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Fermat-Pell equation and the numbers of the form $w^2 + (w+1)^2$

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1. Introduction

In the present paper we obtain recursive formulae for the determination of all non-negative (that is $X \geq 0$ and $Y \geq 0$) integral solutions of

$$(F) \qquad X^2 - dY^2 = C \quad (d \neq \square,\ C \neq 0),$$

where $d \neq \square$ (non-square) is a natural number and C is an integer $\neq 0$ (Theorem 2.3 for $C > 0$ and Theorem 2.4 for $C < 0$ below). Also, we obtain same recursive formulae (Theorem 2.6 below).

The special case $d = 2$ and $C = 2k^2 - 1$, $k = 0, 1, 2, \dots,$ of (F) constitute the connecting link with the numbers of the form

$$N(w) \equiv w^2 + (w+1)^2$$

(for $w = (X-1)/2$, we have $N(w) = Y^2 + k^2$).

In a fortcoming paper these recursive formulae will be used in the special case $d = 2$ and $C = 2k^2 - 1$ for the complete determination of all composite numbers of the form $w^2 + (w+1)^2$.

The expression $x_1 + y_1\sqrt{d}$ will always denote the fundamental solution of

$$(P) \qquad x^2 - dy^2 = 1 \quad (d \neq \square).$$

Also, $x_n + y_n\sqrt{d}$ $n = 0, 1, \dots,$ will denote the sequece of all non-negative integral solutions of (P). These solutions are given in [3, p. 439] by the following recursive formulae:

$$x_{n+1} = 2x_1x_n - x_{n-1}, \quad \text{where} \quad x_0 = 1 \quad \text{and} \quad x_1 = x_1, \tag{1.1}$$

$$y_{n+1} = 2x_1y_n - y_{n-1}, \quad \text{where} \quad y_0 = 0 \quad \text{and} \quad y_1 = y_1, \tag{1.2}$$

Let G be the group of all integral solutions of (P). Let $Z \equiv X + Y\sqrt{d}$ be an integral solution of (F). Consider the class

$$A \equiv \{Zz \mid z \in G\}$$

of solutions of (F) represented by Z. Define

$$\bar{A} \equiv \{-\bar{Z}z \mid z \in G\}.$$

Then $\bar{A}$ constitutes a class of solutions of (F) represented by $-\bar{Z}$. This class $\bar{A}$ is called *conjugate* class of A. lf $A \neq \bar{A}$ then A is called *genuine* or *not ambiguous* class. If $A = \bar{A}$, then A is called *ambiguous* [cf. 2, p. 205].

Let $Z^* = X^* + Y^*\sqrt{d}$ be the *fundamental solution* (as defined in NAGELL in [2, p. 205]) of (F) belonging to the class A, then

$$A = \{Z^*z \mid z \in G\} \quad \text{and} \quad \bar{A} = \{-\bar{Z}^*z \mid z \in G\}.$$

Theorem 1.1. *The Diophantine equation (F) has a finite number of classes of solutions. The fundamental solutions of all such classes are determined by the following (equivalent) inequalities in case $C > 0$*

$$0 < |X^*| \leq \sqrt{(x_1+1)C/2}, \tag{1.3}$$

$$0 \leq Y^* \leq (y_1/\sqrt{2(x_1+1)}\,)\sqrt{C} \tag{1.4}$$

and by the followinq (equivalent) inequalities in case $C < 0$

$$0 \leq |X^*| \leq \sqrt{(x_1-1)(-C)/2}, \tag{1.5}$$

$$0 < Y^* \leq (y_1/\sqrt{2(x_1-1)}\,)\sqrt{(-C)}. \tag{1.6}$$

Moreover, A consists of all elements of the form

$$X + Y\sqrt{d} = (X^* + Y^*\sqrt{d})(x + y\sqrt{d}),$$

where $x + y\sqrt{d}$ ranges over the set of all integral solutions of (P).

The Diophantine equation (F) has no solution at all when it has no solution satisfying the inequalities (1.3) *and* (1.4) *or* (1.5) *and* (1.6) *respectively.*

PROOF. See Theorem 109, in [2] (cf. also [1], [4] and [5]).

In case $C > 0$ the recursive description of all non-negative integral solutions of (F) belonging to a class of solutions A, is given by Theorem 2.1.

Its proof is based on Proposition 1.2. In the sequel A will always denote an arbitrarily chosen fixious class of solutions of (F) and $X^* + Y^*\sqrt{d}$ its fundamental solution.

Proposition 1.2. *Consider the Diophantine equation (F), $C > 0$. Let A be a class of solutions with $X^* > 0$. Let*

$$X_n + Y_n\sqrt{d} \equiv (X^* + Y^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \text{for all} \quad n = 0, 1, \dots,$$
$$X'_n + Y'_n\sqrt{d} \equiv (X^* - Y^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \text{for all} \quad n = 1, 2, \dots .$$

Then the set of all non-negative integral solutions of (F) belonging to A consists of all pairs $(X_n,\ Y_n)$, while the set of all non-negative (positive) integral solutions of (F) belonging to $\bar{A}$ consists of all pairs $(X'_n,\ Y'_n)$.

Proof. By Theorem 1.1 the class A consists of all elements having one of the following typical forms:

$$\begin{aligned}
(X^* + Y^*\sqrt{d})(x_n + y_n\sqrt{d}) &= (x_nX^* + dy_nY^*) + (y_nX^* + x_nY^*)\sqrt{d} \\
&\equiv X_n + Y_n\sqrt{d}, \\
(X^* + Y^*\sqrt{d})(-x_n - y_n\sqrt{d}) &= -X_n - Y_n\sqrt{d}, \\
(X^* + Y^*\sqrt{d})(-x_n + y_n\sqrt{d}) &= -(x_nX^* - dy_nY^*) + (y_nX^* - x_nY^*)\sqrt{d} \\
&\equiv -X'_n + Y'_n\sqrt{d}, \\
(X^* + Y^*\sqrt{d})(x_n - y_n\sqrt{d}) &= X'_n - Y'_n\sqrt{d}.
\end{aligned}$$

Also, $\bar{A}$ consists of all elements having one of the following typical forms:

$$\begin{aligned}
(-X^* + Y^*\sqrt{d})(x_n + y_n\sqrt{d}) &= -X'_n - Y'_n\sqrt{d}, \\
(-X^* + Y^*\sqrt{d})(-x_n - y_n\sqrt{d}) &= X'_n + Y'_n\sqrt{d}, \\
(-X^* + Y^*\sqrt{d})(-x_n + y_n\sqrt{d}) &= X_n - Y_n\sqrt{d}, \\
(-X^* + Y^*\sqrt{d})(x_n - y_n\sqrt{d}) &= -X_n + Y_n\sqrt{d}.
\end{aligned}$$

The following hold true:

$$X_n = x_nX^* + dy_nY^* > 0, \tag{1.7}$$

$$Y_n = y_nX^* + x_nY^* \geq 0, \tag{1.8}$$

$$X'_n = x_nX^* - dy_nY^* > 0. \tag{1.9}$$

The last equality holds true because $x_n > y_n\sqrt{d}$ and $X^* > Y^*\sqrt{d}$.
It will be proved that:

$$Y'_n = y_nX^* - x_nY^* > 0 \quad \text{for every} \quad n = 1, 2, \dots . \tag{1.10}$$

In fact, by (1.4) we deduce that

$$Y^{*^2} \le (y_1^2 C)/(2(x_1+1)) < y_1^2 C \le y_n^2 (X^{*^2} - dY^{*^2}) \quad \text{for every} \quad n \ge 1,$$

that is

$$(y_n X^*)^2 - (x_n Y^*)^2 > 0.$$

Hence

$$y_n X^* - x_n Y^* > 0, \qquad \text{that is} \qquad Y'_n > 0.$$

From (1.7), (1.8), (1.9) and (1.10) follows the desired conclusion.

In the sequel X_n, X'_n, Y_n, Y'_n will have the same meaning as in Proposition 1.2.

2. Study of the generalized Fermat equation

Theorem 2.1. *Consider the Diophantine equation (F), $C > 0$. Let A be a class of solutions with $X^* > 0$. Then the sequence of all non-negative integral solutions of (F) belonging to A is determined by the following recursive formulae:*

$$(2.1) \qquad X_{n+1} = 2x_1 X_n - X_{n-1}, \text{ where } X_0 = X^* \text{ and } X_1 = x_1 X^* + dy_1 Y^*,$$

$$(2.2) \qquad Y_{n+1} = 2x_1 Y_n - Y_{n-1}, \quad \text{where } Y_0 = Y^* \text{ and } Y_1 = y_1 X^* + x_1 Y^*.$$

Also, the sequence of all non-negative (positive) integral solutions of (F) belonging to $\bar{A}$ is determined by the following recursive formulae:

$$(2.3) \qquad X'_{n+1} = 2x_1 X'_n - X'_{n-1}, \text{ where } X'_0 = X^* \text{ and } X'_1 = x_1 X^* - dy_1 Y^*,$$

$$(2.4) \qquad Y'_{n+1} = 2x_1 Y'_n - Y'_{n-1}, \quad \text{where } Y'_0 = -Y^* \text{ and } Y'_1 = y_1 X^* - x_1 Y^*.$$

Proof. It is easily seen, because of Proposition 1.2, that the non-negative solutions of A and $\overline{A}$ satisfy the recursive formulae (2.1), (2.2) and (2.3), (2.4) respectively. We now use Proposition 1.2 to prove the reverse side of the theorem. It will be proved that

$$(2.5) \qquad X_n + Y_n\sqrt{d} = (X^* + Y^*\sqrt{d})(x_n + y_n\sqrt{d}) = (x_n X^* + dy_n Y^*) + (y_n X^* + x_n Y^*)\sqrt{d}$$

for all $n = 0, 1, \ldots$.

Clearly (2.5) is true for $n = 0, 1$. Suppose that (2.5) holds true for every index less than $n+1$. (Induction hypothesis). It will be proved that

(2.5) holds true for $n+1$. In fact;

$$\begin{aligned} X_{n+1}+Y_{n+1}\sqrt{d} &= 2x_1X_n - X_{n-1} + (2x_1Y_n - Y_{n-1})\sqrt{d} \\ &= (X^*+Y^*\sqrt{d})(x_n+y_n\sqrt{d})(2x_1-(x_1-y_1\sqrt{d})) \\ &= (X^*+Y^*\sqrt{d})(x_{n+1}+y_{n+1}\sqrt{d}). \end{aligned}$$

Evidently $X_n > 0$ and $Y_n \geq 0$. Hence, every pair $(X_n,\ Y_n)$ is a non-negative integral solution of (F) belonging to A.

In a similar way to the proof of (2.5) it can be proved that

$$X'_n + Y'_n\sqrt{d} = (X^* - Y^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \text{for all} \quad n = 1, 2, \ldots .$$

Furthermore, by (1.9) and (1.10), we deduce that $X'_n > 0$ and $Y'_n > 0$ for all $n = 1, 2, \ldots$. Hence, every pair $(X'_n,\ Y'_n)$ is a non-negative (positive) integral solution of (F) belonging to $\bar{A}$.

The set of *all* non-negative integral solutions of (F), for $C > 0$, is determined in Theorem 2.3 whose proof is based (inter alia) on Proposition 2.2. A similar determination for $C < 0$ is described in Theorem 2.4.

Proposition 2.2. *Consider the Diophantine equation* (F), $C > 0$. *Let* A *be a class of solutions with* $X^* > 0$. *Then the following hold true:*

(i) $Y_{n+1} > Y_n \geq 0$ *for every* $n = 0, 1, \ldots$.

(ii) Let $Y^* > 0$. *Then* $Y'_{n+1} \geq Y_n > Y'_n > 0$ *for every* $n = 1, 2, \ldots$.

(iii) Let $Y^* = 0$. *Then* $Y_n = Y'_n$ *for every* $n = 0, 1, \ldots$.

(iv) Let A *be genuine. Then*

$$Y'_{n+1} > Y_n > Y'_n > 0 \quad \textit{for all} \quad n = 1, 2, \ldots .$$

(v) Let A *be ambiguous. Then for every* m *there exists* n *such that:*

$$X'_m = X_n \quad \textit{and} \quad Y'_m = Y_n.$$

Proof. i)

$$\begin{gathered} Y_{n+1} = y_{n+1}X^* + x_{n+1}Y^* = (x_1y_n + x_ny_1)X^* + (x_1x_n + dy_1y_n)Y^* \\ = y_n(x_1X^* + dy_1Y^*) + x_n(y_1X^* + x_1Y^*) > y_nX^* + x_nY^* = Y_n \geq 0, \\ \text{that is} \quad Y_{n+1} > Y_n \geq 0 \quad \text{for every } n = 0, 1, \ldots . \end{gathered}$$

ii)

$$Y'_{n+1} = y_n(x_1X^* - dy_1Y^*) + x_n(y_1X^* - x_1Y^*). \tag{2.6}$$

Also, $-X^* + Y^*\sqrt{d}$ is the fundamental solution of $\bar{A}$, while,

$$x_1X^* - dy_1Y^* + (y_1X^* - x_1Y^*)\sqrt{d} = (-X^* + Y^*\sqrt{d})(-x_1 - y_1\sqrt{d})$$

is (by (1.9) and (1.10)) a positive integral solution of (F) belonging to $\bar{A}$. Hence, by Definition of fundamental solution, we obtain:

$$(2.7) \quad y_1X^* - x_1Y^* \geq Y^* \quad \text{(and equivalently} \quad x_1X^* - dy_1Y^* \geq X^*).$$

From (2.6) and (2.7) we deduce:

$$Y'_{n+1} \geq Y_n > Y'_n > 0 \quad \text{for every} \quad n = 1, 2, \dots .$$

iii) By the definition of Y_n and Y'_n.
iv) It will be proved that

$$(2.8) \qquad x_1X^* - dy_1Y^* > X^* \quad \text{and} \quad y_1X^* - x_1Y^* > Y^*.$$

In fact; by (2.7) we have:

$$x_1X^* - dy_1Y^* \geq X^* > 0 \quad \text{and} \quad y_1X^* - x_1Y^* \geq Y^* \geq 0.$$

Assume that (2.8) is not true. Then

$$x_1X^* - dy_1Y^* = X^* \quad \text{and} \quad y_1X^* - x_1Y^* = Y^*,$$

i.e.
$$(X^* - Y^*\sqrt{d})(x_1 + y_1\sqrt{d}) = X^* + Y^*\sqrt{d},$$

which condradicts the assumption, because A is genuine. Hence (2.8) holds true. Thus by (2.6) we obtain:

$$Y'_{n+1} > Y_n > Y'_n > 0.$$

v) Evident by the assumption $A = \bar{A}$.

Theorem 2.3. *Consider the Diophantine equation (F), $C > 0$. Let $X_r^* + Y_r^*\sqrt{d}$, where $r = 1, 2, \dots, m$, be the only integral solutions of (F) such that:*

$$0 < X_r^* \leq \sqrt{(x_1+1)C/2} \quad \textit{and} \quad 0 \leq Y_r^* \leq y_1\sqrt{C}/\sqrt{2(x_1+1)}.$$

Let

$$X_n + Y_n\sqrt{d} \equiv (X_r^* + Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \textit{for all} \quad n = 0, 1, \dots ,$$
$$X'_n + Y'_n\sqrt{d} \equiv (X_r^* - Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \textit{for all} \quad n = 1, 2, \dots .$$
(For a typical r).

Then the set of all non-negative integral solutions of (F) consists of all pairs $(X_n,\ Y_n)$ together with all pairs $(X'_n,\ Y'_n)$ for all respective genuine classes A_r in addition to all pairs $(X_n,\ Y_n)$ for all respective ambiguous *classes B_r. Moreover, X_n, Y_n, X'_n and Y'_n are determined by the following recursive formulae:*

$$\begin{aligned} X_{n+1} &= 2x_1X_n - X_{n-1} \quad \text{for} \quad n = 1, 2, \dots \quad \text{with} \\ &\qquad X_0 = X_r^*,\ X_1 = x_1X_r^* + dy_1Y_r^* \quad \text{and} \quad r = 1, 2, \dots, m. \\ Y_{n+1} &= 2x_1Y_n - Y_{n-1} \quad \text{for} \quad n = 1, 2, \dots \quad \text{with} \\ &\qquad Y_0 = Y_r^*,\ Y_1 = y_1X_r^* + x_1Y_r^* \quad \text{and} \quad r = 1, 2, \dots, m. \\ X'_{n+1} &= 2x_1X'_n - X'_{n-1} \quad \text{for} \quad n = 1, 2, \dots \quad \text{with} \\ &\qquad X'_0 = X_r^*,\ X'_1 = x_1X_r^* - dy_1Y_r^* \quad \text{and} \quad r = 1, 2, \dots, m. \\ Y'_{n+1} &= 2x_1Y'_n - Y'_{n-1} \quad \text{for} \quad n = 1, 2, \dots \quad \text{with} \\ &\qquad Y'_0 = -Y_r^*,\ Y'_1 = y_1X_r^* - x_1Y_r^* \quad \text{and} \quad r = 1, 2, \dots, m. \end{aligned}$$

PROOF. By using Proposition 2.2 and Theorems 1.1 and 2.1.

Theorem 2.4. *Consider the Diophantine equation (F), $C < 0$. Let $X_r^* + Y_r^*\sqrt{d}$, where $r = 1, 2, \dots, m$, be the only integral solutions of (F) such that:*

$$0 \le X_r^* \le \sqrt{(x_1 - 1)(-C)/2} \quad \text{and} \quad 0 < Y_r^* \le y_1\sqrt{(-C)}/\sqrt{2(x_1 - 1)}.$$

Let

$$\begin{aligned} X_n + Y_n\sqrt{d} &\equiv (X_r^* + Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \text{for all} \quad n = 0, 1, \dots, \\ X''_n + Y''_n\sqrt{d} &\equiv (-X_r^* + Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \\ &\qquad \text{for all} \quad n = 1, 2, \dots. \quad \text{(For a typical } r\text{)} \end{aligned}$$

Then the set of all non-negative integral solutions of (F) consists of all pairs $(X_n,\ Y_n)$ together with all pairs $(X''_n,\ Y''_n)$ for all respective genuine classes A_r in addition to all pairs $(X_n,\ Y_n)$ for all respective ambiguous classes B_r. Moreover, X_n, Y_n, X''_n and Y''_n are determined by the following recursive formulae:

$$\begin{aligned} X_{n+1} &= 2x_1X_n - X_{n-1} \quad \text{for} \quad n = 1, 2, \dots \quad \text{with} \\ &\qquad X_0 = X_r^*,\ X_1 = x_1X_r^* + dy_1Y_r^* \quad \text{and} \quad r = 1, 2, \dots, m. \\ Y_{n+1} &= 2x_1Y_n - Y_{n-1} \quad \text{for} \quad n = 1, 2, \dots \quad \text{with} \\ &\qquad Y_0 = Y_r^*,\ Y_1 = y_1X_r^* + x_1Y_r^* \quad \text{and} \quad r = 1, 2, \dots, m. \\ X''_{n+1} &= 2x_1X''_n - X''_{n-1} \quad \text{for} \quad n = 1, 2, \dots \quad \text{with} \\ &\qquad X''_0 = -X_r^*,\ X''_1 = -x_1X_r^* + dy_1Y_r^* \quad \text{and} \quad r = 1, 2, \dots, m. \end{aligned}$$

$$Y''_{n+1} = 2x_1 Y''_n - Y''_{n-1} \quad \textit{for} \quad n = 1, 2, \ldots \quad \textit{with}$$
$$Y''_0 = Y^*_r,\ Y''_1 = -y_1 X^*_r + x_1 Y^*_r \quad \textit{and} \quad r = 1, 2, \ldots, m.$$

PROOF. Similar to the proof of Theorem 2.3.

Our next Theorem 2.5 provides a recursive determination of all Y^2 for the elements $X + Y\sqrt{d}$ comprising the set of all *absolutely* distinct solutions of a class of (F). [Any two solutions $X + Y\sqrt{d}$ and $X' + Y'\sqrt{d}$ of (F) are considered as *absolutely* the same whenever $|X| = |X'|$ and $|Y| = |Y'|$]. A similar recursive determination of all $Y^2 + k^2$, for a fixed integer k (and Y etc. as above) is provided by Theorem 2.6 whose proof is a direct consequence of that of Theorem 2.5.

Theorem 2.5. *Consider the Diophantine equation (F). Let*

$$X_n + Y_n\sqrt{d} \equiv (X^* + Y^*\sqrt{d})(x_1 + y_1\sqrt{d})^n,$$
$$X'_n + Y'_n\sqrt{d} \equiv (X^* - Y^*\sqrt{d})(x_1 + y_1\sqrt{d})^n \quad \textit{for all} \quad n = 0, 1, \ldots.$$

Let $P_n \equiv Y_n^2$ and $P'_n \equiv {Y'_n}^2$ for all $n = 0, 1, \ldots$. Then the numbers P_n, P'_n are determined by the following recursive formulae:

$$P_{n+1} = 2x_2 P_n - P_{n-1} + 2y_1^2 C, \ \textit{where } P_0 = {Y^*}^2 \textit{ and} \quad (2.9)$$
$$P_1 = (x_1 Y^* + y_1 X^*)^2,$$

$$P'_{n+1} = 2x_2 P'_n - P'_{n-1} + 2y_1^2 C, \ \textit{where} \quad (2.10)$$
$$P'_0 = {Y^*}^2 \textit{ and } P'_1 = (y_1 X^* - x_1 Y^*)^2.$$

PROOF. First we prove that the numbers P_n, P'_n satisfy the above mentioned recursive formulae. Let $Z^* = X^* + Y^*\sqrt{d}$, $z_n = (x_1 + y_1\sqrt{d})^n = z_1^n = x_n + y_n\sqrt{d}$, $Z_n = Z^* z_n$ and $Z'_n = \bar{Z}^* z_n$.

The following hold true:

$$Z_n^2 = {Z^*}^2 z_1^{2n} = {Z^*}^2 z_{2n} \quad \text{and} \quad {Z^*}^2 = {X^*}^2 + d{Y^*}^2 + 2X^*Y^*\sqrt{d}.$$

Let $X_2^* \equiv {X^*}^2 + d{Y^*}^2$ and $Y_2^* \equiv 2X^*Y^*$. Then

$$(X_n + Y_n\sqrt{d})^2 = (X_2^* + Y_2^*\sqrt{d})(x_{2n} + y_{2n}\sqrt{d}).$$

Hence

$$X_n^2 + dY_n^2 = X_2^* x_{2n} + dY_2^* y_{2n}. \quad (2.11)$$

Also,

$$X_n^2 - dY_n^2 = C.$$

Therefore

$$2dY_n^2 = X_2^* x_{2n} + dY_2^* y_{2n} - C.$$

Let

(2.12) $$Q_{2n} \equiv X_2^* x_{2n} + dY_2^* y_{2n}.$$

But $P_n = Y_n^2$, then

(2.13) $$Q_{2n} = 2dP_n + C.$$

Also,

$$z_{m+2} = z_m z_2 \quad \text{and} \quad z_{m-2} = z_m \bar{z}_2.$$

Hence we deduce:

(2.14) $$x_{m+2} = 2x_2 x_m - x_{m-2} \quad \text{and} \quad y_{m+2} = 2x_2 y_m - y_{m-2}.$$

From (2.12) and (2.14) we obtain:

(2.15) $$Q_{2n+2} = 2x_2 Q_{2n} - Q_{2n-2}.$$

By (2.13) we have:

$$2x_2 Q_{2n} - Q_{2(n-1)} = 2dP_{n+1} + C,$$

that is

$$2x_2(2dP_n + C) - 2dP_{n-1} - C = 2dP_{n+1} + C,$$

and so

$$P_{n+1} = 2x_2 P_n - P_{n-1} + C(x_2 - 1)/d.$$

Also $x_2 = x_1^2 + dy_1^2$, that is $(x_2 - 1)/d = 2y_1^2$. Hence

$$P_{n+1} = 2x_2 P_n - P_{n-1} + 2Cy_1^2.$$

In a similar way as above we deduce:

$$P'_{n+1} = 2x_2 P'_n - P'_{n-1} + 2Cy_1^2.$$

Also, the initial conditions $P_0 = {Y^*}^2$ etc are proved directly by the definitions of P_n and P'_n for $n = 0, 1$.

Consider now the sequences P_n, P'_n defined by (2.9) and (2.10). We shall prove that $P_n = Y_n^2$ and $P'_n = {Y'}_n^2$. Clearly

(2.16) $$P_n = Y_n^2$$

is true for $n = 0, 1$. Suppose that (2.16) holds true for every index less than $n+1$. (Induction hypothesis). It will be proved that (2.16) holds true for $n+1$. In fact;

$$2y_1^2 = (x_2 - 1)/d,$$

hence

$$2dP_{n+1} = 2x_2 2dP_n - 2dP_{n-1} + 2(x_2 - 1)C.$$

Hence, by the induction hypothesis, we have

$$2dP_{n+1} + C = 2x_2(X_n^2 + dY_n^2) - (X_{n-1}^2 + dY_{n-1}^2). \tag{2.17}$$

The following holds true:

$$x_{2n+2} = 2x_2 x_{2n} - x_{2n-2} \quad \text{and} \quad y_{2n+2} = 2x_2 y_{2n} - y_{2n-2} \tag{2.18}$$

From (2.11), (2.17) and (2.18) we obtain:

$$2dP_{n+1} + C = X_{n+1}^2 + dY_{n+1}^2.$$

Thus we deduce:

$$P_{n+1} = Y_{n+1}^2.$$

In a similar way as above we deduce that:

$$P'_n = {Y'}_n^2 \quad \text{for every } n = 0, 1, \dots .$$

Theorem 2.6. *Consider the Diophantine equation* (F). *Let* $R_n \equiv Y_n^2 + k^2$ *and* $R'_n \equiv {Y'_n}^2 + k^2$, *where* k *is a fixed integer. Then the numbers* R_n, R'_n *are determined by the following recursive formulae:*

$$R_{n+1} = 2x_2 R_n - R_{n-1} - 2k^2(x_2 - 1) + 2y_1^2 C,$$

where $R_0 = {Y^*}^2 + k^2$ *and* $R_1 = (y_1 X^* + x_1 Y^*)^2 + k^2$.

$$R'_{n+1} = 2x_2 R'_n - R'_{n-1} - 2k^2(x_2 - 1) + 2y_1^2 C,$$

where $R'_0 = {Y^*}^2 + k^2$ *and* $R'_1 = (y_1 X^* - x_1 Y^*)^2 + k^2$.

Proof. It is actually a direct consequence of the proof of Theorem 2.5.

3. An application of Theorem 2.6

A special case of Theorem 2.6 ($d = 2$ and $C = 2k^2 - 1$, $k = 0, 1, 2, \dots$) is the following

Theorem 3.1. *The Diophantine equation*

$$(F_k) \qquad X^2 - 2Y^2 = 2k^2 - 1, \quad \text{where } k = 0, 1, \dots .$$

has at least one class of solutions A. Moreover, if $R_n \equiv Y_n^2 + k^2$ and $R'_n \equiv {Y'}_n^2 + k^2$, then the numbers R_n, R'_n are determined by the following recursive formulae:

$$(3.1) \qquad R_{n+1} = 34R_n - R_{n-1} - 8(2k^2 + 1) \quad \text{for all} \quad n = 1, 2, \dots$$

with $R_0 = {Y^*}^2 + k^2$ and $R_1 = (2X^* + 3Y^*)^2 + k^2$.

$$(3.2) \qquad R'_{n+1} = 34R'_n - R'_{n-1} - 8(2k^2 + 1) \quad \text{for all} \quad n = 1, 2, \dots$$

with $R'_0 = {Y^*}^2 + k^2$ and $R'_1 = (2X^* - 3Y^*)^2 + k^2$.

Proof. It suffices to prove the existence of the class A. The other assertions are evident by Theorem 2.6, since $3 + 2\sqrt{2}$ is the fundamental solution of $x^2 - 2y^2 = 1$.

The fundamental solution of (F_0) is $1 + \sqrt{2}$. Also $2k - 1 + (k - 1)\sqrt{2}$ is a solution of (F_k) for $k = 1, 2, \dots$. In fact it is the fundamental solution of its class, since satisfies the inequalities (1.3) and (1.4). This proves the Theorem.

Let $X + Y\sqrt{2}$ be a non-negative integral solution of (F_k) (see Theorem 2.3 for $k \geq 1$ or Theorem 2.4 for $k = 0$). Hence, we have $X^2 = 2(Y^2 + k^2) - 1 \geq 1$ and so X is an old natural number. In case $X + Y\sqrt{2}$ is the fundamental solution of (F_0) or (F_1) we have $X = 1$. We set $N(w) \equiv w^2 + (w + 1)^2$; If $w = (X - 1)/2$ [w is an integer > 0 if $X + Y\sqrt{2}$ is not the fundamental solution of (F_0) or (F_1)] it follows that $N(w) = Y^2 + k^2$. Hence, by Theorem 3.1 the numbers R_n, R'_n, $n = 1, 2, \dots$ [see (3.1) and (3.2)] are of the form $w^2 + (w + 1)^2$.

Example. We consider the Diophantine equation

$$(F_0) \qquad X^2 - 2Y^2 = -1.$$

From Theorems 2.4 and 3.1 we obtain: $X^* + Y^*\sqrt{2} = 1 + \sqrt{2}$ and

$$R_{n+1} = 34R_n - R_{n-1} - 8 = Y_{n+1}^2, \quad n = 1, 2, \dots \quad \text{with}$$
$$R_0 = 1 \text{ and } R_1 = 25.$$

[It follows that $R_1 = 25$, $R_2 = 841$, $R_3 = 28561$, $R_4 = 970225$, $R_5 = 32959081$, $R_6 = 1119638521$, $R_7 = 38034750625$, $R_8 = 1292061882721$, $R_9 = 43892069261881, \dots$]

The numbers $R_n = 1, 2, \dots,$ are square (composite) numbers of the form $w^2 + (w+1)^2$.

Remark. Let $X^* + Y^*\sqrt{2}$ be the fundamental solution of a class A of integral solutions of (F_k), with $X^* > 0$. If A is genuine, then (by Proposition 2.2, (iv) and Theorem 3.1) $R'_n < R_n < R'_{n+1}$ for all $n = 1, 2, \dots$. But if A is ambiguous, then for every m there exists n such that $R'_m = R_n$.

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