that (II) hold, and the proof of the Theorem is complete.

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