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Watson integral transforms on new spaces of functions and distributions

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Abstract. In this paper we investigate Watson integral transformations in new spaces of functions and distributions. In the procedure developed the Mellin integral transformation plays an essential role. Our investigation includes many important well-known special cases: H-transformation, Krätzel transformation, Riemann-Liouville and Weyl fractional integrals, amongst others.

1 Introduction

The study of integral transformations in spaces of generalized functions has been an active area of work in the last years. As it is well-known there exist two ways to define integral transforms of distributions, the adjoint and the kernel methods. The adjoint method has been employed by A.H. Zemanian [21], J.M. Mendez [10] and R.S. Pathak [11] amongst others. The kernel method was used by A.H. Zemanian [20], E.L. Koh and A.H. Zemanian [9] and L.S. Dube and J.N. Pandey [6].

In [2] and [3] J.A. Barrios and J.J. Betancor defined the $\boldsymbol{K}_\nu$ transformation and the Krätzel transformation of generalized functions through the adjoint method. They developed a technique, inspired by the papers of A. Schuitman ([14] and [15]) and B.L.J. Braaksma and A. Schuitman [4], where the Mellin integral transformation plays an important role.

In this paper we modify the procedure employed in [2] and [3] to define new integral transformations on spaces of generalized functions. We shall investigate the integral transform defined by

$$W(\phi)(x) = \int_0^\infty k(xt)\phi(t)dt \tag{1}$$

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where $k(x) = \frac{1}{2\pi\, i} \int_{c-i\infty}^{c+i\infty} x^{-s} \mathcal{K}(s) ds$ for some $c \in \mathbb{R}$, $\mathcal{K}(s)$ being a suitable meromorphic function with real poles. The integral transforms in (1) are usually called Watson transformations. In Section 2 we introduce new Fréchet spaces of functions and we analyze the behaviour of the Mellin transformation over them. The W-transformation, when $\mathcal{K}(s)$ has a potential growth as $|\,\mathrm{Im}\ s| \to \infty$, is investigated in Section 3 on the spaces which were introduced in the previous paragraph. The integral transformation (1) of distributions is defined in Section 4 by using the adjoint method. In Section 5 we consider new Fréchet function spaces on which the W-transformation is a homeomorphism when $\mathcal{K}(s)$ has an exponential growth as $|\,\mathrm{Im}\ s| \to \infty$. To finish we list in Section 6 some known integral transformations that can be seen as special cases of the theory developed.

Throughout this paper we will denote by $\boldsymbol{I}$ the open interval $(0\,,\,\infty)$. As usual the space $\mathcal{D}\,(\boldsymbol{I})$ consists of all complex valued smooth functions having compact support on $\boldsymbol{I}$ and we represent by $\mathcal{E}\,(\boldsymbol{I})$ the spaces of complex valued smooth functions on $\boldsymbol{I}$. $\mathcal{D}\,(\boldsymbol{I})$ and $\mathcal{E}\,(\boldsymbol{I})$ are endowed with the usual topologies ([21]).

2 Some new function spaces

In this Section we introduce new function spaces on which we shall define the Watson transformation (1) in the next paragraph.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $inf\,\{a_n - a_{n+1} : n \in \mathbb{N}\} > 0$. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $inf\,\{b_{n+1} - b_n : n \in \mathbb{N}\} > 0$. Moreover we assume that $a_1 < b_1$.

The space $\mathcal{A}\,(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty})$ is constituted by all those complex valued smooth functions $\phi(x)\,,\, x \in \boldsymbol{I}$, for which the quantity

$$\gamma_{\ell\,,\,k}^{m}(\phi) \;=\; \sup_{x\in \boldsymbol{I}} \left| x^{m(a_{k+1}-b_{\ell+1})} \prod_{i=1}^{\ell} \left(x^{b_{i+1}-b_i+1} \frac{d}{dx} \right) \cdot \right.$$
$$\left. \cdot \left(x^{b_1 - a_{k+1}} \prod_{j=1}^{k} \left(x^{a_{j+1}-a_j+1} \frac{d}{dx} \right) (x^{a_1} \phi(x)) \right) \right|$$

is finite for every $l, k \in \mathbb{N}$ and $m = 0, 1$. Here and throughout this paper $\displaystyle\prod_{i=1}^{0}$ is understood as 1. The space $\mathcal{A}\,(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty})$ is endowed with the topology generated by the family of seminorms $\left\{\gamma_{\ell\,,\,k}^{m}\right\}_{\ell,k\in\mathbb{N}\,,\,m=0,1}$. By using a standard procedure (A.H. Zemanian [21], p. 131, and A. Zayed

[19]) we can prove that $\mathcal{A}\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right)$ is a Fréchet space. Moreover the inclusions

$$\mathcal{D}\left(\boldsymbol{I}\right) \subset \mathcal{A}\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right) \subset \mathcal{E}\left(\boldsymbol{I}\right)$$

hold and each of these inclusions is continuous.

Let $\epsilon > 0$ be such that $b_1 - a_1 > 2\epsilon$, $a_{n+1} + \epsilon < a_n$ and $b_n + \epsilon < b_{n+1}$, $n \in \mathbb{N}$. We introduce the space $\mathcal{B}_\epsilon\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right)$ that consists of all the meromorphic functions $\Phi(s)$ satisfying the following two conditions:

(i) Φ is holomorphic in $\mathbb{C} - \left(\{a_n\}_{n=1}^{\infty}\ \cup\ \{b_n\}_{n=1}^{\infty}\right)$ and $\Phi(s)$ has at most simple poles in $s = a_n$ *and* $s = b_n$, $n \in \mathbb{N}$, and

(ii) $$\omega_{\ell,k}^{\epsilon}\left(\Phi\right) = \sup_{s \in V_\epsilon(k,\ell)} \left| \prod_{i=1}^{\ell} \left(s - b_i\right) \prod_{j=1}^{k} \left(s - a_j\right) \Phi(s) \right| < \infty,$$

where $V_\epsilon(k,\ell) = \{s \in \mathbb{C}\ :\ a_{k+1} + \epsilon \leq \operatorname{Re}\, s \leq b_{\ell+1} - \epsilon\}$, for every $l, k \in \mathbb{N}$.

We consider in $\mathcal{B}_\epsilon\left(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}\right)$ the topology defined by the family of seminorms $\left\{\omega_{\ell,k}^{\epsilon}\right\}_{\ell,k \in \mathbb{N}}$. Thus $\mathcal{B}_\epsilon\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right)$ is a Fréchet space. Moreover $\mathcal{B}_\epsilon\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right)$ is continuously contained in $\mathcal{H}\left(\mathbb{C} - \left(\{a_n\}_{n=1}^{\infty} \cup \{b_n\}_{n=1}^{\infty}\right)\right)$ (the space of holomorphic functions in $\mathbb{C} - \left(\{a_n\}_{n=1}^{\infty} \cup \{b_n\}_{n=1}^{\infty}\right)$, where $\mathcal{H}\left(\mathbb{C} - \left(\{a_n\}_{n=1}^{\infty} \cup \{b_n\}_{n=1}^{\infty}\right)\right)$ is equipped as usual, with the topology of the uniform convergence on the compact subsets of $\mathbb{C} - \left(\{a_n\}_{n=1}^{\infty} \cup \{b_n\}_{n=1}^{\infty}\right)$. As it is easy to see, the space $\mathcal{B}_\epsilon(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty})$ does not depend on ϵ provided that $\epsilon > 0,\ b_1 - a_1 > 2\epsilon, a_n > a_{n+1} + \epsilon$ and $b_{n+1} > b_n + \epsilon$, $n \in \mathbb{N}$. Hence, in the sequel we will write $\mathcal{B}\left(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}\right)$, $\omega_{\ell,k}$ and $V(k,\ell)$ instead of $\mathcal{B}_\epsilon\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right)$, $\omega_{\ell,k}^{\epsilon}$ and $V_\epsilon(k,\ell)$.

As it is well-known the Mellin transform $\mathcal{M}\phi$ of ϕ is defined by

$$\left(\mathcal{M}\phi\right)(s) = \int_0^\infty t^{s-1}\phi(t)dt\,,\ s \in \Omega,$$

where Ω is a subset of the complex plane. The Mellin integral transformation plays an important role in our study. In the following we shall prove that $\mathcal{M}$ is a homeomorphism between the function spaces that we have introduced at the beginning of the paragraph.

Proposition 1. *The Mellin integral transformation is a homeomorphism from* $\mathcal{A}\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right)$ *onto* $\mathcal{B}\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right)$.

PROOF. Let ϕ be in $\mathcal{A}\left(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty}\right)$. It is not hard to see that the function

$$\Phi(s) = (\mathcal{M}\phi)(s) = \int_0^\infty t^{s-1}\phi(t)dt \tag{2}$$

is holomorphic in $\{s \in \mathbb{C} : a_1 < \text{Re } s < b_1\}$. Moreover the integral on the right side of (2) is absolutely convergent in $\{s \in \mathbb{C} : a_1 < \text{Re } s < b_1\}$.

By partial integration we can obtain

$$\Phi(s) = \frac{-1}{s-a_1}\int_0^\infty x^{s-a_2-1}\left(x^{a_2-a_1+1}\frac{d}{dx}\right)\left(x^{a_1}\phi(x)\right)dx\,, \qquad a_1 < \text{Re } s < b_1\,, \tag{3}$$

and the last integral is absolutely convergent in $a_2 < \text{Re } s < b_1$. Hence the function Φ is holomorphically extended to $\{s \neq a_1 : a_2 < \text{Re } s < b_1\}$ by the function on the right hand side of (3). The extended function, that will be denoted again by Φ, has in $s = a_1$ at most a simple pole.

By repeating the argument we can extend $\Phi(s)$ to a meromorphic function in the complex plane that has in $s = a_n$ and $s = b_n$, $n \in \mathbb{N}$, at most simple poles. Moreover we get

$$\Phi(s) = \frac{(-1)^{k+l}}{\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)} \cdot \int_0^\infty x^{s-b_{\ell+1}-1}\prod_{i=1}^{\ell}\left(x^{b_{i+1}-b_i+1}\frac{d}{dx}\right)\cdot$$

$$\cdot\left(x^{b_1-a_{k+1}}\prod_{j=1}^{k}\left(x^{a_{j+1}-a_j+1}\frac{d}{dx}\right)\left(x^{a_1}\phi(x)\right)\right)dx\,,$$

$$s \in \{s \in \mathbb{C}/a_{k+1} < \text{Re } s < b_{\ell+1}\} - \left(\{a_n\}_{n=1}^{k} \cup \{b_n\}_{n=1}^{\ell}\right),\ l,k \in \mathbb{N}.$$

Hence, for every $s \in V(k,\ell)$

$$\left|\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,\Phi(s)\right| \leq \int_0^1 x^{\text{Re } s-a_{k+1}-1}dx\ \gamma^1_{\ell,k}(\phi) +$$

$$+\int_1^\infty x^{\text{Re } s-b_{\ell+1}-1}dx\ \gamma^0_{\ell,k}(\phi) = \frac{1}{\text{Re } s-a_{k+1}}\ \gamma^1_{\ell,k}(\phi) +$$

$$+\frac{1}{b_{\ell+1}-\text{Re } s}\ \gamma^0_{\ell,k}(\phi) \ \leq\ \frac{1}{\epsilon}\left[\gamma^1_{\ell,k}(\phi)+\gamma^0_{\ell,k}(\phi)\right],\quad l,k \in \mathbb{N}.$$

Then

$$\omega_{\ell,k}(\Phi) \ \leq\ \frac{1}{\epsilon}\left[\,\gamma^1_{\ell,k}(\phi)+\gamma^0_{\ell,k}(\phi)\,\right],\ l,k \in \mathbb{N}.$$

Thus we conclude that $\mathcal{M}$ is a continuous mapping from $\mathcal{A}(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty})$ into $\mathcal{B}(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty})$.

Let now Φ be in $\mathcal{B}(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty})$. We define the function

$$(\mathcal{T}\Phi)(x) = \phi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Phi(s)ds\,,\ x \in \boldsymbol{I},$$

where $c \in (a_1, b_1)$. The Cauchy residues theorem allows to see that the integral defining ϕ is not depending on $c \in (a_1, b_1)$.

By differentiating under the integral sign we obtain

$$\begin{gathered} x^{m(a_{k+1}-b_{\ell+1})}\prod_{i=1}^{\ell}\left(x^{b_{i+1}-b_i+1}\frac{d}{dx}\right) \\ \left(x^{b_1-a_{k+1}}\prod_{j=1}^{k}\left(x^{a_{j+1}-a_j+1}\frac{d}{dx}\right)(x^{a_1}\phi(x))\right) = \\ = \frac{(-1)^{k+\ell}}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{m(a_{k+1}-b_{\ell+1})-s+b_{\ell+1}}\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,\Phi(s)ds\,, \\ x\in\boldsymbol{I},\quad l,k\in\mathbb{N},\quad m=0,1. \end{gathered}$$

Note that the differentiation under the integral sign is justified because $\Phi \in \mathcal{B}(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty})$.

Assume now $m = 0$ and $l, k \in \mathbb{N}$. If $x \in (0,1)$ then

$$\left|\prod_{i=1}^{\ell}\left(x^{b_{i+1}-b_i+1}\frac{d}{dx}\right)\left(x^{b_1-a_{k+1}}\prod_{j=1}^{k}\left(x^{a_{j+1}-a_j+1}\frac{d}{dx}\right)(x^{a_1}\phi(x))\right)\right| \le$$

$$\le \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{|(c+it-b_{\ell+1})(c+it-b_{\ell+2})|}dt\ \omega_{\ell+2,k}(\Phi) \tag{4}$$

because $c < b_{\ell+1}$.

On the other hand, for every $x \in [1, \infty)$ and $R > 0$ we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\Gamma_R}\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,x^{-s+b_{\ell+1}}\Phi(s)ds = \\ &= \mathrm{Re}\,s\left[\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,x^{-s+b_{\ell+1}}\Phi(s)\,;\,s=b_{\ell+1}\right] \end{aligned} \tag{5}$$

Figure 1

where Γ_R is the closed path in Figure 1, and c_1 is chosen in the real interval $(b_{\ell+1}\,,\,b_{\ell+2})$.

Moreover, denoting by $\boldsymbol{L}_{R,\alpha}$ the path having the parametrization $s(t)=t+i\alpha R$, $t\in[c,c_1]$ for $\alpha=1,-1$, one has

$$\left|\int_{\boldsymbol{L}_{R,\alpha}}\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,x^{-s+b_{\ell+1}}\Phi(s)ds\right|\leq$$

$$\leq\frac{1}{R^2}\int_c^{c_1}x^{-t+b_{\ell+1}}dt\;\omega_{\ell+2,k}\,(\Phi)\to 0,$$

$$\text{as } R\to\infty,\quad\text{for }\ \alpha=1,-1.$$

Hence, by taking $R\to\infty$ in (5) we conclude that

$$\frac{1}{2\pi\,i}\int_{c-i\infty}^{c+i\infty}x^{-s+b_{\ell+1}}\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,\Phi(s)ds=$$

$$=\frac{1}{2\pi\,i}\int_{c_1-i\infty}^{c_1+i\infty}x^{-s+b_{\ell+1}}\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,\Phi(s)ds-\tag{6}$$

$$-\operatorname{Re}s\left[\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,x^{-s+b_{\ell+1}}\Phi(s)\,;\,s=b_{\ell+1}\right].$$

Also, since $c_1 > b_{\ell+1}$, we get

(7)
$$\left|\int_{c_1-i\infty}^{c_1+i\infty} x^{-s+b_{\ell+1}} \prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,\Phi(s)ds\right| \leq$$
$$\leq \int_{-\infty}^{+\infty} \frac{1}{|(c_1+it-b_{\ell+1})\,(c_1+it-b_{\ell+2})|}dt\ \omega_{\ell+2,k}(\Phi)\ ,\ x\in[1\,,\infty).$$

Moreover,

(8)
$$\left|\operatorname{Re}s\left[\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,x^{-s+b_{\ell+1}}\Phi(s)\,;s=b_{\ell+1}\right]\right| =$$
$$= \lim_{s\to b_{\ell+1}}\left|\prod_{i=1}^{\ell+1}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,x^{-s+b_{\ell+1}}\Phi(s)\right| \leq \omega_{\ell+1,k}(\Phi)\,,\ x\in[1,\infty).$$

By combining (4), (6), (7) and (8) we deduce that

(9)
$$\gamma^0_{\ell,k}(\phi) \leq\ M_1\left[\omega_{\ell+2,k}(\Phi)\ +\ \omega_{\ell+1,k}(\Phi)\right]$$

for a certain $M_1 > 0$.

By proceeding in a similar way we can find a positive constant M_2 such that

(10)
$$\gamma^1_{\ell,k}(\phi) \leq\ M_2\left[\omega_{\ell+2,k}(\Phi)\ +\ \omega_{\ell+1,k}(\Phi)\right].$$

From (9) and (10) we infer that $\mathcal{T}$ is a continuous mapping from $\mathcal{B}(\{a_n\}_{n=1}^{\infty},\{b_n\}_{n=1}^{\infty})$ into $\mathcal{A}(\{a_n\}_{n=1}^{\infty},\{b_n\}_{n=1}^{\infty})$.

To finish the proof it is sufficient to take into account that $\mathcal{T}\circ\mathcal{M}(\phi) = \phi$, $\phi\in\mathcal{A}(\{a_n\}_{n=1}^{\infty},\{b_n\}_{n=1}^{\infty})$ and $\mathcal{M}\circ\mathcal{T}(\Phi)=\Phi$, $\Phi\in\mathcal{B}(\{a_n\}_{n=1}^{\infty}\,,\ \{b_n\}_{n=1}^{\infty})$. (I. N. Sneddon [17], p. 273). □

We now investigate a multiplier mapping between spaces of type $\mathcal{B}(\{a_n\}_{n=1}^{\infty}\ ,\ \{b_n\}_{n=1}^{\infty})$ that will be useful in the sequel.

Proposition 2. *Let $\{a_{n,i}\}_{n=1}^{\infty}$ be a sequence of real numbers such that $inf\,\{a_{n,i}-a_{n+1,i}\,:\,n\in\mathbb{N}\} > 0$ and let $\{b_{n,i}\}_{n=1}^{\infty}$ be a sequence of real numbers such that $inf\,\{b_{n+1,i}-b_{n,i}\,:\,n\in\mathbb{N}\} > 0$, $i=1,2$. Assume also that $a_{1,i} < b_{1,i}\,,\ i=1,2$.*

If $\mathcal{K}(s)$ is a meromorphic function in the complex plane satisfying

(i) *$\mathcal{K}(s)$ has simple zeros in $s=1-a_{n,1}$ and $s=1-b_{n,1}\,,\ n\in\mathbb{N}$,*

(ii) *$\mathcal{K}(s)$ has its singularities at most in $s = a_{n,2}$ and $s = b_{n,2}$, $n \in \mathbb{N}$; moreover such singularities are simple poles, and*

(iii) *for every compact subset $\boldsymbol{J}$ of $\mathbb{R}$ there exist $M_{\boldsymbol{J}} > 0$, $Y_{\boldsymbol{J}} > 0$ and $\alpha_{\boldsymbol{J}} \in \mathbb{R}$ such that*

$$|\mathcal{K}(s)| \leq M_{\boldsymbol{J}} \, |\mathrm{Im}\; s|^{\alpha_{\boldsymbol{J}}} \;, \quad \textit{for} \, |\,\mathrm{Im}\; s| > Y_{\boldsymbol{J}} \quad \textit{and}\ \mathrm{Re}\; s \in \boldsymbol{J},$$

then the mapping $\mathcal{T}_{\mathcal{K}}(\Phi)(s) = \mathcal{K}(s)\Phi(1-s)$ is a continuous linear mapping from $\mathcal{B}(\{a_{n,1}\}_{n=1}^{\infty}, \{b_{n,1}\}_{n=1}^{\infty})$ into $\mathcal{B}(\{a_{n,2}\}_{n=1}^{\infty}, \{b_{n,2}\}_{n=1}^{\infty})$.

Proof. Let Φ be in $\mathcal{B}(\{a_{n,1}\}_{n=1}^{\infty}, \{b_{n,1}\}_{n=1}^{\infty})$. It is clear that the function $\mathcal{K}(s)\Phi(1-s)$ is a meromorphic function in $\mathbb{C}$ having at most simple poles in $s = a_{n,2}$ and $s = b_{n,2}$, $n \in \mathbb{N}$.

Let $\ell, k \in \mathbb{N}$ and choose a small enough $\epsilon > 0$. We put $V_{\epsilon}^{i}(k,\ell) = \{s \in \mathbb{C} : a_{k+1,i} + \epsilon \leq \mathrm{Re}\; s \leq b_{\ell+1,i} - \epsilon\}$, $i = 1, 2$. Two nonnegative integer numbers γ, β are choosen such that

$$\omega_{\epsilon}^{2}(k,\ell) = \left\{s \in \mathbb{C} : 1 - s \in V_{\epsilon}^{2}(k,\ell)\right\} \subset V_{\epsilon}^{1}(\beta,\gamma)$$

and $\ell + k + \alpha < \gamma + \beta$, where α is the positive constant $\alpha_{\boldsymbol{J}}$ given in (iii) with $\boldsymbol{J} = [\epsilon + 1 - b_{\ell+1,2}, \epsilon + 1 - a_{k+1,2}]$.

Then by virtue of the conditions imposed on the function $\mathcal{K}$ there exists a positive constant M for which

$$\sup_{s \in V_{\epsilon}^{2}(k,\ell)} \left| \prod_{i=1}^{\ell} (s - b_{i,2}) \prod_{j=1}^{k} (s - a_{j,2}) \, \mathcal{T}_{\mathcal{K}}(\Phi)(s) \right| \leq$$

$$\leq \sup_{s \in \omega_{\epsilon}^{2}(k,\ell)} \left| \frac{\displaystyle\prod_{i=1}^{\ell} (1 - b_{i,2} - s) \prod_{j=1}^{k} (1 - a_{j,2} - s) \, \mathcal{K}(1-s)}{\displaystyle\prod_{i=1}^{\gamma} (s - b_{i,1}) \prod_{j=1}^{\beta} (s - a_{j,1})} \right| \cdot$$

$$\cdot \sup_{s \in V_{\epsilon}^{1}(\beta,\gamma)} \left| \prod_{i=1}^{\gamma} (s - b_{i,1}) \prod_{j=1}^{\beta} (s - a_{j,1}) \, (\Phi)(s) \right| \leq$$

$$\leq M \sup_{s \in V_{\epsilon}^{1}(\beta,\gamma)} \left| \prod_{i=1}^{\gamma} (s - b_{i,1}) \prod_{j=1}^{\beta} (s - a_{j,1}) \, (\Phi)(s) \right| .$$

Thus the proof of this Proposition is complete. □

An immediate consequence of Proposition 2 is the following

Corollary 1. *Let* $\{a_{n,i}\}_{n=1}^{\infty}$ *and* $\{b_{n,i}\}_{n=1}^{\infty}$, $i = 1, 2$, *be as in Proposition 2. If* $\mathcal{K}(s)$ *is a meromorphic function in the complex plane satisfying*

(i) $\mathcal{K}(s)$ *has simple zeros in* $s = 1 - a_{n,1}$ *and* $s = 1 - b_{n,1}$, $n \in \mathbb{N}$,

(ii) $\mathcal{K}(s)$ *has its singularities in* $s = a_{n,2}$ *and* $s = b_{n,2}$, $n \in \mathbb{N}$; *moreover such singularities are simple poles, and*

(iii) *For every compact subset* $\boldsymbol{J}$ *of* $\mathbb{R}$ *there exist* $M_{\boldsymbol{J}} > 0$, $Y_{\boldsymbol{J}} > 0$, $\alpha_{\boldsymbol{J}} > 0$ *and* $\beta_{\boldsymbol{J}} > 0$ *such that*

$$\frac{1}{M_{\boldsymbol{J}}} |\mathrm{Im}\ s|^{\beta_{\boldsymbol{J}}} \leq |\mathcal{K}(s)| \leq M_{\boldsymbol{J}} |\mathrm{Im}\ s|^{\alpha_{\boldsymbol{J}}} , \quad \textit{for } |\mathrm{Im}\ s| > Y_{\boldsymbol{J}} \ \textit{ and } \mathrm{Re}\ s \in \boldsymbol{J},$$

then the mapping $\mathcal{T}_{\mathcal{K}}(\Phi)(s) = \mathcal{K}(s)\Phi(1-s)$ *is a homeomorphism from* $\mathcal{B}(\{a_{n,1}\}_{n=1}^{\infty}, \{b_{n,1}\}_{n=1}^{\infty})$ *onto* $\mathcal{B}(\{a_{n,2}\}_{n=1}^{\infty}, \{b_{n,2}\}_{n=1}^{\infty})$.

3 The Watson integral transformation

In this section we investigate the Watson integral transformation (1) on the spaces of type $\mathcal{A}(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty})$ and their duals when the Mellin transform $\mathcal{K}(s) = \mathcal{M}\{k\}(s)$ of k satisfies suitable smoothness and growth conditions. The main result of this paragraph is the following

Theorem 1. *Let* $\{a_{n,i}\}_{n=1}^{\infty}$ *and* $\{b_{n,i}\}_{n=1}^{\infty}$, $i = 1, 2$, *and* $\mathcal{K}(s)$ *be as in Corollary 1. Assume also that* $a = \max\{1 - b_{1,1}, a_{1,2}\} < \min\{1 - a_{1,1}, b_{1,2}\} = b$ *and that for every compact subset* $\boldsymbol{J}$ *of* (a, b) *there exist* $M_{\boldsymbol{J}} > 0$, $Y_{\boldsymbol{J}} > 0$ *and* $\alpha_{\boldsymbol{J}} < -1$ *such that*

$$|\mathcal{K}(s)| \leq M_{\boldsymbol{J}} |\mathrm{Im}\ s|^{\alpha_{\boldsymbol{J}}} , \quad \textit{for } |\mathrm{Im}\ s| > Y_{\boldsymbol{J}} \quad \textit{and } \mathrm{Re}\ s \in \boldsymbol{J}.$$

Then the Watson transformation (1), where

$$k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \mathcal{K}(s) ds , \quad x \in \boldsymbol{I}, \tag{11}$$

with $a < c < b$, *is a homeomorphism from* $\mathcal{A}(\{a_{n,1}\}_{n=1}^{\infty}, \{b_{n,1}\}_{n=1}^{\infty})$ *onto* $\mathcal{A}(\{a_{n,2}\}_{n=1}^{\infty}, \{b_{n,2}\}_{n=1}^{\infty})$.

Proof. Note firstly that the integral in (11) is not depending on c provided that $a < c < b$.

Let $\phi \in \mathcal{A}(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty})$. We have

$$W(\phi)(y) = \frac{1}{2\pi i} \int_0^{\infty} \phi(x) \int_{c-i\infty}^{c+i\infty} (xy)^{-s} \mathcal{K}(s) ds\, dx , \quad y \in \boldsymbol{I} ,$$

where $a < c < b$.

By virtue of the Fubini Theorem we can deduce

$$W(\phi)(y) = \frac{1}{2\,\pi\, i} \int_{c-i\infty}^{c+i\infty} \mathcal{K}(s) y^{-s} \int_0^\infty \phi(x) x^{-s} dx\, ds\,, \quad y \in \boldsymbol{I}\,,$$

because $\int_0^\infty |\phi(x)|\, x^{-c} dx < \infty$ when $a < c < b$.

Hence we can be write

$$W(\phi)(y) = \mathcal{M}^{-1} \circ \mathcal{T}_{\mathcal{K}} \circ \mathcal{M}(\phi)(y)\,, \quad y \in \boldsymbol{I}, \tag{12}$$

where as usual $\mathcal{M}$ denotes the Mellin transformation and $\mathcal{T}_{\mathcal{K}}$ is the mapping studied in Section 2.

The desired result follows now from (12) as an immediate consequence of Proposition 1 and Corollary 1. □

Next we establish a Parseval equality for the transformation W.

Proposition 3. *Let* $\{a_{n,i}\}_{n=1}^\infty$ *and* $\{b_{n,i}\}_{n=1}^\infty$, $i = 1, 2$, *and* $\mathcal{K}(s)$ *be as in Proposition 2. Assume also that* $a = \max\{1 - b_{1,1}\,,\, 1 - b_{1,2}\} < \min\{1 - a_{1,1}\,,\, 1 - a_{1,2}\} = b$ *and that for every compact subset* $\boldsymbol{J}$ *of* (a, b) *there exist* $M_{\boldsymbol{J}} > 0$, $Y_{\boldsymbol{J}} > 0$ *and* $\alpha_{\boldsymbol{J}} < -1$ *such that* $|\mathcal{K}(s)| \leq M_{\boldsymbol{J}}\, |\mathrm{Im}\ s|^{\alpha_{\boldsymbol{J}}}$, *for* $|\,\mathrm{Im}\ s| > Y_{\boldsymbol{J}}$ *and* $\mathrm{Re}\ s \in \boldsymbol{J}$. *If* W *denotes the integral transformation* (1), *where* $k(x) = \dfrac{1}{2\pi\, i} \displaystyle\int_{c-i\infty}^{c+i\infty} x^{-s} \mathcal{K}(s) ds$ *with* $a < c < b$, *then*

$$\int_0^\infty \phi_1(x) W(\phi_2)(x) dx = \int_0^\infty W(\phi_1)(x) \phi_2(x) dx \tag{13}$$

for every $\phi_i \in \mathcal{A}(\{a_{n,i}\}_{n=1}^\infty\,,\, \{b_{n,i}\}_{n=1}^\infty)$, $i = 1, 2$.

Proof. Let $\phi_i \in \mathcal{A}(\{a_{n,i}\}_{n=1}^\infty\,,\, \{b_{n,i}\}_{n=1}^\infty)$, $i = 1, 2$. Since $k(x) = \dfrac{1}{2\pi\, i} \displaystyle\int_{c-i\infty}^{c+i\infty} x^{-s} \mathcal{K}(s) ds$ where $a < c < b$, and this integral is not depending on c provided that $a < c < b$, for every c, $a < c < b$ there exists $M_c > 0$ for which

$$\begin{aligned} &\int_0^\infty \int_0^\infty |\phi_1(x) \phi_2(y) k(xy)|\, dx\, dy \leq \\ &\leq M_c \int_0^\infty x^{-c}\, |\phi_1(x)|\, dx \int_0^\infty x^{-c}\, |\phi_2(x)|\, dx. \end{aligned}$$

Therefore $\int_0^\infty \int_0^\infty |\phi_1(x) \phi_2(y) k(xy)|\, dx\, dy < \infty$ because $\int_0^\infty x^{-c}\, |\phi_i(x)|\, dx < \infty$, $i = 1, 2$, provided that $a < c < b$.

By invoking the Fubini Theorem we conclude that

$$\int_0^\infty \phi_1(x) W(\phi_2)(x) dx = \int_0^\infty \phi_1(x) \int_0^\infty k(xy)\phi_2(y) dy\, dx =$$
$$= \int_0^\infty \phi_2(y) \int_0^\infty k(xy)\phi_1(x) dx\, dy = \int_0^\infty \phi_2(y) W(\phi_1)(y) dy. \qquad \square$$

We define the generalized W-transformation on $\mathcal{A}(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty)'$, the dual space of $\mathcal{A}(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty)$, as the transpose of the classical W transform. An immediate consequence of Theorem 1 is the following

Theorem 2. *Let $\{a_{n,i}\}_{n=1}^\infty$ and $\{b_{n,i}\}_{n=1}^\infty$, $i = 1, 2$, $\mathcal{K}$ and k be as in Theorem 1. Then the generalized W transform $W'f$ of $f \in \mathcal{A}(\{a_{n,2}\}_{n=1}^\infty, \{b_{n,2}\}_{n=1}^\infty)'$ given by*

$$\langle W'f , \phi\rangle = \langle f , W\phi\rangle , \ \phi \in \mathcal{A}(\{a_{n,1}\}_{n=1}^\infty , \{b_{n,1}\}_{n=1}^\infty) \tag{14}$$

defines a homeomorphism from $\mathcal{A}(\{a_{n,2}\}_{n=1}^\infty, \{b_{n,2}\}_{n=1}^\infty)'$ onto $\mathcal{A}(\{a_{n,1}\}_{n=1}^\infty, \{b_{n,1}\}_{n=1}^\infty)'$ if $\mathcal{A}(\{a_{n,i}\}_{n=1}^\infty , \{b_{n,i}\}_{n=1}^\infty)'$ is endowed either with its weak $*$ or its strong topology.

Next we give sufficient conditions for $\mathcal{A}(\{a_{n,2}\}_{n=1}^\infty , \{b_{n,2}\}_{n=1}^\infty)$ to be a subspace of $\mathcal{A}(\{a_{n,1}\}_{n=1}^\infty , \{b_{n,1}\}_{n=1}^\infty)'$.

Proposition 4. *Let $\{a_{n,i}\}_{n=1}^\infty$ and $\{b_{n,i}\}_{n=1}^\infty$ be as in Proposition 2. Suppose also that $a_{1,1} + a_{1,2} < 1$ and $b_{1,1} + b_{1,2} > 1$. Then $\mathcal{A}(\{a_{n,1}\}_{n=1}^\infty, \{b_{n,1}\}_{n=1}^\infty) \subset \mathcal{A}(\{a_{n,2}\}_{n=1}^\infty , \{b_{n,2}\}_{n=1}^\infty)'$ in the following sense : every $\phi \in \mathcal{A}(\{a_{n,1}\}_{n=1}^\infty, \{b_{n,1}\}_{n=1}^\infty)$ defines an element of $\mathcal{A}(\{a_{n,2}\}_{n=1}^\infty, \{b_{n,2}\}_{n=1}^\infty)'$ through*

$$\langle \phi , \psi\rangle = \int_0^\infty \phi(x)\psi(x) dx , \ \psi \in \mathcal{A}(\{a_{n,2}\}_{n=1}^\infty , \{b_{n,2}\}_{n=1}^\infty) .$$

Proof. Let $\phi \in \mathcal{A}(\{a_{n,1}\}_{n=1}^\infty, \{b_{n,1}\}_{n=1}^\infty)$. For every $\psi \in \mathcal{A}(\{a_{n,2}\}_{n=1}^\infty, \{b_{n,2}\}_{n=1}^\infty)$ we have

$$\left|\int_0^\infty \phi(x)\psi(x) dx\right| \le \int_0^1 x^{-a_{1,1}-a_{1,2}} dx \sup_{t\in \boldsymbol{I}} |t^{a_{1,1}}\phi(t)| . \sup_{t\in \boldsymbol{I}} |t^{a_{1,2}}\psi(t)| +$$
$$+ \int_1^\infty x^{-b_{1,1}-b_{1,2}} dx \sup_{t\in \boldsymbol{I}} \left|t^{b_{1,1}}\phi(t)\right| \sup_{t\in \boldsymbol{I}} \left|t^{b_{1,2}}\psi(t)\right| .$$

Hence there exists a positive constant M such that

$$\left|\int_0^\infty \phi(x)\psi(x)dx\right| \leq M\left(\sup_{t\in \boldsymbol{I}}\left|t^{a_{1,2}}\psi(t)\right| + \sup_{t\in \boldsymbol{I}}\left|t^{b_{1,2}}\psi(t)\right|\right),$$
$$\text{for every } \ \psi \in \mathcal{A}\left(\{a_{n,2}\}_{n=1}^\infty\,,\ \{b_{n,2}\}_{n=1}^\infty\right).$$

Therefore $\phi \in \mathcal{A}\left(\{a_{n,2}\}_{n=1}^\infty\,,\ \{b_{n,2}\}_{n=1}^\infty\right)'$. □

According to Proposition 4 for every $\phi \in \mathcal{A}\left(\{a_n\}_{n=1}^\infty\,,\ \{b_n\}_{n=1}^\infty\right)$ two W transforms of ϕ can be defined: the classical transform (1) and the generalized transform (14). We now prove that under suitable conditions the classical W transform of $\phi \in \mathcal{A}\left(\{a_{n,1}\}_{n=1}^\infty\,,\ \{b_{n,1}\}_{n=1}^\infty\right)$ given by (1) is equal (in the sense of equality in $\mathcal{A}\left(\{a_{n,2}\}_{n=1}^\infty\,,\ \{b_{n,2}\}_{n=1}^\infty\right)'$) to the generalized W transform of ϕ as given in (14).

Proposition 5. *Assume that* $\{a_{n,i}\}_{n=1}^\infty$ *and* $\{b_{n,i}\}_{n=1}^\infty$, $i = 1, 2$, $\mathcal{K}$ *and* k *satisfy the conditions in Theorem 1 and Proposition 3. Suppose also that* $a_{1,1} + a_{1,2} < 1$ *and* $b_{1,1} + b_{1,2} > 1$. *Then for every* $\phi \in \mathcal{A}\left(\{a_{n,1}\}_{n=1}^\infty\,,\ \{b_{n,1}\}_{n=1}^\infty\right)$

$$\langle W\phi\,,\,\psi\rangle = \langle\phi\,,\,W\psi\rangle\ ,\ \psi \in \mathcal{A}\left(\{a_{n,1}\}_{n=1}^\infty\,,\ \{b_{n,1}\}_{n=1}^\infty\right).$$

Proof. Let $\phi \in \mathcal{A}\left(\{a_{n,1}\}_{n=1}^\infty\,,\ \{b_{n,1}\}_{n=1}^\infty\right)$. By virtue of Theorem 1, $W\phi \in \mathcal{A}\left(\{a_{n,2}\}_{n=1}^\infty\,,\ \{b_{n,2}\}_{n=1}^\infty\right)$. Hence by Proposition 4, $W\phi \in \mathcal{A}\left(\{a_{n,1}\}_{n=1}^\infty\,,\ \{b_{n,1}\}_{n=1}^\infty\right)'$ and

$$\langle W\phi\,,\,\psi\rangle = \int_0^\infty W(\phi)(x)\psi(x)dx\,,\ \psi \in \mathcal{A}\left(\{a_{n,1}\}_{n=1}^\infty\,,\ \{b_{n,1}\}_{n=1}^\infty\right).$$

Moreover, according to (13)

$$\langle W\phi,\psi\rangle = \int_0^\infty \phi(x)W(\psi)(x)dx = \langle\phi, W\psi\rangle\,,\ \psi\in\mathcal{A}\left(\{a_{n,1}\}_{n=1}^\infty, \{b_{n,1}\}_{n=1}^\infty\right),$$

and the proof is finished. □

4 Other Watson type transformations

In the previous Section we have investigated Watson transforms whose kernel k has a Mellin transform $\mathcal{K}(s)$ having a potential growth when $|\mathrm{Im}\ s|$ is large. Motivated by the papers of A. Schuitman [15] and J.A. Barrios and J.J. Betancor ([2] and [3]) we now extend our previous results to Watson transformations whose kernel k is such that $\mathcal{K}(s) = \mathcal{M}(k)(s)$ admits an exponential growth when $|\mathrm{Im}\ s|$ is large. Since the proofs of the new results are essentially similar to those of the results in Section 2 and 3 we shall omit the proofs here.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\inf\{a_n - a_{n+1}\colon n \in \mathbb{N}\} > 0$ and let $\{b_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\inf\{b_{n+1} - b_n\colon n \in \mathbb{N}\} > 0$. Also assume that $a_1 < b_1$. For every $\theta \in (0\,,\pi)$ we define the set

$$G_\theta = \{x \in \mathbb{C} \,:\, |\arg x| \le \theta\}$$

where as usual $\arg x$ denotes the principal argument of $x \in \mathbb{C}$. By G_θ° we will denote the interior set of G_θ. Note that 0 does not belong to G_θ. The space $\mathcal{A}^\theta\left(\{a_n\}_{n=1}^{\infty}\,,\,\{b_n\}_{n=1}^{\infty}\right)$ consists of all those functions $\phi(x)$, $x \in G_\theta$, which satisfy the following two conditions:

(i) ϕ is holomorphic in G_θ° and $\frac{d^m}{dx^m}\phi$ can be continuously extended to G_θ for every $m \in \mathbb{N}$, and

(ii) the quantity

$$\gamma_{\ell\,,\,k}^{\theta\,,\,m}(\phi) \;=\; \sup_{x\in G_\theta}\left| x^{m(a_{k+1}-b_{\ell+1})}\prod_{i=1}^{\ell}\left(x^{b_{i+1}-b_i+1}\frac{d}{dx}\right)\cdot \right.$$
$$\left.\cdot\left(x^{b_1-a_{k+1}}\prod_{j=1}^{k}\left(x^{a_{j+1}-a_j+1}\frac{d}{dx}\right)\left(x^{a_1}\phi(x)\right)\right)\right|$$

is finite for every $\ell, k \in \mathbb{N}$ and $m = 0, 1$.

$\mathcal{A}^\theta\left(\{a_n\}_{n=1}^{\infty}\,,\,\{b_n\}_{n=1}^{\infty}\right)$ is endowed with the topology induced by the family of seminorms $\left\{\gamma_{\ell\,,\,k}^{\theta\,,\,m}\right\}_{\ell\,,\,k\,\in\mathbb{N},\,m=0,1}$. Thus $\mathcal{A}^\theta\left(\{a_n\}_{n=1}^{\infty}\,,\,\{b_n\}_{n=1}^{\infty}\right)$ is a Fréchet space.

Let $\epsilon > 0$ such that $b_1 - a_1 > 2\epsilon$, $a_{n+1} + \epsilon < a_n$ and $b_n + \epsilon < b_{n+1}$, $n \in \mathbb{N}$. $\mathcal{B}_\epsilon^\theta\left(\{a_n\}_{n=1}^{\infty}\,,\,\{b_n\}_{n=1}^{\infty}\right)$ is constituted by all meromorphic functions Φ in the complex plane such that

(i) Φ has at most simple poles in $s = a_n$ and $s = b_n$, $n \in \mathbb{N}$, and $\Phi(s)$ is holomorphic in $\mathbb{C} - \left(\{a_n\}_{n=1}^{\infty}\,\cup\,\{b_n\}_{n=1}^{\infty}\right)$,

(ii) for every $\ell, k \in \mathbb{N}$,

$$\omega_{\ell,k}^{\theta,\epsilon}(\Phi) = \sup_{s\in V_\epsilon(k,\ell)}\left|\prod_{i=1}^{\ell}(s-b_i)\prod_{j=1}^{k}(s-a_j)\,\Phi(s)e^{\theta|\operatorname{Im}\,s|}\right| < \infty \tag{15}$$

where $V_\epsilon(k,\ell)$ is understood as in Section 2.

We consider in $\mathcal{B}_\epsilon^\theta\left(\{a_n\}_{n=1}^{\infty}\,,\,\{b_n\}_{n=1}^{\infty}\right)$ the topology generated by the system of seminorms $\left\{\omega_{\ell,k}^{\theta,\epsilon}\right\}_{\ell,k\in\mathbb{N}}$. Thus $\mathcal{B}_\epsilon^\theta\left(\{a_n\}_{n=1}^{\infty}\,,\,\{b_n\}_{n=1}^{\infty}\right)$ is a

Fréchet space. As in Section 2, it can be seen that the space $\mathcal{B}_\epsilon^\theta(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty)$ is not depending on ϵ provided that $b_1 - a_1 > 2\epsilon$, $a_{n+1} + \epsilon < a_n$ and $b_n + \epsilon < b_{n+1}$, $n \in \mathbb{N}$. Hence in the sequel we write $\mathcal{B}^\theta(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty)$ and $\omega_{\ell,k}^\theta$ instead of $\mathcal{B}_\epsilon^\theta(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty)$ and $\omega_{\ell,k}^{\theta,\epsilon}$. Moreover, if in (15) we replace $e^{\theta|\operatorname{Im} s|}$ by $e^{\theta|s|}$ then the resulting family of seminorms generates the same topology as $\left\{\omega_{\ell,k}^\theta\right\}_{\ell,k\in\mathbb{N}}$.

The results that we now list are analogous to those established in Section 2.

Proposition 6. *The Mellin integral transformation is a homeomorphism from* $\mathcal{A}^\theta(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty)$ *onto* $\mathcal{B}^\theta(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty)$.

Proposition 7. *Let* $\{a_{n,i}\}_{n=1}^\infty$ *and* $\{b_{n,i}\}_{n=1}^\infty$, $i = 1, 2$, *be as in Proposition 2. If* $\mathcal{K}(s)$ *is a meromorphic function in the complex plane satisfying*

(i) $\mathcal{K}(s)$ *has simple zeros in* $s = 1 - a_{n,1}$ *and* $s = 1 - b_{n,1}$, $n \in \mathbb{N}$,

(ii) $\mathcal{K}(s)$ *is holomorphic in* $\mathbb{C} - (\{a_{n,2}\}_{n=1}^\infty \cup \{b_{n,2}\}_{n=1}^\infty)$ *and* $\mathcal{K}(s)$ *has simple poles in* $s = a_{n,2}$ *and* $s = b_{n,2}$, $n \in \mathbb{N}$,

(iii) *There exists* α, $-\theta < \alpha < \pi - \theta$ *such that for every compact subset* $\boldsymbol{J}$ *of* $\mathbb{R}$ *there exist* $M_{\boldsymbol{J}} > 0$, $Y_{\boldsymbol{J}} > 0$, $\alpha_{\boldsymbol{J}} \in \mathbb{R}$ *and* $\beta_{\boldsymbol{J}} \in \mathbb{R}$ *for which*

$$\frac{1}{M_{\boldsymbol{J}}} |\operatorname{Im} s|^{\beta_{\boldsymbol{J}}} \le |\mathcal{K}(s)| e^{|\operatorname{Im} s|\alpha} \le M_{\boldsymbol{J}} |\operatorname{Im} s|^{\alpha_{\boldsymbol{J}}},$$
$$\text{for } |\operatorname{Im} s| > Y_{\boldsymbol{J}} \quad \text{and} \quad \operatorname{Re} s \in \boldsymbol{J},$$

then the mapping $\mathcal{T}_{\mathcal{K}}(\Phi)(s) = \mathcal{K}(s)\Phi(1-s)$ *is a homeomorphism from* $\mathcal{B}^\theta(\{a_{n,1}\}_{n=1}^\infty, \{b_{n,1}\}_{n=1}^\infty)$ *onto* $\mathcal{B}^{\theta+\alpha}(\{a_{n,2}\}_{n=1}^\infty, \{b_{n,2}\}_{n=1}^\infty)$.

The main result of this Section is the following

Theorem 3. *Let* $\{a_{n,i}\}_{n=1}^\infty$, $\{b_{n,i}\}_{n=1}^\infty$, $i = 1, 2$, *and* $\mathcal{K}$ *be as in Proposition 7. Assume also that* $0 < \alpha < \pi - \theta$ *and* $a = \max\{1 - b_{1,1}, a_{1,2}\} < \min\{1 - a_{1,1}, b_{1,2}\} = b$. *Define* $k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\mathcal{K}(s)ds$, $x \in G_\alpha^\circ$, *with* $a < c < b$. *Then the integral transformation* W^* *defined by*

$$W^*(\phi)(x) = \frac{1}{x}\int_0^{\infty e^{i\xi}} k(u)\phi\left(\frac{u}{x}\right)du, \; x \in G_{\theta+\alpha}^\circ \tag{16}$$

where $|\xi| < \alpha$ *and* $|\arg x - \xi| < \theta$, *is a homeomorphism from* $\mathcal{A}^\theta(\{a_{n,1}\}_{n=1}^\infty, \{b_{n,1}\}_{n=1}^\infty)$ *onto* $\mathcal{A}^{\theta+\alpha}(\{a_{n,2}\}_{n=1}^\infty, \{b_{n,2}\}_{n=1}^\infty)$. *Moreover* $W^*(\phi)(x) = W(\phi)(x)$, $x \in \boldsymbol{I}$.

PROOF. Let ϕ be in $\mathcal{A}^{\theta}\left(\{a_{n,1}\}_{n=1}^{\infty}\,,\ \{b_{n,1}\}_{n=1}^{\infty}\right)$. Note firstly that the integral in (16) is not depending on ξ provided that $|\xi| < \alpha$, $|\arg x - \xi| < \theta$ and $x \in G^{\circ}_{\theta+\alpha}$. In effect, denote by Γ_R, $R > 0$, the path having the parametric representation $z = R\,e^{i\,\varphi}$, $\xi_1 < \varphi < \xi_2$, where $|\xi_i| < \alpha$, $|\arg x - \xi_i| < \theta$, $i = 1, 2$. Then, by taking into account that for every $a < c < b$ there exists $M_c > 0$ such that $|k(x)| \le M_c |x|^{-c}$ for $x \in G_{\xi_2}$, we have

$$\left|\int_{\Gamma_R} k(u)\phi\left(\frac{u}{x}\right) du\right| \le \int_{\xi_1}^{\xi_2} \left|k\left(R\,e^{i\,\varphi}\right)\right| \left|\phi\left(\frac{R\,e^{i\,\varphi}}{x}\right)\right| R\,d\varphi \le$$

$$\le M_c \int_{\xi_1}^{\xi_2} R^{-c+1} \left|\phi\left(\frac{R\,e^{i\,\varphi}}{x}\right)\right| d\varphi,\ c \in (a,b),\ R > 0.$$

Hence there exists $M_c > 0$ such that for every $c \in (a,b)$

$$\left|\int_{\Gamma_R} k(u)\phi\left(\frac{u}{x}\right) du\right| \le M_c \sup_{t\in G_\theta} \left|t^{a_1}\phi(t)\right| R^{-c-a_{1,1}+1}\,,\ R > 0$$

and

$$\left|\int_{\Gamma_R} k(u)\phi\left(\frac{u}{x}\right) du\right| \le M_c \sup_{t\in G_\theta} \left|t^{b_1}\phi(t)\right| R^{-c-b_{1,1}+1}\,,\ R > 0.$$

Therefore we conclude that

$$\int_{\Gamma_R} k(u)\phi\left(\frac{u}{x}\right) du \to 0, as\ R \to 0\ and\ R \to \infty\,.$$

By using now the Cauchy theorem we deduce that

$$\int_0^{\infty\,e^{i\xi_1}} k(u)\phi\left(\frac{u}{x}\right) du = \int_0^{\infty\,e^{i\xi_2}} k(u)\phi\left(\frac{u}{x}\right) du.$$

Thus if $x \in \boldsymbol{I}$ we can choose $\xi = 0$ and then we obtain

$$W^*(\phi)(x) = \frac{1}{x}\int_0^{\infty} k(u)\phi\left(\frac{u}{x}\right) du = \int_0^{\infty} k(ux)\phi(u)du = W(\phi)(x).$$

Also, according to the Fubini Theorem one has

$$\int_0^{\infty\,e^{i\xi}} k(u)\phi\left(\frac{u}{x}\right) du = \tag{17}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{K}(s) \int_0^{\infty\,e^{i\xi}} u^{-s}\phi\left(\frac{u}{x}\right) du\,ds\,,\ x \in G^{\circ}_{\theta+\alpha}.$$

Moreover, by making a change of variables and by employing again the Cauchy Theorem we obtain

$$(18)\quad \int_0^{\infty\, e^{i\xi}} u^{-s}\phi\left(\frac{u}{x}\right) du =$$

$$= x^{1-s}\int_0^{\infty\, e^{i(\xi-\arg\, x)}} z^{-s}\phi(z)dz \;=\; x^{1-s}\int_0^{\infty} z^{-s}\phi(z)dz\,,\quad x\in G^{\circ}_{\theta+\alpha}.$$

Hence by combining (17) and (18) we conclude that

$$W^*(\phi) \;=\; \mathcal{M}^{-1}\circ \mathcal{T}_{\mathcal{K}}\circ \mathcal{M}(\phi)\;.$$

Now the proof follows from Propositions 6 and 7. □

By employing the standard procedure the Watson transform W^* can be defined in $\mathcal{A}^{\theta}\left(\{a_n\}_{n=1}^{\infty}\,,\;\{b_n\}_{n=1}^{\infty}\right)'$, the dual space of $\mathcal{A}^{\theta}(\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty})$.

5 An example and final remarks

The theory developed in the above paragraphs can be applied to study a wide class of integral transformations having as kernel the $\mathcal{H}$-function of Ch. Fox [8].

Let $m,n,p,q\in\mathbb{N}$, being $0\leq m\leq q$, $0\leq n\leq p$ and $p+q\geq 1$. Let $\alpha_j>0$, $a_j\in\mathbb{R}$ $(j=1,\dots,p)$ and $\beta_j>0$, $b_j\in\mathbb{R}$ $(j=1,\dots,q)$. We define the function

$$\mathfrak{H}^{m,n}_{p,q}\left(\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\,\middle|\,s\right)=\frac{\displaystyle\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\displaystyle\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}$$

and the real parameters

$$\alpha=\begin{cases}\max\left\{-\frac{b_j}{\beta_j},\ j=1,\dots,m\right\} & ,\ \text{for}\quad m>0\\ -\infty & ,\ \text{for}\quad m=0\end{cases}$$

$$\beta=\begin{cases}\min\left\{\frac{1-a_j}{\alpha_j},\ j=1,\dots,n\right\} & ,\ \text{for}\quad n>0\\ +\infty & ,\ \text{for}\quad n=0\end{cases}$$

$$\delta=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p}\alpha_j+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j$$

$$\mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \,, \quad \text{and} \quad \nu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j .$$

Here $\sum_{1}^{0}$ is understood as 0 when this sum appears.

If $\alpha < c < \beta$ for every $\delta > 0$ and for $\delta = 0$, being $\mu \neq 0$ and $\nu + \mu c - \frac{1}{2}(q-p) < -1$ the Fox function is defined by

$$\mathcal{H}\left(\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle|\, x\right) =$$
$$= \frac{1}{2\,\pi\, i}\int_{c-i\infty}^{c+i\infty} x^{-s}\,\mathfrak{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle|\, s\right) ds\,,\ x \in \boldsymbol{I}.$$

By applying well-known properties of the Γ-function (A. Erdelyi [7]) we can see that the function $\mathfrak{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle|\, s\right)$ satisfies the requirements in Theorem 2 and the integral transformation

$$W\left(\phi\right)(x) = \int_0^\infty \mathcal{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle|\, xt\right)\phi(t)dt$$

can be investigated in our spaces for a wide range of values of the parameters.

We finish this paper with two *remarks.*

I. The ideas developed in this paper can be modified to include Watson transformations having other kind of associated functions $\mathcal{K}(s)$. Specifically, associated functions $\mathcal{K}(s)$ should be admitted when

(i) $\mathcal{K}(s)$ is meromorphic in the complex plane,
(ii) $\mathcal{K}(s)$ has real sequences of simple poles (its unique singularities) and simple zeros such that each of them has as unique adherent point $+\infty$ or $-\infty$. Moreover, the distance between two consecutive terms of said sequences is always greater than a positive number, and
(iii) $\mathcal{K}(s)$ satisfies the growth conditions in Theorems 1 or 2.

An example of this situation, the Krätzel integral transformation, is investigated in [3]. Other important integral transforms can be seen as special cases of this investigation: Wright transform [1], Struve transform [13] and Hardy transform [5] amongst others.

II. The procedure investigated here allows to study also integral transforms defined by

$$\mathcal{Y}(\phi)(x) = \int_0^\infty k\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t}\,,\ x \in \boldsymbol{I}. \tag{19}$$

By taking into account that under suitable conditions the formula

$$\mathcal{Y}(\phi) = \mathcal{M}^{-1} \circ \mathcal{T}_{\mathcal{K}} \circ L \circ \mathcal{M}(\phi)$$

holds, where L denotes the mapping defined by

$$L(\Phi)(s) = \Phi(1-s)$$

and $\mathcal{K} = \mathcal{M}(k)$, it is easy to establish results analogous to those proved in Theorems 1, 2 and 3 for the $\mathcal{Y}$-transforms. Note that L defines a homeomorphism from $\mathcal{B}(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty)$ onto $\mathcal{B}(\{1-b_n\}_{n=1}^\infty, \{1-a_n\}_{n=1}^\infty)$. Also, in this case it is possible to make the modifications commented in (I). Important examples of transformations of type (19) are the fractional integrals of Riemann-Liouville and Weyl [12].

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